Dynamic Optimisation: Introduction to Optimal Control

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Summer, 2011
Review of the Theory of Optimal Control I

- Review of basic results in dynamic optimization in continuous time—particularly the *optimal control* approach.
- New mathematical issues: even with a finite horizon, the maximization is with respect to an infinite-dimensional object (an entire function, \( y : [t_0, t_1] \to \mathbb{R} \)).
- Requires brief review of *calculus of variations* and from the theory of optimal control.
- Canonical problem

\[
\max_{x(t), y(t)} W(x(t), y(t)) \equiv \int_0^{t_1} f(t, x(t), y(t)) \, dt
\]

subject to

\[
\dot{x}(t) = g(t, x(t), y(t))
\]

and

\[
y(t) \in \mathcal{Y}(t) \text{ for all } t, \quad x(0) = x_0,
\]
For each $t$, $x(t)$ and $y(t)$ are finite-dimensional vectors (i.e., $x(t) \in \mathbb{R}^{K_x}$ and $y(t) \in \mathbb{R}^{K_y}$, where $K_x$ and $K_y$ are integers).

Refer to $x$ as the state variable, governed by a vector-valued differential equation given behavior of control variables $y(t)$.

End of planning horizon $t_1$ can be infinity.

Function $W(x(t), y(t))$: objective function when controls are $y(t)$ and resulting state variable is summarized by $x(t)$.

Refer to $f$ as the objective function (or the payoff function) and to $g$ as the constraint function.
Consider the following finite-horizon continuous time problem

\[
\max_{x(t), y(t), x_1} \mathcal{W}(x(t), y(t)) \equiv \int_0^{t_1} f(t, x(t), y(t)) \, dt
\]  

subject to

\[
\dot{x}(t) = g(t, x(t), y(t))
\]

and

\[
y(t) \in \mathcal{Y}(t) \text{ for all } t, \quad x(0) = x_0 \text{ and } x(t_1) = x_1.
\]

- \(x(t) \in \mathbb{R}\) is one-dimensional and its behavior is governed by the differential equation (2).
- \(y(t)\) must belong to the set \(\mathcal{Y}(t) \subset \mathbb{R}\).
- \(\mathcal{Y}(t)\) is nonempty and convex.
- Refer to \((x(t), y(t))\) that jointly satisfy (2) and (3) as an *admissible pair*. 
Suppose that \( W(x(t), y(t)) < \infty \) for any admissible pair \((x(t), y(t))\) and that \( t_1 < \infty \).

There is also a terminal value constraint \( x(t_1) = x_1 \), but \( x_1 \) is included as an additional choice variable, is free.

Assume that \( f \) and \( g \) are continuously differentiable functions.

Difficulty lies in two features:

1. Choosing a function \( y : [0, t_1] \rightarrow \mathcal{Y} \) rather than a vector or a finite dimensional object.
2. Constraint is a differential equation, rather than a set of inequalities or equalities.

Hard to know what type of optimal policy to look for: \( y \) may be highly discontinuous function, or hit the boundary.
Variational Arguments III

- In economic problems impose structure to make solutions continuous and Inada conditions ensure they lie in the interior.
- When $y$ is a continuous function of time and lies in the interior of the feasible set: can use variational arguments.
- **Variational principle**: start assuming a continuous solution (function) $\hat{y}$ that lies everywhere in the interior of the set $\mathcal{Y}$ exists.
- Assume $(\hat{x}(t), \hat{y}(t))$ is an admissible pair such that $\hat{y}(\cdot)$ is continuous over $[0, t_1]$ and $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$, and

$$W(\hat{x}(t), \hat{y}(t)) \geq W(x(t), y(t))$$

for any other admissible pair $(x(t), y(t))$.
- Note $x$ is given by (2), so when $y(t)$ is continuous, $\dot{x}(t)$ will also be continuous, so $x(t)$ is continuously differentiable.
- When $y(t)$ is piecewise continuous, $x(t)$ will be, correspondingly, piecewise smooth.
The Hamiltonian and the Maximum Principle I

- More economical way of expressing the Theorem above is to construct the Hamiltonian:

\[ H(t, x, y, \lambda) \equiv f(t, x(t), y(t)) + \lambda(t)g(t, x(t), y(t)). \]  \hspace{1cm} (4)

- Since \( f \) and \( g \) are continuously differentiable, so is \( H \).
Theorem: Maximum Principle

Consider the problem of maximizing (1) subject to (2) and (3), with $f$ and $g$ continuously differentiable. Suppose this problem has an interior continuous solution $\hat{y}(t) \in \text{Int} \mathcal{Y}(t)$ with state variable $\hat{x}(t)$. Then there exists a continuously differentiable function $\lambda(t)$ such that $\hat{y}(t)$ and $\hat{x}(t)$ satisfy the necessary conditions:

\[ x(0) = x_0, \]
\[ H_y(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0 \text{ for all } t \in [0, t_1], \]  
\[ \hat{\lambda}(t) = -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \text{ for all } t \in [0, t_1], \]  
\[ \hat{x}(t) = H_{\lambda}(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \text{ for all } t \in [0, t_1], \]

and $\lambda(t_1) = 0$, with the Hamiltonian $H(t, x, y, \lambda)$ given by (4). Moreover, the Hamiltonian $H(t, x, y, \lambda)$ also satisfies the Maximum Principle that

\[ H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y, \lambda(t)) \text{ for all } y \in \mathcal{Y}(t), \]

for all $t \in [0, t_1]$. 

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For notational simplicity, in equation (7), $\dot{x}(t)$ instead of $\dot{\hat{x}}(t)$ ($= \partial \hat{x}(t) / \partial t$).

Simplified version of the celebrated *Maximum Principle* of Pontryagin:

1. Find optimal solution by looking jointly for a set of “multipliers” (costate variables) $\lambda(t)$ and optimal path of $\hat{y}(t)$ and $\hat{x}(t)$.
2. $\lambda(t)$ is informative about the value of relaxing the constraint (at time $t$): value of an infinitesimal increase in $x(t)$ at time $t$.
3. $\lambda(t_1) = 0$: after the planning horizon, there is no value to having more $x$. Finite-horizon equivalent of *transversality condition*.

Conditions may not be sufficient:

1. May correspond to stationary points rather than maxima.
2. May identify a local rather than a global maximum.
The Hamiltonian and the Maximum Principle III

**Theorem** (Mangasarian’s Sufficient Conditions) Consider the problem of maximizing (1) subject to (2) and (3), with \( f \) and \( g \) continuously differentiable. Define \( H(t, x, y, \lambda) \) as in (4), and suppose that an interior continuous solution \( \hat{y}(t) \in \text{Int} \mathcal{Y}(t) \) and the corresponding path of state variable \( \hat{x}(t) \) satisfy (5)-(7). Suppose also that given the resulting costate variable \( \lambda(t) \), \( H(t, x, y, \lambda) \) is jointly concave in \((x, y)\) for all \( t \in [0, t_1] \), then the \( \hat{y}(t) \) and the corresponding \( \hat{x}(t) \) achieve a global maximum of (1). Moreover, if \( H(t, x, y, \lambda) \) is strictly jointly concave in \((x, y)\) for all \( t \in [0, t_1] \), then the pair \((\hat{x}(t), \hat{y}(t))\) achieves the unique global maximum of (1).
Condition that \( H(t, x, y, \lambda) \) should be concave is rather demanding.

Arrow’s Theorem weakens these conditions.

Define the maximized Hamiltonian as

\[
M(t, x, \lambda) \equiv \max_{y \in \mathcal{Y}(t)} H(t, x, y, \lambda),
\]

with \( H(t, x, y, \lambda) \) itself defined as in (4).

Clearly, the necessary conditions for an interior maximum in (8) is (5).

Therefore, an interior pair of state and control variables \((\hat{x}(t), \hat{y}(t))\) satisfies (5)-(7), then \( M(t, \hat{x}, \lambda) \equiv H(t, \hat{x}, \hat{y}, \lambda) \).
Theorem (Arrow’s Sufficient Conditions) Consider the problem of maximizing (1) subject to (2) and (3), with \( f \) and \( g \) continuously differentiable. Define \( H(t, x, y, \lambda) \) as in \((4)\), and suppose that an interior continuous solution \( \hat{y}(t) \in \text{Int} \mathcal{Y}(t) \) and the corresponding path of state variable \( \hat{x}(t) \) satisfy \((5)-(7)\). Given the resulting costate variable \( \lambda(t) \), define \( M(t, \hat{x}, \lambda) \) as the maximized Hamiltonian as in \((8)\). If \( M(t, \hat{x}, \lambda) \) is concave in \( x \) for all \( t \in [0, t_1] \), then \( \hat{y}(t) \) and the corresponding \( \hat{x}(t) \) achieve a global maximum of \((1)\). Moreover, if \( M(t, \hat{x}, \lambda) \) is strictly concave in \( x \) for all \( t \in [0, t_1] \), then the pair \( (\hat{x}(t), \hat{y}(t)) \) achieves the unique global maximum of \((1)\) and \((\hat{x}(t), \hat{y}(t)) \) are uniquely defined.
Consider the pair of state and control variables \((\hat{x}(t), \hat{y}(t))\) that satisfy the necessary conditions (5)-(7) as well as (2) and (3).

Consider also an arbitrary pair \((x(t), y(t))\) that satisfy (2) and (3) and define \(M(t, x, \lambda) \equiv \max_y H(t, x, y, \lambda)\).

Since \(f\) and \(g\) are differentiable, \(H\) and \(M\) are also differentiable in \(x\).

Denote the derivative of \(M\) with respect to \(x\) by \(M_x\).

Then concavity implies that for all \(t \in [0, t_1]\),

\[
M(t, x(t), \lambda(t)) \leq M(t, \hat{x}(t), \lambda(t)) + M_x(t, \hat{x}(t), \lambda(t)) (x(t) - \hat{x}(t))
\]
Proof of Theorem: Arrow’s Sufficient Conditions II

- Integrating both sides over \([0, t_1]\) yields

\[
\int_0^{t_1} M(t, x(t), \lambda(t)) \, dt \leq \int_0^{t_1} M(t, \hat{x}(t), \lambda(t)) \, dt \\
+ \int_0^{t_1} M_x(t, \hat{x}(t), \lambda(t))(x(t) - \hat{x}(t)) \, dt
\]  

(9)

- Moreover, we have

\[
M_x(t, \hat{x}(t), \lambda(t)) = H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = -\dot{\lambda}(t)
\]  

(10)

- First line follows by an Envelope Theorem type reasoning (since \(H_y = 0\) from equation (5)), while the second line follows from (7).
Next, exploiting the definition of the maximized Hamiltonian, we have

\[
\begin{align*}
\int_0^{t_1} M(t, x(t), \lambda(t)) \, dt &= W(x(t), y(t)) \\
&\quad + \int_0^{t_1} \lambda(t) g(t, x(t), y(t)) \, dt,
\end{align*}
\]

and

\[
\begin{align*}
\int_0^{t_1} M(t, \hat{x}(t), \lambda(t)) \, dt &= W(\hat{x}(t), \hat{y}(t)) \\
&\quad + \int_0^{t_1} \lambda(t) g(t, \hat{x}(t), \hat{y}(t)) \, dt.
\end{align*}
\]
Proof of Theorem: Arrow’s Sufficient Conditions IV

Equation (9) together with (10) then implies

\[ W(x(t), y(t)) \leq W(\hat{x}(t), \hat{y}(t)) \tag{11} \]

\[
+ \int_0^{t_1} \lambda(t) \left[ \begin{array}{c} g(t, \hat{x}(t), \hat{y}(t)) \\ -g(t, x(t), y(t)) \end{array} \right] dt
\]

\[- \int_0^{t_1} \dot{\lambda}(t) (x(t) - \hat{x}(t)) dt.\]

Integrating the last term by parts and using the fact that by feasibility \( x(0) = \hat{x}(0) = x_0 \) and by the transversality condition \( \lambda(t_1) = 0 \), we obtain

\[
\int_0^{t_1} \dot{\lambda}(t) (x(t) - \hat{x}(t)) dt = -\int_0^{t_1} \lambda(t) \left( \dot{x}(t) - \dot{\hat{x}}(t) \right) dt.
\]
Proof of Theorem: Arrow’s Sufficient Conditions V

Substituting this into (11), we obtain

\[ W(x(t), y(t)) \leq W(\hat{x}(t), \hat{y}(t)) \]

\[ + \int_0^{t_1} \lambda(t) \left[ g(t, \hat{x}(t), \hat{y}(t)) - g(t, x(t), y(t)) \right] dt \]

\[ + \int_0^{t_1} \lambda(t) [\dot{x}(t) - \dot{\hat{x}}(t)] dt. \]

Since by definition of the admissible pairs \((x(t), y(t))\) and \((\hat{x}(t), \hat{y}(t))\), we have \(\hat{x}(t) = g(t, \hat{x}(t), \hat{y}(t))\) and \(\dot{x}(t) = g(t, x(t), y(t))\), (12) implies that \(W(x(t), y(t)) \leq W(\hat{x}(t), \hat{y}(t))\) for any admissible pair \((x(t), y(t))\), establishing the first part of the theorem.
If $M$ is strictly concave in $x$, then the inequality in (9) is strict, and therefore the same argument establishes $W(x(t), y(t)) < W(\hat{x}(t), \hat{y}(t))$, and no other $\hat{x}(t)$ could achieve the same value, establishing the second part.
Mangasarian and Arrow Theorems play an important role in the applications of optimal control.

But are not straightforward to check since neither concavity nor convexity of the $g(\cdot)$ function would guarantee the concavity of the Hamiltonian unless we know something about the sign of the costate variable $\lambda(t)$.

In many economically interesting situations, we can ascertain $\lambda(t)$ is everywhere positive.

$\lambda(t)$ is related to the value of relaxing the constraint on the maximization problems; gives another way of ascertaining that it is positive (or negative).

Then checking Mangasarian conditions is straightforward, especially when $f$ and $g$ are concave functions.
Limitations of above:

1. We have assumed that a continuous and interior solution to the optimal control problem exists.
2. So far looked at the finite horizon case, whereas analysis of growth models requires us to solve infinite horizon problems. N
3. Need to look at the more modern theory of optimal control.
The Basic Problem: Necessary and Sufficient Conditions I

- Let us focus on infinite-horizon control with a single control and a single state variable.

- Using the same notation as above, the problem is

\[
\max_{x(t), y(t)} \mathcal{W}(x(t), y(t)) \equiv \int_0^\infty f(t, x(t), y(t)) \, dt \tag{13}
\]

subject to

\[
\dot{x}(t) = g(t, x(t), y(t)), \tag{14}
\]

and

\[
y(t) \in \mathbb{R} \text{ for all } t, \quad x(0) = x_0 \text{ and } \lim_{t \to \infty} x(t) \geq x_1. \tag{15}
\]

- Allows for an implicit choice over the endpoint \(x_1\), since there is no terminal date.

- The last part of (15) imposes a lower bound on this endpoint.
Further simplified by removing feasibility requirement that $y(t)$ should always belong to the set $\mathcal{Y}$, instead simply require to be real-valued.

Have not assumed that the state variable $x(t)$ lies in a compact set.

Call a pair $(x(t), y(t))$ *admissible* if $y(t)$ is Lebesgue-measurable and thus $x(t)$ is absolutely continuous.

Define the *value function*, analog of discrete time dynamic programming:

$$V(t_0, x_0) \equiv \max_{x(t) \in \mathbb{R}, y(t) \in \mathbb{R}} \int_{t_0}^{\infty} f(t, x(t), y(t)) \, dt \quad (16)$$

subject to $\dot{x}(t) = g(t, x(t), y(t))$, $x(t_0) = x_0$

and $\lim_{t \to \infty} x(t) \geq x_1$. 
The Basic Problem: Necessary and Sufficient Conditions III

- \( V(t_0, x_0) \) gives the optimal value of the dynamic maximization problem starting at time \( t_0 \) with state variable \( x_0 \).

- Clearly,

\[
V(t_0, x_0) \geq \int_{t_0}^{\infty} f(t, x(t), y(t)) \, dt
\]

for any admissible pair \((x(t), y(t))\).

- When “max” is not reached, we should be using “sup” instead.

- But we have assumed that all admissible pairs give finite value, so that \( V(t_0, x_0) < \infty \), and our focus throughout will be on admissible pairs \((\hat{x}(t), \hat{y}(t))\) that are optimal solutions to (13) subject to (14) and (15), and thus reach the value \( V(t_0, x_0) \).
Theorem: Infinite-Horizon Maximum Principle

Suppose that problem of maximizing (13) subject to (14) and (15), with \( f \) and \( g \) continuously differentiable, has an interior continuous solution \( \hat{y}(t) \) with corresponding path of state variable \( \hat{x}(t) \). Let \( H(t, x, y, \lambda) \) be given by (4). Then the optimal control \( \hat{y}(t) \) and the corresponding path of the state variable \( \hat{x}(t) \) are such that the Hamiltonian \( H(t, x, y, \lambda) \) satisfies the Maximum Principle, that

\[
H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y, \lambda(t)) \quad \text{for all } y(t),
\]

for all \( t \in \mathbb{R} \). Moreover, the following necessary conditions are satisfied:

\[
H_y(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0, 
\]

\[
\dot{\lambda}(t) = -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)),
\]

\[
\dot{x}(t) = H_\lambda(t, \hat{x}(t), \hat{y}(t), \lambda(t)), \quad \text{with } x(0) = x_0 \text{ and } \lim_{t \to \infty} x(t) \geq x_1,
\]

for all \( t \in \mathbb{R}_+ \).
Notice that whenever an interior continuous optimal solution of the specified form exists, it satisfies the Maximum Principle.

Conditions can be generalized to piecewise continuous functions and non-interior functions, but not necessary in economic publications.

The boundary condition \( \lim_{t \to \infty} x(t) \geq x_1 \) can be generalized to \( \lim_{t \to \infty} b(t) x(t) \geq x_1 \) for some positive function \( b(t) \).

Sufficient conditions to ensure that such a solution exist are somewhat involved

In addition, if the optimal control, \( \hat{y}(t) \), is a continuous function of time, the conditions (18)-(20) are also satisfied.
Most generally, $\hat{y}(t)$ is a Lebesgue measurable function (so it could have discontinuities).

Added generality of allowing discontinuities is somewhat superfluous in most economic applications.

In most economic problems sufficient to focus on the necessary conditions (18)-(20).
Necessary conditions can also be expressed in the form of the so-called Hamilton-Jacobi-Bellman (HJB) equation.

Theorem (Hamilton-Jacobi-Bellman Equations) Let $V(t,x)$ be as defined in (16) and suppose that the hypotheses in the Infinite-Horizon Maximum Principle Theorem hold. Then whenever $V(t,x)$ is differentiable in $(t,x)$, the optimal pair $(\hat{x}(t), \hat{y}(t))$ satisfies the HJB equation. For all $t \in \mathbb{R}$.

$$0 = f(t, \hat{x}(t), \hat{y}(t)) + \frac{\partial V(t, \hat{x}(t))}{\partial t} + \frac{\partial V(t, \hat{x}(t))}{\partial x} g(t, \hat{x}(t), \hat{y}(t))$$
Proof:

From Lemma, we have that for the optimal pair \((\hat{x}(t), \hat{y}(t))\),

\[
V(t_0, x_0) = \int_{t_0}^{t} f(s, \hat{x}(s), \hat{y}(s)) \, ds + V(t, \hat{x}(t)) \quad \text{for all } t.
\]

Differentiating this with respect to \(t\) and using the differentiability of \(V\) and Leibniz’s rule,

\[
f(t, \hat{x}(t), \hat{y}(t)) + \frac{\partial V(t, \hat{x}(t))}{\partial t} + \frac{\partial V(t, \hat{x}(t))}{\partial x} \dot{x}(t) = 0 \quad \text{for all } t.
\]

Setting \(\dot{x}(t) = g(t, \hat{x}(t), \hat{y}(t))\) gives (21).
Note important features:

1. Given that the continuous differentiability of $f$ and $g$, the assumption that $V(t, x)$ is differentiable is not very restrictive, since the optimal control $\hat{y}(t)$ is piecewise continuous.
   
   - From the definition (16), at all $t$ where $\hat{y}(t)$ is continuous, $V(t, x)$ will also be differentiable in $t$.
   
   - Moreover, an envelope theorem type argument also implies that when $\hat{y}(t)$ is continuous, $V(t, x)$ should also be differentiable in $x$.

2. (21) is a partial differential equation, since it features the derivative of $V$ with respect to both time and the state variable $x$. 

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Since in this Theorem there is no boundary, may expect that there should be a transversality condition similar to the condition that \( \lambda (t_1) = 0 \).

Might be tempted to impose a transversality condition of the form

\[
\lim_{t \to \infty} \lambda (t) = 0, \tag{22}
\]

But this is not in general the case. A milder transversality condition of the form

\[
\lim_{t \to \infty} H(t, x, y, \lambda) = 0 \tag{23}
\]

always applies, but is not easy to check.
Theorem (Arrow's Sufficient Conditions for Infinite Horizon)
Consider the problem of maximizing (13) subject to (14) and (15), with \( f \) and \( g \) continuously differentiable. Define \( H(t, x, y, \lambda) \) as in (4), and suppose that a piecewise continuous solution \( \hat{y}(t) \) and the corresponding path of state variable \( \hat{x}(t) \) satisfy (18)-(20). Given the resulting costate variable \( \lambda(t) \), define
\[
M(t, x, \lambda) \equiv \max_{y(t) \in \mathcal{Y}(t)} H(t, x, y, \lambda).
\]
If \( M(t, x, \lambda) \) is concave in \( x \) and \( \lim_{t \to \infty} \lambda(t) (\hat{x}(t) - \bar{x}(t)) \leq 0 \) for all \( \bar{x}(t) \) implied by an admissible control path \( \bar{y}(t) \), then the pair \((\hat{x}(t), \hat{y}(t))\) achieves the unique global maximum of (13).
Since $x(t)$ can potentially grow without bounds and we require only concavity (not strict concavity), can apply to models with constant returns and endogenous growth.

Both involve the difficult to check condition that

$$\lim_{t \to \infty} \lambda(t)(x(t) - \tilde{x}(t)) \leq 0 \text{ for all } \tilde{x}(t) \text{ implied by an admissible control path } \tilde{y}(t).$$

This condition will disappear when we can impose a proper transversality condition.
Consider the problem of maximizing
\[
\int_{t_1}^{t} H(t, \hat{x}(t), y(t), \lambda(t)) \, dt = \int_{0}^{t_1} \left[ f(t, \hat{x}(t), y(t)) + \lambda(t) g(t, \hat{x}(t), y(t)) \right] \, dt
\]
with respect to the entire function \( y(t) \) for given \( \lambda(t) \) and \( \hat{x}(t) \), where \( t_1 \) can be finite or equal to \( +\infty \).

The condition \( H_y(t, \hat{x}(t), y(t), \lambda(t)) = 0 \) would then be a necessary condition for this alternative maximization problem.

Therefore, the Maximum Principle is implicitly maximizing the sum the original maximand \( \int_{0}^{t_1} f(t, \hat{x}(t), y(t)) \, dt \) plus an additional term \( \int_{0}^{t_1} \lambda(t) g(t, \hat{x}(t), y(t)) \, dt \).

Understanding why this is true provides much of the intuition for the Maximum Principle.
Economic Intuition II

- Let $V(t, \hat{x}(t))$ be the value of starting at time $t$ with state variable $\hat{x}(t)$ and pursuing the optimal policy from then on.

- We will see that
  $$\lambda(t) = \frac{\partial V(t, \hat{x}(t))}{\partial x}.$$  

  Consequently, $\lambda(t)$ is the (shadow) value of relaxing the constraint (14) by increasing the value of $x(t)$ at time $t$.

- Moreover, recall that $\dot{x}(t) = g(t, \hat{x}(t), y(t))$, so that the second term in the Hamiltonian is equivalent to $\int_0^{t_1} \lambda(t) \dot{x}(t) \, dt$.

- This is clearly the shadow value of $x(t)$ at time $t$ and the increase in the stock of $x(t)$ at this point.

- Can think of it $x(t)$ as a “stock” variable in contrast to the control $y(t)$, which corresponds to a “flow” variable.
Therefore, maximizing (24) is equivalent to maximizing instantaneous returns as given by the function $f(t, \hat{x}(t), y(t))$, plus the value of stock of $x(t)$, as given by $\lambda(t)$, times the increase in the stock, $\dot{x}(t)$.

Thus essence of the Maximum Principle is to maximize the flow return plus the value of the current stock of the state variable.

Turn to the interpreting the costate equation,

$$
\dot{\lambda}(t) = -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = -f_x(t, \hat{x}(t), \hat{y}(t)) - \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t)).
$$

Since $\lambda(t)$ is the value of the stock of the state variable, $x(t)$, $\dot{\lambda}(t)$ is the appreciation in this stock variable.

A small increase in $x$ will change the current flow return plus the value of the stock by $H_x$, and the value of the stock by the amount $\dot{\lambda}(t)$.

This gain should be equal to the depreciation in the value of the stock, $-\dot{\lambda}(t)$. 
Economic Intuition IV

- Otherwise, it would be possible to change the \( x(t) \) and increase the value of \( \int_0^\infty H(t, x(t), y(t)) \).
- Second and complementary intuition for the Maximum Principle comes from the HJB equation (21).
- Consider an exponentially discounted problem, 
  \[ f(t, x(t), y(t)) = \exp(-\rho t) f(x(t), y(t)). \]
- Law of motion of the state variable given by an autonomous differential equation, i.e., 
  \[ g(t, x(t), y(t)) = g(x(t), y(t)). \]
- In this case:
  1. if an admissible pair \((\hat{x}(t), \hat{y}(t))_{t \geq 0}\) is optimal starting at \( t = 0 \) with initial condition \( x(0) = x_0 \), it is also optimal starting at \( s > 0 \), starting with the same initial condition,
  2. that is, \((\hat{x}(t), \hat{y}(t))_{t \geq s}\) is optimal for the problem with initial condition \( x(s) = x_0 \).
- In view of this, define \( V(x) \equiv V(0, x) \), value of pursuing the optimal plan \((\hat{x}(t), \hat{y}(t))\) starting with initial condition \( x \), evaluated at \( t = 0 \).
Economic Intuition V

Since \((\hat{x}(t), \hat{y}(t))\) is an optimal plan irrespective of the starting date, we have that \(V(t, x(t)) \equiv \exp(-\rho t) \, V(x(t))\).

Then, by definition,

\[
\frac{\partial V(t, x(t))}{\partial t} = -\exp(-\rho t) \rho V(x(t)).
\]

Moreover, let \(\dot{V}(x(t)) \equiv (\partial V(t, x(t)) / \partial x) \dot{x}(t)\).

Substituting these expressions into (21) and noting that \(\dot{x}(t) = g(\hat{x}(t), \hat{y}(t))\), we obtain the “stationary” form of the Hamilton-Jacobi-Bellman

\[
\rho V(x(t)) = f(\hat{x}(t), \hat{y}(t)) + \dot{V}(x(t)). \tag{25}
\]

Can be interpreted as a “no-arbitrage asset value equation”

Think of \(V\) as the value of an asset traded in the stock market and \(\rho\) as the required rate of return for (a large number of) investors.
Economic Intuition VI

- Return on the assets come from two sources.
- First, “dividends,” the flow payoff \( f(\hat{x}(t), \hat{y}(t)) \).
- If this dividend were constant and equal to \( d \), and there were no other returns, \( V = d/\rho \) or
  \[ \rho V = d. \]
- Returns also come from capital gains or losses (appreciation or depreciation of the asset), \( \dot{V} \).
- Therefore, instead of \( \rho V = d \), we have
  \[ \rho V(x(t)) = d + \dot{V}(x(t)). \]
- Thus Maximum Principle amounts to requiring that \( V(x(t)) \) and \( \dot{V}(x(t)) \), should be consistent with this no-arbitrage condition.
More on Transversality Conditions: Counterexample I

- Consider the following problem:

\[
\max \int_0^\infty \left[ \log (c(t)) - \log c^* \right] dt
\]

subject to

\[
\dot{k}(t) = \left[ k(t) \right]^\alpha - c(t) - \delta k(t)
\]

\[
k(0) = 1
\]

and

\[
\lim_{t \to \infty} k(t) \geq 0
\]

where \( c^* \equiv \left[ k^* \right]^\alpha - \delta k^* \) and \( k^* \equiv (\alpha / \delta)^{1/(1-\alpha)} \).

- \( c^* \) is the maximum level of consumption that can be achieved in steady state.

- \( k^* \) is the corresponding steady-state level of capital.
The integral converges and takes a finite value (since $c(t)$ cannot exceed $c^*$ forever).

Hamiltonian,

$$H(k, c, \lambda) = \left[ \log c(t) - \log c^* \right] + \lambda \left[ k(t)^\alpha - c(t) - \delta k(t) \right],$$

Necessary conditions (dropping time dependence):

$$H_c(k, c, \lambda) = \frac{1}{c(t)} - \lambda(t) = 0$$

$$H_k(k, c, \lambda) = \lambda(t) \left( \alpha k(t)^{\alpha-1} - \delta \right) = -\dot{\lambda}(t).$$
Any optimal path must feature $c(t) \to c^*$ as $t \to \infty$. This, however, implies

$$\lim_{t \to \infty} \lambda(t) = \frac{1}{c^*} > 0 \text{ and } \lim_{t \to \infty} k(t) = k^*.$$

Thus the equivalent of the standard finite-horizon transversality conditions do not hold.

It can be verified, however, that along the optimal path we have

$$\lim_{t \to \infty} H(k(t), c(t), \lambda(t)) = 0.$$
Theorem **(Transversality Condition for Infinite-Horizon Problems)**

Suppose that problem of maximizing (13) subject to (14) and (15), with \( f \) and \( g \) continuously differentiable, has an interior piecewise continuous solution \( \hat{y}(t) \) with corresponding path of state variable \( \hat{x}(t) \). Suppose moreover that \( V(t, \hat{x}(t)) \) is differentiable in \( x \) and \( t \) for \( t \) sufficiently large and that

\[
\lim_{t \to \infty} \frac{\partial V(t, \hat{x}(t))}{\partial t} = 0.
\]

Let \( H(t, x, y, \lambda) \) be given by (4). Then the optimal control \( \hat{y}(t) \) and the corresponding path of the state variable \( \hat{x}(t) \) satisfy the necessary conditions (18)-(20) and the transversality condition

\[
\lim_{t \to \infty} H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0. \tag{26}
\]
Focus on points where $V(t, x)$ is differentiable in $t$ and $x$ so that the Hamilton-Jacobi-Bellman equation, (21) holds.

Noting that $\frac{\partial V(t, \hat{x}(t))}{\partial x} = \lambda(t)$, this equation can be written as, for all $t$

$$\frac{\partial V(t, \hat{x}(t))}{\partial t} + f(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g(t, \hat{x}(t), \hat{y}(t)) = 0$$

$$\frac{\partial V(t, \hat{x}(t))}{\partial t} + H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0.$$ 

Now take the limit as $t \to \infty$ and use $\lim_{t \to \infty} \frac{\partial V(t, \hat{x}(t))}{\partial t} = 0$ to obtain (26).
Part of the difficulty, especially for transversality condition, comes from not enough structure on $f$ and $g$.

Economically interesting problems often take the following more specific form:

$$\max_{x(t), y(t)} W(x(t), y(t)) \equiv \int_0^\infty \exp(-\rho t) f(x(t), y(t)) \, dt \text{ with } \rho > 0,$$

subject to

$$\dot{x}(t) = g(x(t), y(t)), \quad (28)$$

and

$$y(t) \in \mathbb{R} \text{ for all } t, x(0) = x_0 \text{ and } \lim_{t \to \infty} x(t) \geq x_1. \quad (29)$$

Assume $\rho > 0$, so that there is indeed discounting.

Key: $f$ depends on time only through exponential discounting, $g$ is not a function of time directly.
Discounted Infinite-Horizon Optimal Control III

- Hamiltonian:
  \[ H(t, x(t), y(t), \lambda(t)) = \exp(-\rho t) f(x(t), y(t)) + \lambda(t) g(x(t), y(t)), \]
  \[ = \exp(-\rho t) [f(x(t), y(t)) + \mu(t) g(x(t), y(t))], \]
  where the second line defines
  \[ \mu(t) \equiv \exp(\rho t) \lambda(t). \] (30)

- Hamiltonian depends on time explicitly only through the \( \exp(-\rho t) \) term.

- In fact can work with the current-value Hamiltonian,
  \[ \hat{H}(x(t), y(t), \mu(t)) \equiv f(x(t), y(t)) + \mu(t) g(x(t), y(t)). \] (31)

  “Autonomous” in the sense that it does not directly depend on time.

- Refer to \( f(x, y) \) and \( g(x, y) \) as weakly monotone if each one is monotone in each of its arguments.

- Assume the optimal control \( \hat{y}(t) \) is everywhere a continuous function...
Theorem: Maximum Principle for Discounted Infinite-Horizon Problems I

Suppose that problem of maximizing (27) subject to (28) and (29). Assume that \( f \) and \( g \) are continuously differentiable. Suppose also that the problem has a solution \( \hat{y}(t) \) with corresponding path of state variable \( \hat{x}(t) \). Let \( \hat{H}(\hat{x}, \hat{y}, \mu) \) be the current-value Hamiltonian given by (31). Then the optimal control \( \hat{y}(t) \) and the corresponding path of the state variable \( \hat{x}(t) \) satisfy the following necessary conditions:

\[
\hat{H}_y (\hat{x}(t), \hat{y}(t), \mu(t)) = 0 \quad \text{for all } t \in \mathbb{R}_+, \tag{32}
\]

\[
\rho \mu(t) - \dot{\mu}(t) = \hat{H}_x (\hat{x}(t), \hat{y}(t), \mu(t)) \quad \text{for all } t \in \mathbb{R}_+, \tag{33}
\]

\[
\dot{x}(t) = \hat{H}_\mu (\hat{x}(t), \hat{y}(t), \mu(t)) \quad \text{for all } t \in \mathbb{R}_+, \ x(0) = x_0 \text{ and } \lim_{t \to \infty} x(t) \geq x_1. \tag{34}
\]
Theorem: Maximum Principle for Discounted Infinite-Horizon Problems II

and the transversality condition

$$\lim_{t \to \infty} \exp(-\rho t) \hat{H}(\hat{x}(t), \hat{y}(t), \mu(t)) = 0. \quad (35)$$

Moreover, if $f$ and $g$ are weakly monotone (for example, $f$ could be nondecreasing in $x$ and nonincreasing in $y$, and so on), if there exists $m > 0$ such that $|g_y(t, x(t), y(t))| \geq m$ for all $t$ and for all admissible pairs $(x(t), y(t))$, and if there exists $M < \infty$ such that $|f_y(x, y)| \leq M$ for all $x$ and $y$, then the transversality condition can be strengthened to:

$$\lim_{t \to \infty} [\exp(-\rho t) \mu(t) \hat{x}(t)] = 0. \quad (36)$$
The condition
\[ \lim_{t \to \infty} \left[ \exp(-\rho t) \mu(t) \hat{x}(t) \right] = 0 \]
is the transversality condition used in most economic applications.

However, as the previous theorem shows, it holds only under additional assumptions.

Moreover, only for interior continuous solutions.

Again: does such a solution exist?

It turns out that this question can be answered in most economic problems, because this transversality condition is sufficient for concave problems.
Theorem \textbf{(Sufficient Conditions for Discounted Infinite-Horizon Problems)} Consider the problem of maximizing (27) subject to (28) and (29), with $f$ and $g$ continuously differentiable and weakly monotone. Define $\hat{H}(x, y, \mu)$ as the current-value Hamiltonian as in (31), and suppose that a solution $\hat{y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (32)-(34) and the stronger transversality condition
\[
\lim_{t \to \infty} \left[ \exp(-\rho t) \mu(t) \hat{x}(t) \right] = 0.
\]
Given the resulting current-value costate variable $\mu(t)$, define $M(t, x, \mu) \equiv \max_{y(t) \in Y(t)} \hat{H}(x, y, \mu)$. Suppose that for any admissible pair $(x(t), y(t))$,
\[
\lim_{t \to \infty} \left[ \exp(-\rho t) \mu(t) x(t) \right] \geq 0
\]
and that $M(t, x, \mu)$ is concave in $x$. Then $\hat{y}(t)$ and the corresponding $\hat{x}(t)$ achieve the unique global maximum of (27). If $M$ is strictly concave, then the solution is unique.
General Strategy for Infinite-Horizon Optimal Control Problems

1. Use the necessary conditions given by the Maximum Principle to construct a candidate solution.

2. Check that this candidate solution satisfies the sufficiency conditions.

- This strategy will work in almost all growth models.
Example: Natural Resource I

- Infinitely-lived individual that has access to a non-renewable or exhaustible resource of size 1.
- Instantaneous utility of consuming a flow of resources $y$ is $u(y)$.
- $u : [0, 1] \rightarrow \mathbb{R}$ is a strictly increasing, continuously differentiable and strictly concave function.
- Objective function at time $t = 0$ is to maximize
  \[ \int_{0}^{\infty} \exp(-\rho t) u(y(t)) \, dt. \]
- The constraint is that the remaining size of the resource at time $t$, $x(t)$ evolves according to
  \[ \dot{x}(t) = -y(t), \]
- Also need that $x(t) \geq 0$. 
Example: Natural Resource II

- Current-value Hamiltonian

\[ \hat{H}(x(t), y(t), \mu(t)) = u(y(t)) - \mu(t)y(t). \]

- Necessary condition for an interior continuously differentiable solution \((\hat{x}(t), \hat{y}(t))\). There should exist a continuously differentiable function \(\mu(t)\) such that

\[ u'(\hat{y}(t)) = \mu(t), \]

and

\[ \dot{\mu}(t) = \rho \mu(t). \]

- The second condition follows since neither the constraint nor the objective function depend on \(x(t)\).

- This is the *Hotelling rule* for the exploitation of exhaustible resources.
Example: Natural Resource III

- Integrating both sides of this equation and using the boundary condition,
  \[ \mu(t) = \mu(0) \exp(\rho t). \]
- Now combining this with the first-order condition for \( y(t) \),
  \[ \hat{y}(t) = u^{-1} [\mu(0) \exp(\rho t)]. \]
- \( u^{-1} [\cdot] \) exists and is strictly decreasing since \( u \) is strictly concave.
- Thus amount of the resource consumed is monotonically decreasing over time:
  - because of discounting, preference for early consumption, and delayed consumption has no return.
  - but not all consumed immediately, also a preference for smooth consumption from \( u(\cdot) \) is strictly concave.
Combining the previous equation with the resource constraint,

\[ \dot{x}(t) = -u'^{-1} [\mu(0) \exp(\rho t)] . \]

Integrating this equation and using the boundary condition that \( x(0) = 1 \),

\[ \hat{x}(t) = 1 - \int_0^t u'^{-1} [\mu(0) \exp(\rho s)] \, ds. \]

Since along any optimal path we must have \( \lim_{t \to \infty} \hat{x}(t) = 0 \),

\[ \int_0^{\infty} u'^{-1} [\mu(0) \exp(\rho s)] \, ds = 1. \]

Therefore, \( \mu(0) \) must be chosen so as to satisfy this equation.
Existence of Solutions

- So far, no general result on existence of solutions.
- This can be stated and proved (see book).
- But not so useful for two reasons:
  1. conditions for existence of interior and continuous solutions much more complicated
  2. the strategy of verifying sufficiency conditions much more straightforward.
Neoclassical economy without any population growth and without any technological progress.

Optimal growth problem in continuous time can be written as:

$$\max_{[k(t), c(t)]} \int_{0}^{\infty} \exp(-\rho t) u(c(t)) \, dt,$$

subject to

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

and \( k(0) > 0 \).

\( u : \mathbb{R}_+ \to \mathbb{R} \) is strictly increasing, continuously differentiable and strictly concave.

\( f(\cdot) \) satisfies Assumptions 1 and 2.
The constraint function, \( f(k) - \delta k - c \), is decreasing in \( c \), but may be nonmonotone in \( k \).

But we can restrict attention to \( k(t) \in [0, \bar{k}] \), where \( \bar{k} \) is defined such that \( f'(\bar{k}) = \delta \), so constraint function is also weakly monotone.

Current-value Hamiltonian,

\[
\hat{H}(k, c, \mu) = u(c(t)) + \mu(t)[f(k(t)) - \delta k(t) - c(t)], \tag{37}
\]

Necessary conditions:

\[
\hat{H}_c (k, c, \mu) = u'(c(t)) - \mu(t) = 0
\]
\[
\hat{H}_k (k, c, \mu) = \mu(t)(f'(k(t)) - \delta) = \rho \mu(t) - \dot{\mu}(t)
\]
\[
\lim_{t \to \infty} [\exp(-\rho t) \mu(t) k(t)] = 0.
\]
First necessary condition immediately implies that \( \mu(t) > 0 \) (since \( u' > 0 \) everywhere).

Thus current-value Hamiltonian is sum of two strictly concave functions and is itself strictly concave.

Moreover, since \( k(t) \geq 0 \), for any admissible solution
\[
\lim_{t \to \infty} \left[ \exp(-\rho t) \mu(t) k(t) \right] \geq 0.
\]

Hence a solution that satisfies these necessary conditions in fact gives a global maximum.

Characterizing the solution of these necessary conditions also establishes the existence of a solution in this case.
The q-Theory of Investment I

- Canonical model of investment under adjustment costs, also known as the q-theory of investment.
- Price-taking firm trying to maximize the present discounted value of its profits.
- Firm is subject to “adjustment” costs when it changes its capital stock.
- Let the capital stock of the firm be $k(t)$.
- Firm has access to a production function $f(k(t))$ that satisfies Assumptions 1 and 2.
- Normalize the price of the output of the firm to 1 in terms of the final good at all dates.
- Adjustment costs captured by $\phi(i)$: strictly increasing, continuously differentiable and strictly convex, and satisfies $\phi(0) = \phi'(0) = 0$. 
In some models, the adjustment cost is taken to be $\phi (i/k)$ instead of $\phi (i)$.

Installed capital depreciates at an exponential rate $\delta$.

Firm maximizes its net present discounted earnings with a discount rate equal to the interest rate $r$, constant.

The firm’s problem can be written as

$$\max_{k(t),i(t)} \int_0^\infty \exp (-rt) [f(k(t)) - i(t) - \phi (i(t))] \ dt$$

subject to

$$\dot{k}(t) = i(t) - \delta k(t)$$

and $k(t) \geq 0$, with $k(0) > 0$ given.

Clearly, both the objective function and the constraint function are weakly monotone.

Since $\phi$ is strictly convex, not optimal to make “large” adjustments.
The q-Theory of Investment III

- Current-value Hamiltonian:

\[
\hat{H}(k, i, q) \equiv \left[f(k(t)) - i(t) - \phi(i(t))\right] + q(t) \left[i(t) - \delta k(t)\right],
\]

- Used \( q(t) \) instead of the familiar \( \mu(t) \) for the costate variable.

- Necessary conditions for this problem are standard (suppressing the “\(^\wedge\)” to denote the optimal values):

\[
\begin{align*}
\hat{H}_i(k, i, q) &= -1 - \phi'(i(t)) + q(t) = 0 \\
\hat{H}_k(k, i, q) &= f'(k(t)) - \delta q(t) = rq(t) - \dot{q}(t) \\
\lim_{t \to \infty} \exp(-rt) q(t) k(t) &= 0.
\end{align*}
\]

- First necessary condition implies,

\[
q(t) = 1 + \phi'(i(t)) \text{ for all } t. \quad (39)
\]
Differentiating with respect to time,

\[ \dot{q}(t) = \phi''(i(t)) \dot{i}(t). \]  

(40)

Substituting into the second necessary condition, law of motion for investment:

\[ \dot{i}(t) = \frac{1}{\phi''(i(t))} \left[ (r + \delta) \left(1 + \phi'(i(t))\right) - f'(k(t)) \right]. \]  

(41)

Interesting economic features:

- As \( \phi''(i) \) tends to zero, \( \dot{i}(t) \) diverges, meaning that investment jumps to a particular value.
  - I.e., it can be shown that this value is such that the capital stock immediately reaches its state-state value.
  - As \( \phi''(i) \) tends to zero, \( \phi'(i) \) becomes linear: adjustment costs increase cost linearly and no need for smoothing.
- When \( \phi''(i(t)) > 0 \), smoothing: \( \dot{i}(t) \) will take a finite value, and investment will adjust slowly.
Behavior of investment and capital stock using the differential equations (38) and (41).

There exists a unique steady-state solution with $k > 0$, and involves $i^* = \delta k^*$.

This steady-state level of capital satisfies the first-order condition (corresponding to the right-hand side of (41) being equal to zero):

$$f'(k^*) = (r + \delta) \left(1 + \phi'(\delta k^*)\right).$$

Differs from “modified golden rule:” additional cost means there more investment needed to replenish depreciated capital—term $\phi'(\delta k^*)$.

Since $\phi$ is strictly convex and $f$ is strictly concave and satisfies the Inada conditions, a unique value of $k^*$ satisfies this condition.
Instead of global stability in the $k$-$i$ space, the correct concept here is *saddle-path stability*.

Instead of an initial value constraint, $i(0)$ is pinned down by a boundary condition at “infinity,”

$$\lim_{t \to \infty} \exp(-rt) q(t) k(t) = 0.$$ 

Thus with one state and one control variable, we should have a one-dimensional manifold (a curve) along which capital-investment pairs tend towards the steady state.

This manifold is also referred to as the “stable arm”.

$i(0)$ will then be determined so that the economy starts along this manifold.

If any capital-investment pair were to lead to the steady state, we would not know how to determine $i(0)$; “indeterminacy” of equilibria.
Mathematically, saddle-path stability involves the number of negative eigenvalues to be the same as the number of state variables.

**Theorem** Consider the following linear differential equation system

$$\dot{x}(t) = Ax(t) + b$$  \hspace{1cm} (42)

with initial value $x(0)$, where $x(t) \in \mathbb{R}^n$ for all $t$ and $A$ is an $n \times n$ matrix. Let $x^*$ be the steady state of the system given by $Ax^* + b = 0$. Suppose that $m \leq n$ of the eigenvalues of $A$ have negative real parts. Then there exists an $m$-dimensional subspace $M$ of $\mathbb{R}^n$ such that starting from any $x(0) \in M$, the differential equation (42) has a unique solution with $x(t) \rightarrow x^*$. 

Sugata Bag (Delhi School of Economics)
The q-Theory of Investment VIII

Theorem  Consider the following nonlinear autonomous differential equation

\[ \dot{x}(t) = G(x(t)) \]  \hspace{1cm} (43)

where \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and suppose that \( G \) is continuously differentiable, with initial value \( x(0) \). Let \( x^* \) be a steady-state of this system, given by \( F(x^*) = 0 \). Define

\[ A = DG(x^*) , \]

and suppose that \( m \leq n \) of the eigenvalues of \( A \) have negative real parts and the rest have positive real parts. Then there exists an open neighborhood of \( x^* \), \( B(x^*) \subset \mathbb{R}^n \) and an \( m \)-dimensional manifold \( M \subset B(x^*) \) such that starting from any \( x(0) \in M \), the differential equation (43) has a unique solution with \( x(t) \rightarrow x^* \).
Figure investigates the transitional dynamics in the q-theory of investment.

Adjustment costs discourage large values of investment: firm cannot adjust to its steady-state level immediately.

Diminishing returns imply benefit of increasing $k$ is greater when $k$ is low.

As capital accumulates and $k(t) > k(0)$, the benefit of boosting the capital stock declines and the firm also reduces investment.

Initial investment $i(0)$ is the unique optimum. Why? *Sufficiency Theorem*. 
The q-Theory of Investment X

- Alternative popular approach, use Figure.
- Consider, for example, \( i'(0) > i(0) \) as the initial level:
  - \( i(t) \) and \( k(t) \) would tend to infinity.
  - \( q(t)k(t) \) would tend to infinity at a rate faster than \( r \), violating the transversality condition, \( \lim_{t \to \infty} \exp(-rt)q(t)k(t) = 0 \).
  - Along a trajectory starting at \( i'(0), k(t)/k(t) > 0 \), and thus we have
    \[
    \frac{d(q(t)k(t))/dt}{q(t)k(t)} \geq \frac{\dot{q}(t)}{q(t)} = \frac{i(t)\phi''(i(t))}{1 + \phi'(i(t))} = r + \delta - f'(k(t))/(1 + \phi'(i(t))),
    \]
  - Second line uses (39) and (40), while third line substitutes from (41).
  - As \( k(t) \to \infty \), we have that \( f'(k(t)) \to 0 \), implying that
    \[
    \lim_{t \to \infty} \exp(-rt)q(t)k(t) \geq \lim_{t \to \infty} \exp(-rt)\exp((r + \delta)t) = \lim_{t \to \infty} \exp(\delta t)
    \]
    violating the transversality condition.
In contrast, if we start with \( i'' (0) < i (0) \) as the initial level:

- \( i (t) \) would tend to 0 in finite time
- \( k (t) \) would also tend towards zero (though not reaching it in finite time).
- After the time where \( i (t) = 0 \), we also have \( q (t) = 1 \) and thus \( \dot{q} (t) = 0 \) (from (39)).
- Moreover, by the Inada conditions, as \( k (t) \to 0 \), \( f' (k (t)) \to \infty \).
- Consequently, after \( i (t) \) reaches 0, the necessary condition \( \dot{q} (t) = (r + \delta) q (t) - f' (k (t)) \) is violated (though care necessary, since at the boundary this condition is no longer necessary).
“q-theory” aspects (Tobin): value of an extra unit of capital divided by its replacement cost is a measure of the “value of investment”.

When this ratio is high, the firm would like to invest more.

In steady state, firm will settle where this ratio is 1 or close to 1.

Costate variable $q(t)$ measures Tobin’s q.

Denote the current (maximized) value of the firm when it starts with a capital stock of $k(t)$ by $V(k(t))$.

Same arguments as above imply that

$$V'(k(t)) = q(t),$$

$q(t)$ measures exactly by how much one dollar increase in capital will raise the value of the firm.
In steady state, we have $\dot{q}(t) = 0$, so that $q^* = f'(k^*) / (r + \delta)$, which is approximately equal to 1 when $\phi'(\delta k^*)$ is small.

Out of steady state, $q(t)$ can be significantly greater than this amount, signaling that there is need for greater investments.

Tobin’s $q$, or alternatively the costate variable $q(t)$, will play the role of signaling when investment demand is high.

Proxies for Tobin’s $q$ can be constructed using stock market prices and book values of firms.

When stock market prices are greater than book values, this corresponds a high Tobin’s $q$.

But whether this is a good approach is intensely debated:

- Tobin’s $q$ does not contain all the relevant information when there are irreversibilities or fixed costs of investment,
- What is relevant is the “marginal $q$,” but typically only measure “average $q$.”
Conclusions

- Basic ideas of optimal control may be a little less familiar than those of discrete time dynamic programming, but used in much of growth theory and in other areas of macroeconomics.
- Moreover, some problems become easier in continuous time.
- The most powerful theorem in optimal control, Pontryagin’s Maximum Principle, is as much an economic result as a mathematical result.
- Maximum Principle has a very natural interpretation both in terms of maximizing flow returns plus the value of the stock, and also in terms of an asset value equation for the value of the maximization problem.