Chapter 15

Dynamic Optimization

Several of the applications of constrained optimization presented in Chapter 11 are two-period discrete-time optimization problems. The objective function in these intertemporal consumption problems is the discounted sum of utility in each period. The intertemporal constraints in these problems link actions taken in the one period with actions taken in the other period. For example, if consumption is higher in the first period, then, all else equal, the budget constraint requires that it is lower in the second period. These intertemporal constraints make these **dynamic optimization** problems more complicated than a simple sequence of one-period optimization problems. Nevertheless, we are able to obtain solutions to these problems by simply applying the standard techniques of constrained optimization after setting up the problem in an appropriate manner. As shown in the first section of this chapter, the extension of these methods to settings of more than two periods is relatively straightforward.

The extension of these techniques to continuous time dynamic optimization problems is not as straightforward. The greater part of this chapter is devoted to techniques for finding the optimal time path of variables in a continuous time framework. In particular, we consider dynamic optimization problems in which the constraints include one or more differential equations. The technique for solving these types of problems, which is called **optimal control theory,** was developed in the 1950s by the Soviet mathematician L.S. Pontryagin and his associates and, independently, by the American mathematician Richard Bellman.[1] Optimal control theory allows consideration of the optimal time path of a set of variables rather than just the identification of a stationary equilibrium. By way of analogy to the material on differential equations in Chapter 14, a full characterization of the solution to many types of dynamic problems requires the determination of the path to the steady state, as well as the steady state itself.

In this chapter we introduce the elements of optimal control theory in order to provide a working knowledge of its basic results.[2] In Section 15.1 we set up and solve a

[1] L. S. Pontryagin et al., *The Mathematical Theory of Optimal Processes*, K. N. Trirogoff, trans. (New York: Interscience Publishers, 1962) and Richard Bellman, *Dynamic Programming,* (Princeton: Princeton University Press, 1957).

[2] A full exposition of dynamic optimization is beyond the scope of this book. A reader interested in more detail and more general results than those presented here can consult Chapter 8 of Akira Takayama, *Mathematical Economics,* 2d ed. (Cambridge, England: Cambridge University Press, 1985), 600–719 and the references therein.

discrete time dynamic optimization problem and interpret its solution. In Section 15.2 we present a continuous time version of the discrete time problem and present the conditions that solve this problem. Section 15.3 concludes this chapter with some extensions of the basic results presented in the previous section and some economic applications. An appendix to the chapter presents a heuristic derivation of the necessary conditions for a continuous time dynamic optimization problem.

15.1 DYNAMIC OPTIMIZATION IN DISCRETE TIME

Discrete time dynamic optimization problems can be solved with the Lagrange multiplier method presented in Chapter 11. In this section we extend the results from the two-period model, which are discussed in Chapter 11, to the case where there are more general functions and the problem includes more than two periods. The intuition and results from this discrete time analysis will serve us well when we turn to the continuous time case in the next section.

Consider the problem facing the managers of a firm who wish to maximize total profits from time $t = 0$ to time $t = T$. Profits at any moment t depend upon two variables, the amount spent on labor during that moment and the capital available to the firm at the beginning of the period. The amount spent on labor is called the **control variable,** which we denote as c_t. The value of the control variable can be selected each period. Control variables take the form of flows, such as the flow of consumption or the flow of labor services. Given a choice of the time path of the control variable, the managers cannot independently choose the time path of capital, which is the **state variable,** since the amount spent on labor implicitly determines the amount of revenues left over for investment in capital. State variables are stocks, such as the stock of capital, the stock of available oil, or the stock of gold. In this problem, k_t represents the stock of capital from the end of period $t - 1$ to the end of period t.

We define an instantaneous profit function $\pi(c_t, k_t, t)$, which is assumed to be twice-differentiable and concave for all values of its arguments. The profit function describes an instantaneous flow of profits, and, therefore, total profits accrued over an entire period equals the value of the profit function during that period times the length of the period, which we denote as m.[3] The goal of the managers of the firm is to maximize the profit of the firm over the time period $t = 0$ to T. The profit over this period is represented by the sum

$$\sum_{t=0}^{T} m \cdot \pi(c_t, k_t, t). \tag{15.1}$$

This problem also includes an intertemporal constraint on the accumulation of capital. Investment in period t, which is the change in the capital stock from period t to period $t + 1$ (absent depreciation), is assumed to be determined by the function

$$k_{t+1} - k_t = m \cdot f(c_t, k_t, t), \tag{15.2}$$

[3] We use instantaneous functions here, which must be multiplied by m to give a value over a period, in anticipation of extending this example to the continuous time case in the next section.

where $f(c_t, k_t, t)$, like $\pi(c_t, k_t, t)$, is an "instantaneous" function and must be multiplied by m to get the cumulative change in capital for an entire period. We also assume that there is an initial given stock of capital equal to $\underline{k}$ and that, at the end of period T, the capital stock must equal $\bar{k}$. Thus the Lagrangian function associated with this problem (see Chapter 11) is

$$L = \sum_{t=0}^{T}[m \cdot \pi(c_t, k_t, t) - \lambda_t(k_{t+1} - m \cdot f(c_t, k_t, t) - k_t)]$$
$$- \mu_0(k_0 - \underline{k}) - \mu_T(k_T - \bar{k}), \tag{15.3}$$

where the λ_t's represent $T + 1$ Lagrange multipliers on the constraint imposed by the capital accumulation equation and μ_0 and μ_T represent the Lagrange multipliers on the constraints due to the initial condition and the terminal condition, respectively. The λ_t's are called **costate variables** or, more descriptively, **dynamic Lagrange multipliers.** As discussed in Chapter 11, the solution to this problem requires simultaneously solving first-order conditions. These first-order conditions include

$$\frac{\partial L}{\partial c_t} = m \cdot [\pi_c(c_t, k_t, t) + \lambda_t \cdot f_c(c_t, k_t, t)] = 0 \quad \text{for} \quad t = 0, 1, \ldots, T, \tag{15.4}$$

$$\frac{\partial L}{\partial \lambda_t} = k_{t+1} - m \cdot f(c_t, k_t, t) - k_t = 0 \quad \text{for} \quad t = 0, 1, \ldots, T, \tag{15.5}$$

$$\frac{\partial L}{\partial \mu_0} = k_0 - \underline{k} = 0, \quad \text{and} \tag{15.6}$$

$$\frac{\partial L}{\partial \mu_T} = k_T - \bar{k} = 0, \tag{15.7}$$

where $\pi_c(c_t, k_t, t)$ and $f_c(c_t, k_t, t)$ represent the partial derivatives of the respective functions with respect to c_t. The first-order condition (15.4) requires that c_t in each period maximizes the term in square brackets in (15.3), given the amount of capital for that period. The first-order condition (15.5) represents the difference equation that governs the evolution of the capital stock, (15.2).[4] The first-order conditions (15.6) and (15.7) require that the initial condition and the terminal condition are satisfied.

The final first-order condition for this problem requires us to take the partial derivative of L with respect to the capital stock. Both k_t and k_{t+1} appear in the Lagrangian function. To obtain the first-order condition with respect to the capital stock in any one period, it is useful to note that

$$\sum_{t=0}^{T} - \lambda_t(k_{t+1} - k_t) = -\lambda_0(k_1 - k_0) - \lambda_1(k_2 - k_1) - \cdots - \lambda_T(k_{T+1} - k_T)$$
$$= \left[\sum_{t=1}^{T} k_t(\lambda_t - \lambda_{t-1})\right] + k_0\lambda_0 - k_{T+1}\lambda_T.$$

[4] In Chapter 11 we discussed the equivalence of the condition $\partial L/\partial\lambda$ and the condition that the equality constraint holds exactly.

This result allows us to rewrite (15.3) as

$$L = \sum_{t=1}^{T}[m \cdot \pi(c_t, k_t, t) + \lambda_t \cdot m \cdot f(c_t, k_t, t) + k_t(\lambda_t - \lambda_{t-1})] + k_0\lambda_0 - k_{T+1}\lambda_T$$
$$+ m \cdot \pi(c_0, k_0, 0) + \lambda_0 \cdot m \cdot f(c_0, k_0, 0) - \mu_0(k_0 - \underline{k}) - \mu_T(k_T - \bar{k}). \qquad (15.8)$$

Then we can more easily obtain the first-order condition with respect to k_t, which is

$$\frac{\partial L}{\partial k_t} = m \cdot [\pi_k(c_t, k_t, t) + \lambda_t \cdot f_k(c_t, k_t, t)] + (\lambda_t - \lambda_{t-1}) = 0 \quad \text{for}$$
$$t = 1, 2, \ldots, T, \qquad (15.9)$$

where $\pi_k(c_t, k_t, t)$ and $f_k(c_t, k_t, t)$ represent the partial derivatives of the functions with respect to k_t.

We can interpret the first-order condition (15.9) by remembering that the Lagrange multiplier represents the effect of a small change in the constraint on the optimal value of the objective function.[5] In this context, the Lagrange multiplier λ_t represents the amount by which the maximum attainable value of the sum of profits is increased if an additional unit of capital is obtained at the end of the tth period at no cost in foregone spending on labor (that is, as if the capital were a pure gift). Thus λ_t is the shadow price of capital, and it reflects the marginal value of capital at the end of the tth period. Therefore $-(\lambda_t - \lambda_{t-1})$ represents the rate at which capital depreciates in value. The first-order condition (15.9) requires that the depreciation in value of capital over an interval equals the sum of its contribution to profits during that interval, $m \cdot \pi_k(c_t, k_t, t)$, plus its contribution to enhancing the value of the capital stock at the end of the interval, $m \cdot \lambda_t \cdot f_k(c_t, k_t, t)$. As noted by Robert Dorfman, along the optimal time path, "a unit of capital loses value or depreciates as time passes at the rate at which its potential contribution to profits becomes its past contribution."[6]

An alternative way to present the first-order conditions (15.4), (15.5), and (15.9) involves the **Hamiltonian** function

$$H(c_t, k_t, \lambda_t, t) = \pi(c_t, k_t, t) + \lambda_t \cdot f(c_t, k_t, t). \qquad (15.10)$$

The product $m \cdot H(c_t, k_t, \lambda_t, t)$ represents the total value of activity in period t since it is the sum of total profits earned over that period, $m \cdot \pi(c_t, k_t, t)$, plus the amount of capital accumulated during that period, $m \cdot f(c_t, k_t, t)$, times the marginal value of capital at that time, λ_t.

The Lagrangian function (15.3) can be rewritten with this Hamiltonian function as

$$L = \sum_{t=0}^{T}[m \cdot H(c_t, k_t, \lambda_t, t) - \lambda_t(k_{t+1} - k_t)] - \mu_0(k_0 - \underline{k}\ ) - \mu_T(k_T - \bar{k}). \qquad (15.11)$$

[5] See the discussion in Chapter 11. The following interpretation draws from Robert Dorfman, "An economic interpretation of optimal control theory," *American Economic Review* 59, no. 5 (December 1969): 817–831.

[6] Ibid., 821.

Along the optimal path where the Hamiltonian takes its maximum value, which we denote as $H(c_t^*, k_t^*, \lambda_t, t)$, the first-order condition (15.4) shows that

$$\frac{\partial L}{\partial c_t} = m \cdot \frac{\partial H(c_t^*, k_t^*, \lambda_t, t)}{\partial c_t} = 0 \quad \text{for} \quad t = 0, 1, \ldots, T. \tag{15.12}$$

The Envelope Theorem allows us to ignore terms that arise from the chain rule. For example, the term

$$\frac{\partial H(c_t^*, k_t^*, \lambda_t, t)}{\partial k_t^*} \cdot \frac{\partial k_t^*}{\partial c_t}$$

equals zero because, along the optimum path, $\partial k_t / \partial c_t = 0$. Since the Hamiltonian represents the total value of activity, the first-order condition (15.12) can be interpreted as requiring that consumption in each period maximizes the total value of activity in that period. The first-order condition (15.5) can be written as

$$\frac{\partial L}{\partial \lambda_t} = m \cdot \frac{\partial H(c_t^*, k_t^*, \lambda_t, t)}{\partial \lambda_t} - (k_{t+1} - k_t) = 0 \quad \text{for} \quad t = 0, 1, \ldots, T \tag{15.13}$$

when the partial derivative of the Hamiltonian is evaluated along the optimal path. This just restates the capital accumulation constraint (15.2) since, along the optimal path, $\partial H(c_t^*, k_t^*, \lambda_t, t)/\partial \lambda_t = f(c_t^*, k_t^*, t)$. Finally, substituting the Hamiltonian into the Lagrange function as it is written in (15.8) and taking the partial derivative with respect to k_t along the optimal path, we have the restatement of (15.9) as

$$\frac{\partial L}{\partial k_t} = m \cdot \frac{\partial H(c_t^*, k_t^*, \lambda_t, t)}{\partial k_t} + (\lambda_t - \lambda_{t-1}) = 0 \quad \text{for} \quad t = 1, 2, \ldots, T. \tag{15.14}$$

This problem has a specified terminal value k_T. An alternative situation arises when the terminal value can be chosen optimally. In this case the term $-\mu_T(k_T - \bar{k})$ is not included in the Lagrangian functions (15.3) and (15.8), and the first-order condition (15.7) is omitted. In its place is the first-order condition

$$\lambda_T k_T = 0, \tag{15.7$'$}$$

which is known as the **transversality condition.** This condition meets the requirement that

$$\frac{\partial L}{\partial k_{T+1}} = -\lambda_T = 0$$

if k_T does not equal zero. The intuition behind the transversality condition is that if there is any capital left at the end of the relevant time period ($k_T \neq 0$), then its shadow price must be zero.[7] Alternatively, if there is a positive shadow price at the end of the relevant time period, then there must be no capital left. If the transversality condition

[7] The relevant time period in this problem ends with period T; but the state equation dictates a value for k_{T+1} given k_T, and the optimal value c_T^*.

is not met, then profits could have been higher had there been a different allocation of spending between labor and capital.

These first-order conditions constitute the **maximum principle** for discrete time dynamic optimization problems. As we will see in the next section, there is a similar set of conditions that constitute the maximum principle for problems in continuous time.

Exercises 15.1

1. Consider a firm with the profit function

$$\Pi_t = F(K_t, L_t) - w_t L_t - I_t,$$

where Π_t is profits, K_t is capital, L_t is labor, w_t is the (exogenous) wage, and I_t is investment. Subscripts refer to the time period. The production function, $F(K_t, L_t)$ has continuous first partial derivatives with $F_i > 0$ and $F_{ii} < 0$ for $i = K, L$. The managers of the firm wish to maximize the present discounted value of profits from the present (period 0) to period T, which equals

$$V_0 = \sum_{s=0}^{T} \left(\frac{1}{1+r} \right)^s \Pi_s.$$

The capital accumulation constraint facing the firm is

$$K_{t+1} = (1 - \delta)K_t + I_t,$$

where δ is the depreciation rate. The firm initially has no capital, and in period T the firm must have capital equal to $\underline{K}$. Set up the Hamiltonian for this problem. Solve for the optimal amount of labor to hire and the optimal amount of investment in each period.

2. Consider a variant of question 1 in which there is a cost to installing capital, C_t, which is

$$C_t = \frac{\gamma I_t^2}{2K_t}$$

and therefore the total cost of investment is

$$I_t + C_t = I_t + \frac{\gamma I_t^2}{2K_t}.$$

The firm's profit in any period is

$$\Pi_t = F(K_t, L_t) - w_t L_t - I_t - \frac{\gamma I_t^2}{2K_t}.$$

Assume that the capital accumulation equation is

$$K_{t+1} = K_t + I_t$$

that, as in the previous example, $K_0 = 0$, and that at the terminal period T, $K_T = \underline{K}$.

(a) Set up the Hamiltonian in this case.

(b) Does the first-order condition for labor differ from that in question 1?

(c) Solve for the first-order conditions for capital and investment.

(d) Denoting the Lagrange multiplier λ_t, define $q_t = \lambda_t(1 + r)^t$. Show that the optimal solution includes the condition

$$q_t(1 + r) = F_K(K_{t+1}, L_{t+1}) + \frac{\gamma I_{t+1}^2}{K_{t+1}^2} + q_{t+1}.$$

3. In Chapter 11 we discussed a two-period intertemporal consumption problem. Consider a T-period version of that problem. A consumer attempts to maximize, at time 0,

$$\sum_{t=0}^{T} \beta^t U(C_t),$$

where C_t is consumption in period $t(C_t > 0)$, β is a subjective discount rate $(1 > \beta > 0)$, and the utility function has the properties

$$U'(C_t) > 0, U''(C_t) < 0, U(0) = -\infty.$$

The consumer enjoys an exogenous sequence of income $\{Y_s\}_{t=0}^{T}$ and faces the budget constraint

$$(1 + R)B_t + Y_t = C_t + B_{t+1},$$

where R is the real interest rate and B_t is the consumer's portfolio of bonds in period t. We assume $B_0 = 0$ and require $B_{T+1} = 0$.

(a) Set up the Hamiltonian function for this problem.

(b) Use the first-order conditions to solve for the optimal relationship between marginal utility in any two adjacent periods.

(c) Compare this solution to the one obtained for the two-period problem in Chapter 11.

4. Consider an extension of question 3 in which consumers must produce their income using capital which they accumulate through investment. The budget constraint in any period t is

$$(1 + R)B_t + F(K_t) = C_t + B_{t+1} + I_t,$$

where $F(K_t)$ is a production function with a continuous first derivative, $F' > 0$ and also $F'' < 0$, and I_t is investment in period t. The capital accumulation equation is

$$K_{t+1} = I_t + K_t.$$

Again consider this problem from period 0 to period T with $B_0 = 0$, $K_0 = 0$, $B_T = 0$, and $K_T = \underline{K}$, and assume the representative consumer wishes to maximize, at time 0,

$$\sum_{t=0}^{T} \beta^t U(C_t),$$

where C_t is consumption in period $t(C_t > 0)$, β is a subjective discount rate $(1 > \beta > 0)$, and the utility function has the properties

$$U'(C_t) > 0, \quad U''(C_t) < 0, \quad \text{and} \quad U(0) = -\infty.$$

(a) Set up the Hamiltonian in this case. The Hamiltonian must include a set of Lagrange multipliers for capital and another set of Lagrange multipliers for bonds.

(b) Use the first-order conditions to determine the condition for the optimal amount of capital each period, a condition which will use $F'(K_t)$.

15.2 OPTIMAL CONTROL THEORY

Optimal control theory shows how to solve continuous time maximization problems in which the objective function includes an integral and the constraints include a differential equation. In this section we build on the intuition developed in the discrete time problems discussed in the previous section to present the main results of optimal control theory. Although a formal proof of the continuous time maximum principle is well beyond the scope of this book, the appendix to this chapter presents a heuristic argument for the necessary conditions for an optimal time path.

The continuous time version of the problem facing the managers of the firm discussed in the previous section involves the optimal selection of a time path of the control variable $c(t)$, which represents the flow of expenditures on labor. In general, the path of the control variable in an optimal control problem must be **piecewise continuous.** This condition requires that the path of the control variable is continuous but for (possibly) some finite number of jump discontinuities and that the size of the jumps is finite. This problem also includes a constraint on the accumulation of the state variable, capital $k(t)$. The state variable in an optimal control problem must be continuous and **piecewise differentiable,** that is, the path can have a finite number of "corners" or points where its derivative is not defined. Figure 15.1(a) illustrates the time path of a piecewise continuous variable, and Figure 15.1(b) illustrates the time path of a piecewise differentiable variable.

The profit of the firm over the time period $t = 0$ to T is a function of both the control variable and the state variable, as well as time itself, and is given by the integral

$$\int_{t=0}^{T} \pi(c(t), k(t), t)\, dt, \tag{15.15}$$

where $\pi(c(t), k(t), t)$ is the instantaneous profit function. This integral is the continuous time analogue to the sum in (15.1). The continuous time analogue to the difference equation (15.2) that describes the evolution of the capital stock is the differential equation

$$\dot{k}(t) = f(c(t), k(t), t), \tag{15.16}$$

where $\dot{k}(t) \equiv dk(t)/dt$. The functions $\pi(c(t), k(t), t)$ and $f(c(t), k(t), t)$ are assumed to be continuous in all their arguments and to have continuous first-order partial

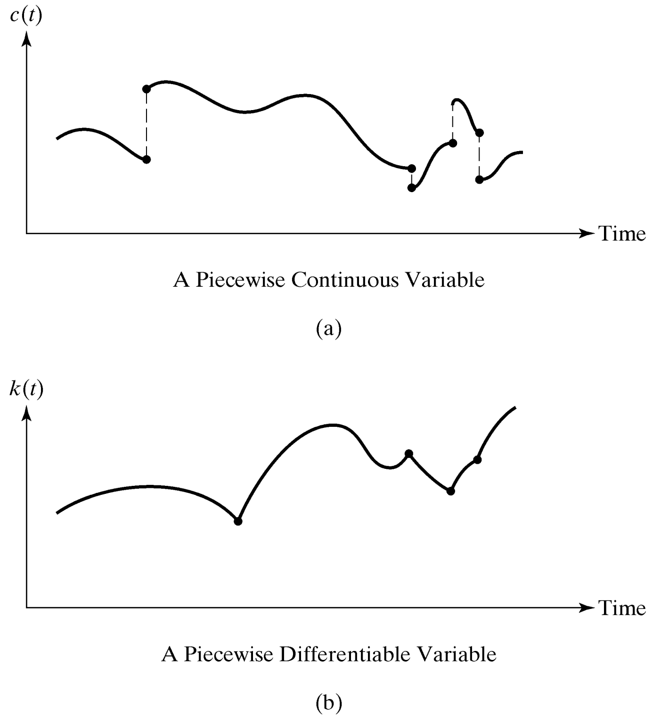

FIGURE 15.1 Piecewise Continuous and Piecewise Differentiable Variables

derivatives with respect to $k(t)$ and t, but not necessarily with respect to $c(t)$. The initial condition mirrors that in the problem in Section 15.1, with

$$k(0) = \underline{k}. \tag{15.17}$$

We consider two possible cases with respect to the terminal condition, the fixed terminal condition case where

$$k(T) = \overline{k}$$

as well as the case where $k(T)$ is chosen optimally.

The **maximum principle** (also known as **Pontryagin's maximum principle**) provides a set of conditions that are necessary for obtaining the maximum value of the objective function subject to the constraints.

The Maximum Principle If $c(t)$ and $k(t)$ maximize

$$\int_{t=0}^{T} \pi(c(t), k(t), t)\, dt$$

subject to the state equation

$$\dot{k}(t) = f(c(t), k(t), t),$$

where

- $\pi(c(t), k(t), t)$ is piecewise continuous,
- $f(c(t), k(t), t)$ is continuous and piecewise differentiable,
- both $\pi(c(t), k(t), t)$ and $f(c(t), k(t), t)$ have continuous first-order partial derivatives with respect to $k(t)$ and t, and
- the initial condition $k(0) = \underline{k}$ holds,

then, for the Hamiltonian function

$$H(c(t), k(t), \lambda(t), t) = \pi(c(t), k(t), t) + \lambda(t) \cdot f(c(t), k(t), t),$$

the following maximum principle conditions must be met at the optimal values $c^*(t)$ and $k^*(t)$:

(i) $H(c^*(t), k^*(t), \lambda(t), t) \geq H(c(t), k^*(t), \lambda(t), t)$ for all t

(ii) $\dfrac{\partial H(c^*(t), k^*(t), \lambda(t), t)}{\partial k(t)} = -\dot{\lambda}(t)$

(iii) $\dfrac{\partial H(c^*(t), k^*(t), \lambda(t), t)}{\partial \lambda(t)} = \dot{k}(t)$

(iv) $k(0) = k$,

where the notation used in conditions (ii) and (iii) means that the partial derivatives are evaluated along the optimal path.

One of the following conditions must also be met:

(v) If the terminal value of the state variable must equal $\overline{k}$, then $k(T) = \overline{k}$.

(vi) If the terminal value of the state variable is not given, then $\lambda(T) = 0$. ■

We also note the following.

- If $c^*(t)$ is in the interior (rather than at a boundary) of the admissable region for the control variable and if the Hamiltonian is continuously differentiable in $c(t)$, then condition (i) implies

$$\frac{\partial H(c^*(t), k^*(t), \lambda(t), t)}{\partial c(t)} = 0.$$

- We can solve a problem that requires the minimization of the integral of an instantaneous objective function, $v(c(t), k(t), \lambda(t), t)$, in the same fashion as the maximum problem by simply defining a function

$$w(c(t), k(t), \lambda(t), t) = -v(c(t), k(t), \lambda(t), t)$$

and finding the maximum of the problem by using the function $w(c(t), k(t), \lambda(t), t)$.

We illustrate the use of the maximum principle with the following example, which has a simple geometric interpretation. Consider the problem of choosing the optimum path of the control $c(t)$ to maximize

$$\int_{t=0}^{10} -\sqrt{1 + c(t)^2} \; dt$$

subject to the constraints

$$\begin{aligned} \frac{dk(t)}{dt} &= c(t), \\ k(0) &= 4, \quad \text{and} \\ k(10) &= 24, \end{aligned}$$

where $k(t)$ is the state variable. We form the Hamiltonian

$$H(c(t), k(t), \lambda(t)) = -\sqrt{1 + c(t)^{\,2}} + \lambda(t) \cdot c(t).$$

The maximum principle shows that the first order-conditions are

$$\frac{\partial H(c(t), k(t), \lambda(t))}{\partial c(t)} = -\frac{c(t)}{\sqrt{1 + c(t)^2}} + \lambda(t) = 0, \tag{15.18}$$

$$\frac{\partial H(c(t), k(t), \lambda(t))}{\partial k(t)} = 0 = -\dot{\lambda}(t), \tag{15.19}$$

$$\frac{\partial H(c(t), k(t), \lambda(t))}{\partial \lambda(t)} = c(t) = \dot{k}(t), \tag{15.20}$$

$$k(0) = 4, \quad \text{and} \tag{15.21}$$

$$k(10) = 24. \tag{15.22}$$

The condition (15.19) shows that the costate variable is constant since $\dot{\lambda}(t) = 0$. This result, combined with (15.18), shows that the control variable $c(t)$ is also constant, and, therefore, we denote it as c. To solve for the optimal value c, we integrate condition (15.20) to obtain

$$\begin{gathered} \int_{t=0}^{10} \frac{dk}{dt}\, dt = \int_{t=0}^{10} c \cdot dt \\ k(10) - k(0) = c \cdot 10 - c \cdot 0 = 10 \cdot c. \end{gathered}$$

The conditions (15.21) and (15.22) show that $k(10) - k(0) = 24 - 4 = 20$, and, therefore, the optimal value of the control variable is $c = 20/10 = 2$.

This example has the geometric interpretation of finding the minimum distance between the two points (0, 4) and (10, 24) in a graph of the function $k = f(t)$. Figure 15.2 presents these two points along with an arbitrary function $f(t)$. The distance between the line segment connecting any two points i and $i + m$ on the function $f(t)$, is given by the Pythagorean Theorem to be

$$s_i = \sqrt{(\Delta t)^2 + (\Delta k(i))^2},$$

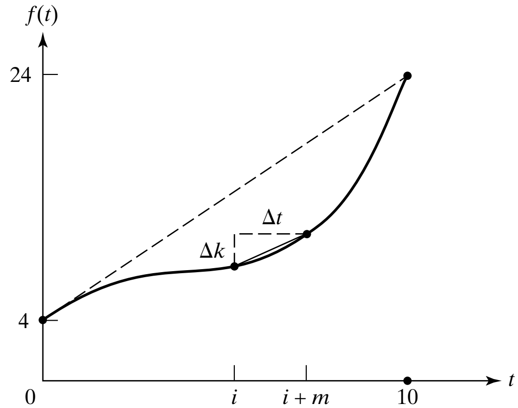

FIGURE 15.2 A Minimum Distance Problem

where $\Delta t = (i + m) - i = m$ and $\Delta k(i) = k(i + m) - k(i)$. This is equal to

$$s_i = \left(\sqrt{1 + \left(\frac{\Delta k(i)}{\Delta t}\right)^2}\right) \cdot \Delta t.$$

If the distance 0 to 10 is divided up into M sections, then the total distance of the function $f(t)$ can be approximated by

$$\sum_{i=0}^{M} s_i = \sum_{i=0}^{M}\left(\sqrt{1 + \left(\frac{\Delta k(i)}{\Delta t}\right)^2}\right) \cdot \Delta t.$$

In the limit, as $\Delta t \to 0$, this sum equals the integral

$$\int_{t=0}^{10} \sqrt{1 + \left(\frac{dk}{dt}\right)^2} \cdot dt.$$

Given the state equation $dk(t)/dt = c(t)$, the integrand is $\sqrt{1 + c(t)^2}$. The solution to the optimal control problem shows that the optimal value of c is the constant 2. Therefore the function that minimizes the distance is the straight line

$$k(t) = 2t + A,$$

where A is a constant. The terminal condition shows that $k(10) = 24$, and, therefore, $A = 24 - 20 = 4$. Thus the function that minimizes the distance between the points is

$$k(t) = 2t + 4.$$

As you may have suspected, and as the maximum principle confirms, the shortest distance between two points is a straight line.

The conditions that provide for a maximum in the continuous time case closely resemble those of the discrete-time case. While a complete proof of these conditions is beyond the scope of this book, the discussion in the appendix provides heuristic support for the maximum principle.

Sufficient Conditions

In the discussion of the static optimization problems in Chapters 10 and 11, we show that the concavity of the objective function is a sufficient condition for identifying an interior critical point as a maximum. In a dynamic optimization framework, O. L. Mangasarian shows that if the functions $\pi(c(t), k(t), t)$ and $f(c(t), k(t), t)$ are both concave in $c(t)$ and $k(t)$ and if the optimal solution has $\lambda(t) \geq 0$ for the case where $f(c(t), k(t), t)$ is nonlinear in $c(t)$ or $k(t)$, then the necessary conditions are also sufficient conditions. More generally, Kenneth Arrow and Mordecai Kurz show that if $H^*(k(t), \lambda(t), t)$ is the maximum of the Hamiltonian with respect to $c(t)$, given $k(t)$, $\lambda(t)$, and t, then, if $H^*(k(t), \lambda(t), t)$ is concave in $k(t)$, for given $\lambda(t)$ and t, the necessary conditions are also sufficient. This more general result, however, requires checking the concavity of the derived function $H^*(k(t), \lambda(t), t)$ rather than the functions that are basic to the problem, $\pi(c(t), k(t), t)$ and $f(c(t), k(t), t)$.[8]

Exercises 15.2

1. Consider the following problem involving the control variable $y(t)$ and the state variable $x(t)$. The objective is to maximize

$$\int_{t=0}^{40} -\frac{y(t)^2}{2}\, dt,$$

where the state variable is governed by the differential equation

$$\dot{x}(t) = y(t);$$

$x(0) = 20$ and $x(40) = 0$, and $t = 40$ is the terminal time.

(a) Set up the Hamiltonian.

(b) Find the conditions that satisfy the maximum principle.

(c) Solve for the explicit solution for the control variable, $y(t)$.

2. Consider the optimal control problem involving the control variable $y(t)$ and the state variable $x(t)$. The objective is to maximize

$$\int_{t=0}^{10} -\left(2x(t)y(t) + y(t)^2\right) dt$$

when the state variable is determined by

$$\dot{x}(t) = y(t);$$

the initial value of the state variable is $x(0) = 10$, while its terminal value is $x(10) = 100$.

(a) Set up the Hamiltonian.

(b) Find the conditions that satisfy the maximum principle.

(c) Solve for the explicit solution for the control variable.

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[8] O. L. Mangasarian, "Sufficient conditions for the optimal control of nonlinear systems," *Journal of SIAM Control,* 4 (February 1966): 139–152 and Kenneth Arrow and Mordecai Kurz, *Public Investment, the Rate of Return, and Optimal Fiscal Policy* (Baltimore: Johns Hopkins University Press, 1970).

3. Consider the optimal control problem in which we want to find the maximum value of

$$\int_{t=0}^{1} \ln\left(y(t)\right) dt$$

where $y(t)$ is the control variable and $x(t)$ is the state variable. The state variable has the initial value $x(0) = 0$ and the terminal value $x(1) = 30$; its value over time is determined by

$$\dot{x}(t) = -3y(t).$$

(a) Set up the Hamiltonian.
(b) Find the conditions that satisfy the maximum principle.
(c) Solve for the explicit solution for the control variable.

4. The managers of the firm Orange Computers want to maximum profits over the next five years before the firm goes public. This goal corresponds to maximizing

$$\int_{t=0}^{5} \left(K(t) - K(t)^2 - \frac{I(t)^2}{2}\right) dt,$$

where $K(t)$ is the capital stock of the firm and $I(t)$ is its investment. Because of the nature of the computer industry, capital depreciates very quickly as new technologies come on line, and therefore the capital accumulation equation is

$$\frac{dK(t)}{dt} = I(t) - \frac{K(t)}{2}.$$

The firm initially has no capital, and at the time the firm goes public, the managers want $K(5) = 10$.
(a) Set up the Hamiltonian for this problem.
(b) Show that the solution is saddlepath stable. What are the two characteristic roots of the system?

5. Show that the solution to the optimal control problem of maximizing

$$\int_{t=0}^{T} -\gamma \frac{y(t)^2}{2}\, dt$$

subject to

$$\dot{x}(t) = \alpha x(t) + \beta y(t)$$

is saddlepath stable. In this problem, $y(t)$ is the control variable, $x(t)$ is the state variable, the parameters α, β, and γ are all positive, and there is an initial condition and a terminal condition for the state variable.

15.3 EXTENSIONS AND APPLICATIONS OF OPTIMAL CONTROL THEORY

This section continues our discussion of optimal control theory by linking the maximum principle to the older technique of the calculus of variations, as well as by extending the framework discussed in several different ways. This section includes several economic applications that illustrate these extensions.

The Calculus of Variations

Optimal control theory is a generalization of an older technique called the calculus of variations. The calculus of variations shows how to solve the problem of maximizing the integral

$$\int_{t=a}^{b} f(\dot{x}(t), x(t), t)\, dt$$

subject to the conditions $x(a) = \underline{x}$ and $x(b) = \overline{x}$. We can easily express this in terms of an optimal control problem by defining

$$u(t) = \dot{x}(t)$$

and rewriting the integral as

$$\int_{t=a}^{b} f(u(t), x(t), t)\, dt.$$

Consider $u(t)$ as the control variable and $x(t)$ as the state variable. The Hamiltonian for this problem is

$$H(u(t), x(t), \lambda(t)) = f(u(t), x(t), t) + \lambda(t)u(t).$$

The first-order conditions from the maximum principle then show that the solution to this problem includes

$$\frac{\partial H(u(t), x(t), t)}{\partial u(t)} = \frac{\partial f(u(t), x(t), t)}{\partial u(t)} + \lambda(t) = 0$$

and

$$\frac{\partial H(u(t), x(t), t)}{\partial x(t)} = \frac{\partial f(u(t), x(t), t)}{\partial x(t)} = -\dot{\lambda}(t).$$

Taking the derivative with respect to time of the first condition, we have

$$\frac{d}{dt}\left(\frac{\partial f(u(t), x(t), t)}{\partial u(t)}\right) + \dot{\lambda}(t) = 0.$$

Recalling that $\dot{x}(t) = u(t)$, we can solve to remove $\dot{\lambda}(t)$ to obtain **Euler's equation** for the solution to this calculus of variation problem

$$\frac{\partial f(\dot{x}(t), x(t), t)}{\partial x(t)} = \frac{d}{dt}\left(\frac{\partial f(\dot{x}(t), x(t),t)}{\partial \dot{x}(t)}\right).$$

This formula is a necessary condition for the maximization of the integral given previously.

The minimum distance in Section 15.2 can be solved with Euler's equation. Let $f(\dot{x}(t), x(t), t) = -\sqrt{1 + (\dot{y}(t))^2}$, so $\dot{x}(t) = \dot{y}(t)$. Then Euler's equation shows that the solution to the minimum distance problem is

$$\frac{\partial\left(-\sqrt{1 + (y^{\cdot}(t))^2}\right)}{\partial y(t)} = \frac{d}{dt}\left(\frac{\partial\left(-\sqrt{1 + (y^{\cdot}(t))^2}\right)}{\partial y^{\cdot}(t)}\right).$$

Evaluating this equation, we have

$$0 = \frac{d}{dt}\left(\frac{\dot{y}(t)}{\sqrt{1 + (\dot{y}(t))^2}}\right),$$

and, therefore, $\dot{y}(t)$ is constant. Since $\dot{y}(t)$ is what we call $c(t)$ in the discussion of the maximum principle in the previous section, we see that this result from the calculus of variations is identical to the maximum principle conditions (15.18) and (15.19). The solution using the Euler equation technique from this point onward proceeds in the same fashion as discussed in Section 15.2.

Current-Value Hamiltonian

Many of the dynamic optimization problems studied in economics involve the discounted present value of the path of a function. For example, consider the present discounted profits of the firm described in Section 15.1 when the instantaneous profit function is $\pi(c(t), k(t))$. The time-zero value of profits over the period 0 to T, discounted at the rate ρ, is

$$\int_{t=0}^{T} e^{-\rho t}\pi(c(t), k(t))\,dt, \tag{15.23}$$

where the effect of time on the value of the instantaneous profit function can be separated out with the explicit discounting factor $e^{-\rho t}$. Assume the state equation for this problem is again (15.16). Then the Hamiltonian of this function, constructed as outlined previously, is

$$H(c(t), k(t), \lambda(t), t) = e^{-\rho t}\pi(c(t), k(t)) + \lambda(t) \cdot f(c(t), k(t), t).$$

The costate variable $\lambda(t)$ represents the shadow price of capital in time-zero units.

Alternatively, we can define

$$q(t) = \lambda(t)e^{\rho t}.$$

The costate variable $q(t)$ is the shadow-price of capital at moment t in time-t units and is called the **current-value shadow price.** In this case we can write the Hamiltonian as

$$\begin{aligned} H(c(t), k(t), \lambda(t), t) &= e^{-\rho t}[\pi(c(t), k(t)) + q(t) \cdot f(c(t), k(t), t)] \\ &= e^{-\rho t}\hat{H}(c(t), k(t), q(t), t), \end{aligned}$$

where $\hat{H}(c(t), k(t), q(t), t)$ is called the **current-value Hamiltonian** and is equal to the expression in square brackets above. The condition (i) required by the maximum principle in terms of the current value Hamiltonian is that we choose $c^*(t)$ such that

$$\hat{H}(c^*(t), k^*(t), q(t), t) > \hat{H}(c(t), k^*(t), q(t), t)$$

for all t, which is equivalent, under appropriate conditions, to

$$\frac{\partial \hat{H}(c^*(t), k^*(t), q(t), t)}{\partial c} = 0.$$

The first-order condition (ii) presented previously, in terms of the current-value Hamiltonian, can be derived by noting that

$$\frac{\partial H(c^*(t), k^*(t), \lambda(t), t)}{\partial k} = e^{-\rho t}\frac{\partial \hat{H}(c^*(t), k^*(t), q(t), t)}{\partial k}$$

and

$$-\frac{d\lambda(t)}{dt} = -\frac{dq(t)\cdot e^{-\rho t}}{dt} = -\dot{q}(t)e^{-\rho t} + \rho q(t)e^{-\rho t}.$$

Thus, in terms of the current-value Hamiltonian and the current-value shadow price, the first-order condition (ii) is

$$\frac{\partial \hat{H}(c^*(t), k^*(t), q(t), t)}{\partial k} = \rho q(t) - \dot{q}(t).$$

This condition has an interesting economic interpretation. The costate variable $q(t)$ represents the price of capital in terms of current profits. The instantaneous change in this variable, $\dot{q}(t)$ is the capital gain (that is, the change in the price of capital). The marginal contribution of capital to profits, $\hat{H}_k$, represents the dividend associated with capital. The discount factor ρ represents the rate of return on an alternative asset. Thus this condition requires that, along the optimal path,

$$\frac{\hat{H}_k}{q(t)} + \frac{\dot{q}(t)}{q(t)} = \rho$$

or that the overall rate of return to capital, which is the sum of the "dividend rate" $\hat{H}_k/q(t)$ and the "capital gain rate" $\dot{q}(t)/q(t)$, respectively, equals the rate of return on an alternative asset.

The first-order condition for the current-value Hamiltonian corresponding to condition (iii) given previously is

$$\frac{\partial \hat{H}(c^*(t), k^*(t), q(t), t)}{\partial q(t)} = \dot{k}(t).$$

The transversality condition $\lambda(T)\lambda(T) = 0$ becomes

$$q(T)e^{-\rho T} = 0$$

when the problem is set up in the form of a current-value Hamiltonian.

The current-value Hamiltonian is useful in consumption problems, as shown in the following application.

The Life-Cycle Theory of Consumption

The modern theory of consumption and saving is based upon models in which people choose a path of consumption in order to maximize the discounted value of utility. For example, the life-cycle theory analyzes the optimal consumption pattern of an individual or a family over their lifetime.[9] We use dynamic optimization here to illustrate the life-cycle theory.

We will determine the optimal consumption path for the couple Mr. and Mrs. Best from the time they are married in year "zero" until their simultaneous (and predicted) death at the moment of their fiftieth wedding anniversary. To simplify the analysis, we assume that the Bests have no children and leave no bequests. The discounted utility facing the Bests is

$$\int_{t=0}^{50} 2\sqrt{c(t)}e^{-\rho t}dt,$$

where $c(t)$ is the instantaneous flow of real consumption services. The parameter ρ represents the Bests' **subjective discount rate,** which is the rate at which they discount the utility from future consumption. The Bests' nominal income in any moment, $Y(t)$, is

$$Y(t) = W(t) + iA(t),$$

where $W(t)$ is the flow of nominal wage income, $A(t)$ is the nominal value of the stock of assets owned by the Bests, and i is the nominal interest rate. The Bests accumulate assets at the rate

$$\dot{A}(t) = Y(t) - C(t).$$

Combining the income and asset-accumulation equations, we have

$$\dot{A}(t) = W(t) + iA(t) - C(t).$$

Define the real value of assets as $a(t) = A(t)/p(t)$. This implies that

$$\dot{a}(t) = \frac{\dot{A}(t)}{p(t)} - a(t)\frac{\dot{p}(t)}{p(t)} = \frac{W(t) + iA(t) - C(t)}{p(t)} - a(t)\frac{\dot{p}(t)}{p(t)}.$$

Defining the real variables $w(t) = W(t)/p(t)$ and $c(t) = C(t)/p(t)$ and noting that the Fisher equation shows

$$r = i - \frac{\dot{p}(t)}{p(t)},$$

where r is the real interest rate, we have an equation for the accumulation of real assets,

$$\dot{a}(t) = w(t) + ra(t) - c(t).$$

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[9] The original contribution is Franco Modigliani and Richard Brumberg, "Utility Analysis and Consumption Function: An Interpretation of Cross Section Data," in *Post-Keynesian Economics,* K. Kurihara, ed. (New Brunswick, NJ: Rutgers University Press, 1954). Modigliani won the Nobel Prize in Economics in 1985 for his work on savings behavior, as well as for his work on financial markets.

This equation is the state equation of this model since it describes the evolution of the state variable $a(t)$. We also have the initial condition $a(0) = 0$ and $a(50) = 0$. The time path of wage income, $w(t)$, is assumed to be exogenous.

The current-value Hamiltonian for this problem is

$$\hat{H}(c(t), a(t), q(t)) = 2\sqrt{c(t)} + q(t)(w(t) + ra(t) - c(t)).$$

The conditions for the optimal time path of consumption are[10]

$$\frac{\partial \hat{H}(c(t), a(t), q(t))}{\partial c(t)} = \frac{1}{\sqrt{c(t)}} - q(t) = 0,$$

$$\frac{\partial \hat{H}(c(t), a(t), q(t))}{\partial a(t)} = rq(t) = \rho q(t) - \dot{q}(t),$$

$$\frac{\partial \hat{H}(c(t), a(t), q(t))}{\partial q(t)} = w(t) + ra(t) - c(t) = \dot{a}(t),$$

$$a(0) = 0, \quad \text{and}$$

$$a(50) = 0.$$

These conditions can be manipulated to yield a form of the solution in which the economic interpretation is more transparent. The first condition can be rewritten as

$$c(t)^{-1/2} = q(t).$$

Taking the derivative with respect to time of each side of this expression, we get

$$-\frac{1}{2}c(t)^{-3/2} \cdot \dot{c}(t) = \dot{q}(t).$$

The first-order condition for $\hat{H}_c$ shows that, along the optimal path, $c(t)^{-1/2} = q(t)$. Therefore we can divide the left-hand side of the above expression by $-1/2c(t)^{-1/2}$ and the right-hand side by $-1/2q(t)$ to get

$$\frac{\dot{c}(t)}{c(t)} = -2\frac{\dot{q}(t)}{q(t)}.$$

The first-order condition involving the partial derivative of the Hamiltonian with respect to $a(t)$ can be rewritten as

$$\frac{\dot{q}(t)}{q(t)} = \rho - r.$$

Combining this equation with the previous one, we get

$$\frac{\dot{c}(t)}{c(t)} = 2(r - \rho).$$

The optimality condition shows that the Bests' optimal-consumption path is one in which consumption is constant over time if the real interest rate equals their subjective

[10] We assume that $c(t) > 0$ for all t.

discount rate. The Bests' optimal consumption path is characterized by a steadily rising level of consumption if the real interest rate is greater than their subjective discount rate. The Bests' optimal consumption path is characterized by a steadily decreasing level of consumption if their subjective discount rate is greater than the real interest rate.

The Bests' consumption path must also satisfy an intertemporal budget constraint. Multiplying each side of the third optimality condition by e^{-rt} and then integrating with respect to time, we have, in present value terms, the budget constraint

$$\int_{t=0}^{50} (w(t) + ra(t) - c(t))e^{-rt}dt = \int_{t=0}^{50} \frac{da(t)}{dt}e^{-rt}dt.$$

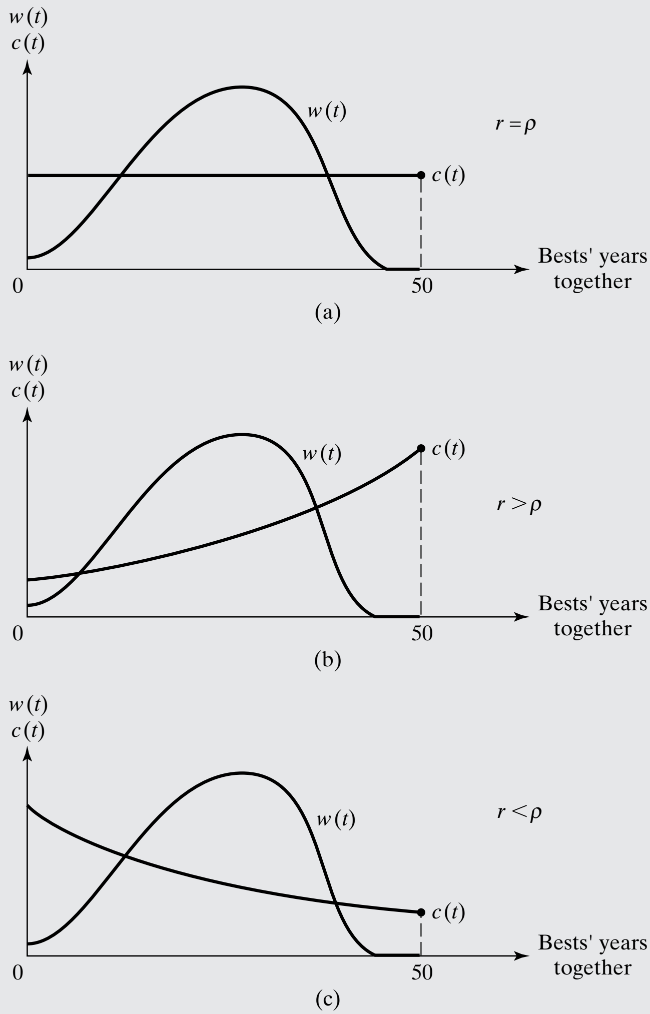

FIGURE 15.3 Optimal Lifetime Consumption Paths

We can further simplify the budget constraint for this problem by using the initial and terminal conditions. These conditions require that

$$\int_{t=0}^{50} \frac{da(t)}{dt} e^{-rt} dt = \int_{t=0}^{50} e^{-rt} da(t) = a(50)e^{-r \cdot 50} - a(0) = 0.$$

The terminal and initial conditions of this problem also require that

$$\int_{t=0}^{50} ra(t)e^{-rt} dt = 0.$$

Therefore the intertemporal budget constraint requires that

$$\int_{t=0}^{50} w(t)e^{-rt} dt = \int_{t=0}^{50} c(t)e^{-rt} dt,$$

that is, that the present value of the lifetime stream of wage income equals the present value of the lifetime stream of consumption.

A typical lifetime pattern of savings and consumption consistent with the life-cycle theory includes borrowing in the early part of one's life, paying back that debt and acquiring assets in mid-life, and living off accumulated savings at the end of one's life. We illustrate this type of pattern by showing the optimal consumption paths for the Bests under three different scenarios concerning the relative size of r and ρ in Figure 15.3(a), (b), and (c). In each of these cases, the path of wages is the same. Each case satisfies the intertemporal budget constraint. Therefore the discounted value of the area under the consumption path equals the discounted value of the area under the wage path. In Figure 15.3(a), where $r = \rho$ consumption is constant, and the Bests borrow money at the beginning of their married lives, pay this back and accumulate assets during the middle of their lives, and live off their savings in their final years. We compare this constant-consumption baseline case to the cases where the real interest rate does not equal the subjective discount rate. In Figure 15.3(b) where $r > \rho$ the Bests' consumption grows steadily over time. As compared to the constant-consumption case, the Bests' borrow less in their early lives and consume more at the end of their lives. Figure 15.3(c) illustrates the case where $r < \rho$, and consumption decreases over time. In this case, as compared to the constant-consumption case, the Bests borrow more at the beginning of their lives together and consume less at the end of their lives.

Infinite Horizon

The planning horizon in the problems previously discussed is finite with a fixed terminal date. In many contexts it is reasonable to consider the relevant planning horizon as the entire future. In this case the objective function, corresponding to (15.15), is

$$\int_{t=0}^{\infty} \pi(c(t), k(t), t)\, dt.$$

The terminal condition in an infinite horizon problem where the state variable must asymptotically approach some value $\bar{k}$ is simply

$$\lim_{t\to\infty}(k(t)) = \bar{k}.$$

The terminal condition in an infinite horizon problem where the state variable can be chosen optimally at all moments is

$$\lim_{t\to\infty}(\lambda(t)k(t)) = 0. \tag{15.24}$$

This transversality condition requires that if $k(t)$ remains nonzero and finite asymptotically, then $\lambda(t)$ must asymptotically approach zero. If $k(t)$ grows forever at some positive rate, then $\lambda(t)$ must approach zero at a faster rate such that the transversality condition is satisfied.

This transversality condition is not necessary when the subjective discount rate equals zero, that is, when, in an integral like (15.23), $\rho = 0$. In this case the necessary transversality condition is

$$\lim_{t\to\infty}(H^*(t)) = 0,$$

where $H^*(t)$ represents the value of the Hamiltonian along the optimal path at time t.[11]

Many problems in economics are cast in terms of an infinite horizon, reflecting the assumption that the appropriate terminal date is in the very distant future. For example, the life spans of corporations exceed the life spans of their current directors. It is also appropriate to cast intertemporal consumption problems in the framework of an infinite horizon if we assume that the current generation values the utility of future generations.

To illustrate this, we modify our optimal consumption example above by assuming Mr. and Mrs. Best have children (who themselves have children, and so on). The Bests value the utility of future generations, and, therefore, the appropriate discounted utility is represented by the improper integral

$$\int_{t=0}^{\infty} 2\sqrt{c(t)}e^{-\rho t}dt.$$

The only other distinction between this problem and the one previously presented is the replacement of the terminal condition $a(50) = 0$ with the transversality condition

$$\lim_{t\to\infty}(\lambda(t)a(t)) = \lim_{t\to\infty}(q(t)e^{\rho t}a(t)) = 0.$$

The solution to this problem includes the same equation for the growth of consumption as that in the finite-horizon case. In this case, however, the intertemporal budget constraint is

$$\int_{t=0}^{\infty}(w(t) + ra(t) - c(t))e^{-rt}dt = \int_{t=0}^{\infty} e^{-rt}da(t).$$

[11] See Phillipe Michel, "On the transversality condition in infinite horizon optimal problems," *Econometrica* 50, no. 4 (July 1982): 975–985.

With the initial condition $a(0) = 0$, this intertemporal budget constraint implies

$$\int_{t=0}^{\infty} w(t)e^{-rt}\,dt = \int_{t=0}^{\infty} c(t)e^{-rt}\,dt.$$

Unlike the previous case where $a(50) = 0$, this case allows for the possibility that the Bests leave a bequest to their children. This would be optimal if the current generation has relatively much higher income than subsequent generations. Alternatively, if later generations have relatively high incomes, then the optimal outcome is one where the current generation leaves a debt that future generations repay.

The following application provides another example of optimal consumption with an infinite time horizon.

Optimal Growth

The framework of the Solow growth model has been used in this book to analyze the optimal long-run savings rate (in the application about the Golden Rule in Chapter 9) as well as the time path of the economy (in the discussions in Chapters 13 and 14). In this application we turn to this model once again to analyze the optimal consumption path for an economy.

As discussed in Chapters 9, 13, and 14, the Solow growth model is based on some simple macroeconomic identities and some assumptions about the aggregate production function for the economy. The basic macroeconomic identity is $Y(t) = C(t) + I(t)$, where $Y(t)$ represents income, $C(t)$ represents consumption, and $I(t)$ represents investment. Another central relationship is that the change in the capital stock equals investment minus depreciation. Modeling depreciation as a constant proportion δ of the capital stock gives us the equation

$$\dot{K}(t) = I(t) - \delta K(t).$$

We frame this model in per capita terms and define income per capita, the capital stock per capita, and consumption per capita as $y(t)$, $k(t)$, and $c(t)$, respectively. Differentiation shows that the instantaneous change in the capital stock per capita at any moment t is

$$\dot{k}(t) = \frac{\dot{K}(t)}{N(t)} - \frac{\dot{N}(t)}{N(t)} \cdot \frac{K(t)}{N(t)}.$$

We assume that population growth $\dot{N}(t)/N(t)$ is constant and equal to n. Combining these relationships, we get the equation

$$\dot{k}(t) = y(t) - c(t) - (n + \delta)k(t).$$

We also assume that production (which equals income) is determined by a Cobb-Douglas production function that can be written in intensive form as $y(t) = k(t)^{\alpha}$ (see the discussion of the Golden Rule in Chapter 9). Thus the state equation in this problem is

$$\dot{k}(t) = k(t)^{\alpha} - c(t) - (n + \delta)k(t).$$

The extension of the Solow model required for analyzing the question of optimal growth requires the inclusion of an objective function. An objective function studied in the literature on optimal growth is

$$U = \int_{t=0}^{T} u(c(t)) \cdot e^{-\rho t} dt,$$

where $u(c(t))$ is the instantaneous utility function associated with the "representative" consumer and ρ is a discount factor with $\rho > 0$. The standard assumptions require that $u'(c(t)) > 0$ (nonsatiation, or utility always increases with an increase in consumption); $u''(c(t)) < 0$ (diminishing marginal utility); and $\lim_{c(t)\to 0} u'(c(t)) = -\infty$ (it is necessary to avoid extremely low levels of consumption). We also require that consumption is strictly positive. In this example we consider the instantaneous utility function

$$u(c(t)) = \ln(c(t)),$$

which has the required characteristics.

The current-value Hamiltonian function for this problem is

$$\hat{H}(c(t), k(t), \lambda(t)) = \ln(c(t)) + q(t) \cdot (k(t)^{\alpha} - c(t) - (n + \delta)k(t)).$$

The maximum principle shows that the first-order conditions are

$$\begin{aligned} \hat{H}_c &= \frac{1}{c(t)} - q(t) = 0, \\ \hat{H}_k &= q(t)(\alpha k(t)^{\alpha-1} - (n + \delta)) = \rho q(t) - \dot{q}(t), \quad \text{and} \\ \hat{H}_q &= k(t)^{\alpha} - c(t) - (n + \delta)k(t) = \dot{k}(t), \end{aligned}$$

and this problem also requires that the initial condition $k(0) = \underline{k}$ and the transversality condition

$$\lim_{t\to\infty} (q(t)e^{-\rho t}k(t)) = 0$$

are satisfied.

The first condition can be rewritten as $c(t)q(t) = 1$. Taking the derivative of each side of this expression with respect to time, we obtain

$$\dot{c}(t) \cdot q(t) + c(t) \cdot \dot{q}(t) = 0.$$

Dividing this expression by $c(t) \cdot q(t)$ and rearranging, we have

$$\frac{\dot{c}(t)}{c(t)} = -\frac{\dot{q}(t)}{q(t)},$$

which is similar to the first-order condition in the life-cycle consumption problem presented earlier.

As in the life-cycle model, we next solve for consumption growth as a function of variables other than the costate variable. To do so, we use the condition $\hat{H}_k = \rho q(t) - \dot{q}(t)$. After some manipulation of this condition, we have

$$\alpha k(t)^{\alpha-1} - (n + \delta) - \rho = -\frac{\dot{q}(t)}{q(t)},$$

which shows that consumption along the optimal path satisfies the condition

$$\frac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1} - (n + \delta) - \rho.$$

This condition, known as the **Keynes-Ramsey Rule,** was derived by Frank Ramsey in a 1928 article that includes an explanation attributed to John Maynard Keynes.[12] This rule implies that consumption increases over time if the marginal product of capital net of population growth and depreciation,

$$\alpha k(t)^{\alpha-1} - n - \delta,$$

is greater than the rate of time preference, ρ. Intuitively, this condition shows that when the marginal product of capital is relatively high, there is a larger benefit from depressing current consumption, which leads to a larger subsequent rate of growth of consumption.

Phase Diagram Depiction

Analysis of solutions to many types of continuous time dynamic optimization problems is aided by the development of an appropriate phase diagram. Here we show how to use a phase diagram to illustrate the solution to the optimal growth problem previously discussed.

The solution to the optimal growth problem can be expressed as two nonlinear differential equations in the two variables: consumption per worker, $c(t)$, and capital per worker, $k(t)$.[13] The two relevant equations are

$$\dot{k}(t) = k(t)^{\alpha} - c(t) - (n + \delta)k(t)$$

and

$$\dot{c}(t) = (\alpha k(t)^{\alpha-1} - (n + \delta) - \rho)c(t).$$

One way to analyze the solution to this problem, or to many other types of dynamic optimization problems, is to use a two-variable phase diagram like the type discussed in Chapter 14.

The two-variable phase diagram for this problem is depicted in Figure 15.4. The vertical axis of this figure is the amount of consumption, and the horizontal axis is the amount of capital. The figure includes a vertical line that shows the set of points where $\dot{c}(t) = 0$. The horizontal intercept of this line is the amount of capital k_G where

$$\frac{\alpha}{k_G^{1-\alpha}} - (n + \delta) - \rho = 0.$$

[12] Frank P. Ramsey, "A mathematical theory of savings," *Economic Journal* 38, no. 152 (December 1928): 543–559. For a discussion of representative agent macroeconomic models that are solved with optimal control theory, see Olivier Blanchard and Stanley Fischer, *Lectures on Macroeconomics* (Cambridge, Mass.: M.I.T. Press, 1989), Chapter 2. Blanchard and Fischer discuss the Keynes-Ramsey Rule on pages 41–43.

[13] Alternatively, the solution can be expressed as two nonlinear differential equations in $k(t)$ and $\lambda(t)$.

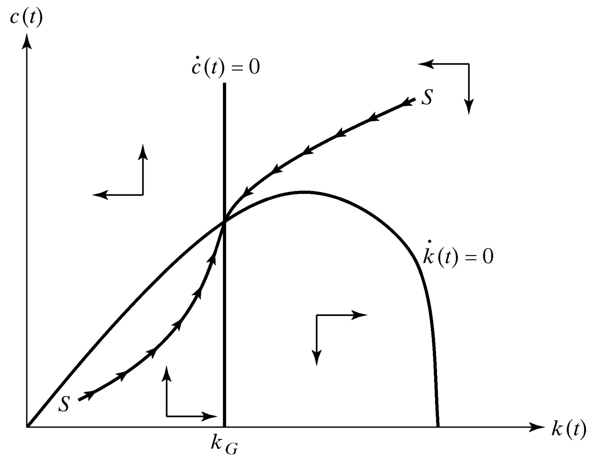

FIGURE 15.4 Phase Diagram of Optimal Consumption

For any amount of capital less k_0 than k_G, consumption rises since

$$\dot{c}(t) = (\alpha k_0^{\alpha - 1} - (n + \delta) - \rho)c(t) > 0,$$

and, for any amount of capital greater than k_G, consumption falls. This result gives us the vertical arrows that depict the forces of motion of the system when $k(t)$ does not equal k_G. These arrows point up to the left of the line $\dot{c}(t) = 0$ and point down to the right of the line $\dot{c}(t) = 0$.

The phase diagram in Figure 15.4 also includes a curve that depicts the points where $\dot{k}(t) = 0$. This curve begins at the point $(0, 0)$ and has a positive slope up to the point where $\alpha k(t)^{\alpha-1} - (n + \delta)$ equals zero and, thereafter, has a negative slope until it crosses the horizontal axis at the point where

$$k(t)^{\alpha} = (n + \delta)k(t).$$

Below this schedule, the level of consumption is less than the level that would cause $\dot{k}(t) = 0$ for the associated level of capital, and, since consumption enters the state equation negatively, the arrows of motion in this region point to the right. Above this schedule, the arrows of motion point to the left since the consumption-capital pairs in this region are associated with a decreasing capital stock per worker.

The intersection of the $\dot{c}(t) = 0$ and the $\dot{k}(t) = 0$ schedules represents the steady state value of consumption per worker and capital per worker. The system of nonlinear differential equations is saddlepath stable. The saddlepath, labeled as *SS* in Figure 15.4, passes through the point that represents the steady state value of consumption per worker and capital per worker. Given any initial positive capital stock, the saddlepath shows the unique path that consumption per worker and capital per worker must follow to be consistent with all the maximum principle conditions, including the transversality condition. The positive slope of the saddlepath indicates that if consumption per worker is growing, then investment per worker is positive (that is, $\dot{k}(t)$ is positive) and conversely.[14]

The Value of the Hamiltonian Over Time

The conditions of the maximum principle enable us to find a simple expression for the value of the Hamiltonian over time when it is evaluated along the optimal path. The total derivative of the Hamiltonian with respect to time, evaluated along its optimal path, equals

$$\frac{dH(c^*(t), k^*(t), \lambda(t), t)}{dt} = H_c^* \cdot \dot{c}(t) + H_k^* \cdot \dot{k}(t) + H_\lambda^* \cdot \dot{\lambda}(t) + \frac{\partial H^*}{\partial t},$$

where H_i^* represents the partial derivative of the Hamiltonian with respect to its ith argument, evaluated at its optimal value. The maximum principle requires that $H_c^* = 0$, $H_k^* = -\dot{\lambda}(t)$, and $H_\lambda^* = \dot{k}(t)$. Therefore

$$\begin{aligned}\frac{dH(c^*(t), k^*(t), \lambda(t), t)}{dt} &= 0 \cdot \dot{c}(t) - \dot{\lambda}(t) \cdot \dot{k}(t) + \dot{k}(t) \cdot \dot{\lambda}(t) + \frac{\partial H^*}{\partial t} \\ &= \frac{\partial H^*}{\partial t}.\end{aligned}$$

In the **nonautonomous case,** where time is a separate argument in the Hamiltonian, as with $H(c(t), k(t), \lambda(t), t)$, the total derivative of the Hamiltonian with respect to time equals its partial derivative with respect to time since the other terms arising from the chain rule cancel out. In the **autonomous case,** where the Hamiltonian does not include time as an explicit and separate argument, although it may include the discounting term $e^{-\rho t}$ as with $H(c(t), k(t), \lambda(t))$, the Hamiltonian is constant along the optimal time path.

This property of the Hamiltonian along the optimal path is used in problems of finding the optimal time to undertake an action, as shown in the following discussion.

Optimal Time

A variant of the maximum principle previously presented can be used for problems in which one component of the solution involves the determination of the terminal time. In problems in which the terminal value of the control $x(T)$ is given, but the terminal time, T, must be determined, the Hamiltonian evaluated along the optimal path must satisfy the condition

$$H(x^*(T), g^*(T), \lambda(T), T) = 0.$$

[14] Saddlepath stability can be shown by a linearization of the solution around the steady state. Linearization through the use of a Taylor series is discussed in Chapter 7. Saddlepath stability is discussed in Chapter 14. For more on the dynamics of this problem and a more detailed analysis of the phase diagram, see Chapter 2 of Blanchard and Fischer, *Lectures on Macroeconomics* or Chapter 2 of Robert Barro and Xavier Sala-i-Martin, *Economic Growth*.

This general result holds in both the nonautonomous and the autonomous cases. In the latter case, we previously show that the Hamiltonian is constant over time. Therefore, in the autonomous case, along the optimal path, the Hamiltonian must satisfy the condition

$$H(x^*(t), g^*(t), \lambda(t)) = 0 \quad \text{for} \quad 0 \leq t \leq T,$$

where the initial time is $t = 0$. This result is used in the following application.

Optimal Depletion of an Exhaustible Resource

Suppose that in your capacity as the manager of a gold mine you must determine the optimal rate at which gold should be extracted from the mine.[15] The instantaneous rate of extractions is $x(t)$, which is nonnegative. With the stock of gold in the mine equal to $g(t)$ at time t

$$x(t) = -\dot{g}(t).$$

The rate of extraction is the control variable in this problem, and the stock of gold is the state variable. The extraction equation is the state equation. We assume that the appropriate planning horizon is the entire future and future payments are discounted at the rate ρ. We also assume that the price of gold follows an exogenous time path with the price equal to $p(t)$ at time t. Finally, we assume that any amount up to a maximum $x_{\max}$ can be extracted from the mine costlessly at any moment. In this case the problem you face is to maximize the present value of the stream of payments

$$\int_{t=0}^{\infty} p(t)x(t)e^{-\rho t}\,dt$$

subject to the state equation

$$\dot{g}(t) = -x(t).$$

The initial stock of gold at time zero, $g(0)$, is

$$g(0) = \underline{G},$$

and we assume that $\underline{G} > x_{\max}$. At some time T, when the mine yields its last ounce of gold,

$$g(T) = 0.$$

This problem involves solving for the optimal T given the exogenous path of prices $p(t)$.

The Hamiltonian for this problem,

$$H(x(t), g(t), \lambda(t)) = p(t)x(t)e^{-\rho t} - \lambda(t)x(t),$$

is autonomous. One of the first-order conditions is

$$\frac{\partial H(x(t), g(t), \lambda(t))}{\partial \lambda(t)} \equiv H_{\lambda} = -x(t) = \dot{g}(t),$$

---

[15] This example is drawn from Colin Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources* (New York: John Wiley, 1976).

which gives us the state equation. Another first-order condition is

$$\frac{\partial H(x(t), g(t), \lambda(t))}{\partial g(t)} \equiv H_g = \dot{\lambda}(t),$$

but, noting that $H_g = 0$, we find that $\dot{\lambda}(t) = 0$ and, therefore, the costate variable λ is constant across time.

The maximum value of the Hamiltonian cannot be found by setting its partial derivative with respect to $x(t)$ equal to zero since the Hamiltonian, which can be written as

$$H(x(t), g(t), \lambda) = [p(t)e^{-\rho t} - \lambda]x(t),$$

is linear in $x(t)$. Instead, we find that the Hamiltonian is maximized if we choose an extraction policy such that

$$\begin{aligned} x(t) &= 0 \quad \text{if} \quad p(t)e^{-\rho t} < \lambda \\ x(t) &= x_{\text{max}} \quad \text{if} \quad p(t)e^{-\rho t} \geq \lambda \end{aligned}$$

since $x(t)$ must be nonnegative.

The condition that the Hamiltonian equals zero for $0 \leq t \leq T$ enables us to solve for λ. At the moment when the mine becomes fully depleted of gold, that is, at time T, when $x(T) = x_{\text{max}}$,

$$H(x(T), g(T), \lambda) = [p(T)e^{-\rho T} - \lambda]x_{\text{max}} = 0,$$

which implies

$$\lambda = p(T)e^{-\rho T}.$$

We use this result in developing the conditions for the maximization of the Hamiltonian.

$$\begin{aligned} x(t) &= 0 \quad \text{if} \quad p(t) < p(T)e^{\rho(t-T)} \\ x(t) &= x_{\text{max}} \quad \text{if} \quad p(t) \geq p(T)e^{\rho(t-T)}. \end{aligned}$$

This is called a **bang-bang solution** since it involves the control being either fully "on" when $x(t) = x_{\text{max}}$ or the control being fully "off" when $x(t) = 0$. The optimal solution may involve several intervals when mining takes place at full capacity, followed by intervals when the mine is left dormant.

Given the terminal date T, we can determine the date at which mining begins, T_S, by using the conditions $g(0) = \underline{G}$ and $g(T) = 0$, along with the state equation

$$\dot{g}(t) = -x(t).$$

In the case where mining continues unabated from T_S to T, these conditions imply

$$\begin{aligned} \underline{G} &= -(g(T) - g(0)) \\ &= -\int_{t=0}^{T} dg \\ &= -\int_{t=0}^{T} \frac{dg}{dt}\, dt \end{aligned}$$

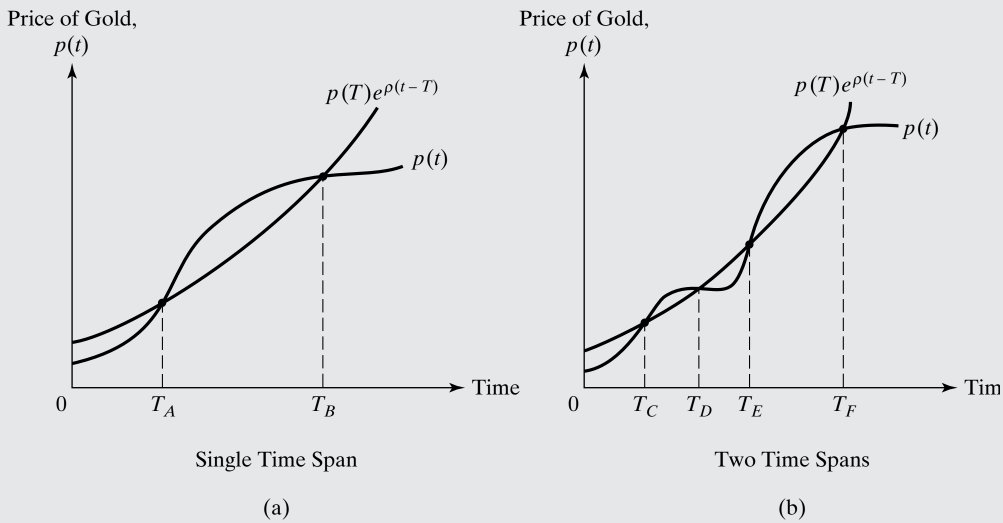

FIGURE 15.5 Phase Diagrams of Optimal Mining Times

$$
\begin{aligned}
&= \int_{t=0}^{T} x(t)\,dt \\
&= \int_{t=0}^{T_S} 0 \cdot dt + \int_{t=T_S}^{T} x_{\max}\,dt \\
&= (T - T_s)x_{\max}.
\end{aligned}
$$

For example, if $\underline{G} = 20$ tons, then, with $x_{\max} = 4$ tons/year, the period T_S to T is five years.

It may be the case that the optimal solution is one where the mine is alternatively opened and closed as the path of prices rises above and then falls below $p(T)e^{\rho(t-T)}$. In a case like this, where the mine is open from time T_C to time T_D, closed for a time, and then reopened from time T_E to time T_F, the analogue to the condition given previously is

$$\underline{G} = \int_{t=T_C}^{T_D} x_{\max}\,dt + \int_{t=T_E}^{T_F} x_{\max}\,dt.$$

Figure 15.5(a) and (b) illustrates two different solutions to this problem corresponding to two different paths of the price of gold. The price path in each of these graphs is depicted by the line labeled $p(t)$ These graphs also include the line representing $p(T)e^{\rho(t-T)}$. An optimal solution is one in which T is chosen such that $p(T)e^{\rho(t-T)}$ intersects the price path for an interval sufficiently long to just exhaust the total supply of gold. In Figure 15.5(a) this interval consists of the single time span $[T_A, T_B]$. In Figure 15.5(b) this interval consists of the two time spans $[T_C, T_D]$ and $[T_E, T_F]$ Each of these diagrams is drawn for the case where $\underline{G} = 20$ and $x_{\max} = 4$ tons/year, and, therefore, $T_B - T_A = (T_D - T_C) + (T_F - T_E) = 5$.

Exercises 15.3

1. In his 1928 paper "A Mathematical Theory of Savings," Frank Ramsey set up a problem in which the utility of future consumption was not discounted since he believed doing so would be "ethically indefensible." Instead, he solved the problem of choosing a capital stock, $k(t)$, to minimize the difference between the utility of consumption, $u(c(t))$, and a "bliss" level of utility, B. A simple version of his optimization problem is to maximize

$$-\int_{t=0}^{\infty} [B - u(c)]\, dt,$$

where

$$u'(c) > 0, \quad u''(c) < 0, \quad \text{and} \quad u(0) = -\infty$$

subject to the constraint

$$c(t) = f(k(t)) - \frac{dk(t)}{dt},$$

where $dk(t)/dt$ represents investment and $f(k(t))$ is a production function with $f' > 0$ and $f'' < 0$.

(a) Use the calculus of variations to show that the first-order condition for this problem is

$$-f'(k(t)) = \frac{d[u'(c(t))/dt]}{u'(c(t))}.$$

(b) Consider the special case of $f(k(t)) = k(t)^{\alpha}$ with $1 > \alpha > 0$ and $u(c(t)) = \ln(c(t))$. Find the first-order condition in this case. Compare it to the Keynes-Ramsey Rule presented in the text.

2. Suppose that Mr. and Mrs. Best receive a wedding gift of $20,000.

(a) How does this alter their lifetime consumption profile in the case where they leave no bequests?

(b) Suppose Mr. and Mrs. Best, having received a wedding gift of $20,000, want to leave a bequest of $20,000. Is their lifetime consumption any different from the case where they received no wedding gift and leave no bequest?

3. You have an inventory of G gold coins. You want to sell these coins at a time that maximizes the integral

$$\int_{t=0}^{\infty} p(t) G e^{-\rho t} dt,$$

where $p(t)$ is the exogenous time path of the price of one gold coin. What is the optimal time to sell these coins? (*Hint*: Consider the gold mine application where $x_{max} > \underline{G}$, and use a graph like the one presented in the text for the gold-mining example.)

Summary

This chapter presents the elements of dynamic optimization and optimal control theory. The discrete time analysis in Section 15.1 is a straightforward extension of the types of two-period problems in Chapter 11. The multiperiod problem presented in this section provides the intuition for the solution to the continuous time problem in Section 15.2. In that section we present the maximum principle, which provides necessary conditions for the identification of the optimal time path. In Section 15.3 we extend the discussion by considering some properties of the optimal solution, alternative frameworks for the problem, and the use of the phase diagram to depict the optimal solution.

Optimal control theory can be thought of as one intertemporal version of the constrained optimization problems discussed in Chapter 11. Constrained optimization problems are at the heart of economic analysis. Likewise, optimal control provides a very important set of tools for those areas in economics in which decisions that are made at different moments in time are central to our understanding of the relevant economic issues.

Appendix to Chapter 15: Heuristic Derivation of Maximum Principle

We can replicate the results from the maximum principle by using the Lagrangian multiplier approach in the special case when there is a continuous first-order partial derivative of the Hamiltonian with respect to the control variable. In this case the Hamiltonian has a continuous first-order partial derivative with respect to $c(t)$, and its maximum value is not at the boundary of the control variable. Under these conditions, consider the problem of maximizing (15.15) subject to the state equation (15.16), with a given initial value $k(0) = \underline{k}$. The Lagrangian-type function corresponding to this is

$$L = \int_{t=0}^{T} \pi(c(t), k(t), t)\, dt - \int_{t=0}^{T} \lambda(t)[\dot{k}(t) - f(c(t), k(t), t)]\, dt - \mu(k(0) - \underline{k}), \quad (15.\text{A1})$$

where $L \equiv L(c(t), k(t), \lambda(t), t, \mu)$; $\lambda(t)$ is the costate variable, which serves as a dynamic Lagrange multiplier; and μ is the Lagrange multiplier associated with the initial condition. We want to write (15.A1) to explicitly include the Hamiltonian function defined above. To do this, we first rewrite the second integral in (15.A1) as

$$-\int_{t=0}^{T} \lambda(t)[\dot{k}(t) - f(c(t), k(t), t)]\, dt = \int_{t=0}^{T} \lambda(t) \cdot f(c(t), k(t), t)\, dt - \int_{t=0}^{T} \lambda(t) \cdot \dot{k}(t)\, dt.$$

Noting that

$$\frac{d[\lambda(t)k(t)]}{dt} = \dot{\lambda}(t)k(t) + \lambda(t)\dot{k}(t),$$

we can write

$$\lambda(t)\dot{k}(t) = \frac{d[\lambda(t)k(t)]}{dt} - \dot{\lambda}(t)k(t).$$

Integrating both sides of this expression from $t = 0$ to $t = T$, we have

$$\int_{t=0}^{T} \lambda(t)\dot{k}(t)\,dt = \int_{t=0}^{T} \frac{d[\lambda(t)k(t)]}{dt}\,dt - \int_{t=0}^{T} \dot{\lambda}(t)k(t)\,dt.$$

Since

$$\int_{t=0}^{T} \frac{d[\lambda(t)k(t)]}{dt}dt = \lambda(T)k(T) - \lambda(0)k(0),$$

we can rewrite the previous expression as

$$\int_{t=0}^{T} \lambda(t)\dot{k}(t)\,dt = \lambda(T)k(T) - \lambda(0)k(0) - \int_{t=0}^{T} \dot{\lambda}(t)k(t)\,dt.$$

We can now write the Lagrangian (15.A1) as

$$L = \int_{t=0}^{T} [\pi(c(t), k(t), t) + \lambda(t)f(c(t), k(t), t)]\,dt + \int_{t=0}^{T} \dot{\lambda}(t)k(t)\,dt$$
$$- \lambda(T)k(T) + \lambda(0)k(0) - \mu(k(0) - \underline{k}\ ).$$

Defining the Hamiltonian

$$H(c(t), k(t), \lambda(t), t) = \pi(c(t), k(t), t) + \lambda(t)f(c(t), k(t), t)$$

enables us to rewrite this Lagrangian as

$$L = \int_{t=0}^{T} [H(c(t), k(t), \lambda(t), t) + \dot{\lambda}(t)k(t)]\,dt$$
$$- \lambda(T)k(T) + \lambda(0)k(0) - \mu(k(0) - \underline{k}). \tag{15.A2}$$

Denote the optimal time paths of the control and state variables as $c^*(t)$ and $k^*(t)$, respectively. Suppose that we deviate from the optimal path for the control by the arbitrary function

$$c(t, \epsilon) = c^*(t) \ + \ \epsilon \cdot p_1(t).$$

The state equation requires that there is some function that shows the corresponding deviation from the optimal path for the state variable

$$k(t, \epsilon) = k^*(t) \ + \ \epsilon \cdot p_2(t).$$

There is also a corresponding deviation from the optimal terminal value

$$k(T, \epsilon) = k^*(T) + \epsilon \cdot dk(T).$$

With the introduction of ϵ, we write the Lagrangian function (15.A2) as

$$L = \int_{t=0}^{T} [H\,(c(t, \epsilon), k(t, \epsilon), \lambda(t), t) + \dot{\lambda}(t)k(t, \epsilon)]\,dt$$
$$-\lambda(T)k(T, \epsilon) + \lambda(0)k(0) - \mu(k(0) - \underline{k}), \tag{15.A3}$$

where now we define $L \equiv L(c(t, \epsilon), k(t, \epsilon), \lambda(t), t, \mu)$. If, in fact, $c^*(t)$ is optimal, then a very small deviation from this path does not affect the value of the Lagrangian function (15.A3); that is,

$$\frac{\partial L(c(t, \epsilon), k(t, \epsilon), \lambda(t), t, \mu)}{\partial \epsilon} = 0.$$

This result allows us to determine the necessary conditions of the maximum principle. The partial derivative of (15.A3) with respect to ϵ, when evaluated at the optimal value of the control and state variables, is comprised of several component parts. The partial derivative will equal zero if each of these component parts equals zero, a result that yields several conditions. One of these conditions is

$$\frac{\partial}{\partial \epsilon}[\lambda(T)k^*(T, \epsilon)] = 0. \tag{15.A4}$$

Using the perturbation equation for the state variable, we see that the condition (15.A4) requires that

$$\frac{\partial}{\partial \epsilon}[\lambda(T)k^*(T, \epsilon)] = \frac{\partial}{\partial \epsilon}[\lambda(T) \cdot (k^*(T) + \epsilon \cdot dk(T))] = \lambda(T)\, dk(T) = 0,$$

which is satisfied by the transversality condition $\lambda(T) = 0$. Another required condition is

$$\frac{\partial}{\partial \epsilon}\left[\int_{t=0}^{T} [H^* + \dot{\lambda}(t)k^*(t, \epsilon)]\, dt\right] = 0, \tag{15.A5}$$

where, to avoid clutter, we define $H^* \equiv H(c^*(t, \epsilon), k^*(t, \epsilon), \lambda(t), t)$. This condition is satisfied when

$$\int_{t=0}^{T}\left[\frac{\partial H^*}{\partial \epsilon} + \frac{\partial \dot{\lambda}(t)k^*(t, \epsilon)}{\partial \epsilon}\right] dt = 0.$$

Using the chain rule and the equations showing the deviation from the optimal path, we find

$$\frac{\partial H^*}{\partial \epsilon} = \frac{\partial H^*}{\partial c} \cdot p_1(t) + \frac{\partial H^*}{\partial k} \cdot p_2(t)$$

and

$$\frac{\partial \dot{\lambda}(t)\, k^*(t, \epsilon)}{\partial \epsilon} = \frac{\partial \dot{\lambda}(t)\, k^*(t, \epsilon)}{\partial k} \cdot p_2(t) = \dot{\lambda}(t) \cdot p_2(t).$$

Thus, for any arbitrary functions $p_1(t)$ and $p_2(t)$, the expression

$$\left[\frac{\partial H^*}{\partial \epsilon} + \frac{\partial \dot{\lambda}(t)k^*(t, \epsilon)}{\partial \epsilon}\right] = \frac{\partial H^*}{\partial c} \cdot p_1(t) + \frac{\partial H^*}{\partial k} \cdot p_2(t) + \dot{\lambda}(t) \cdot p_2(t)$$

equals zero when

$$\frac{\partial H^*}{\partial c} = 0$$

and

$$\frac{\partial H^*}{\partial k} + \dot{\lambda}(t) = 0.$$

These two conditions correspond to condition (i) when there is a continuous first-order partial derivative of the Hamiltonian with respect to the control variable and to condition (ii) of the maximum principle. Condition (iii) of the maximum principle simply requires that the state equation is satisfied.