Lec 4: Mathematical Economics

Quadratic Forms

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Definition: Bilinear Form

Definition

A **bilinear form** on $\mathbb{R}^n$ is a function $L = L(x, y)$ of two arguments $x, y \in \mathbb{R}^n$ which is linear in each argument, i.e. such that

1. $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$;
2. $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$.

Note (1) If $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$, a bilinear form can be written as

$$L(x, y) = \sum_{j,k=1}^{n} a_{jk} x_k y_j = (Ax, y) = y^* Ax$$

where $\mathbf{A}_{nn}$ is determined uniquely by the linear form L.

Note (2): $L(x, y)$ is essentially an inner product.

Quadratic Forms

Fact

A quadratic form in n variables $x_1, ..., x_n$ is an expression

$$F = \sum\sum a_{jk}x_k x_j = a_{11}x_1x_1 + a_{12}x_1x_2 + ... + a_{1n}x_1x_n$$
$$+a_{21}x_2x_1 + ... + a_{2n}x_2x_n +$$
$$......$$
$$+a_{n1}x_nx_1 + ... + a_{nn}x_nx_n.$$

It is called a "quadratic" form because each term $a_{ij}x_ix_j$ contains either the square of a variable or the product of two different variables. A QF on R^n is the "diagonal" of a bilinear form L, i.e. that any quadratic form Q is defined by $Q[x] = L(x, x) = (Ax, x) = x^*Ax$.

Examples

(1) A QF $(a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2)$ in matrix form

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Change of Variables

It is often possible to simplify a QF x^*Ax by a change of variables $x = Sy$ or $y = S^{-1}x$, where S is, of course, a nonsingular matrix. We can go from x to y or from y to x. Substitution of $x = Sy$ into $F = x^*Ax$ gives $F = (Sy)^*A(Sy) = y^*S^*ASy = y^*By$, where $B = S^*AS$.

- Note that if **A** is a symmetric matrix, **B** is also symmetric.
- The $detA$ is called the **discriminant** of the QF x^*Ax.
- If $B = S^*AS$ is **congruent** to A, then the discriminant of the form y^*By is $|B| = |S^*|\ |A|\ |S| = |S|^2\ |A|$;

i.e. under a nonsingular change of the variables $x = Sy$, the discriminant of new QF assumes a magnitude of $|S|^2$ times that of the original form.

- The $\det S$ is often called the **modulus of the transformation** $x = Sy$.

Definition

CONGRUENCE: A square matrix **B** is said to be congruent to the square matrix **A** if there exists a nonsingular matrix **S** such that $B = S^*AS$.

Theorem

If we allow x to vary over all of $\mathbf{E}^n$, then the set of values taken on by $F = x^* Ax$ is called the range of the quadratic form.

Theorem

Under a **nonsingular transformation** *of variables, the* **range** *of a quadratic form remains unchanged.*

Proof.

Suppose, we have the form $F = x^* Ax$ and transform it by the change of variables $x = Sy$ or $y = S^{-1}x$ to obtain the new form $(Sy)^* A(Sy) = y^* S^* ASy = y^* By$.
Now it is only necessary to note that for any x there is a unique y (and, similarly, for any y there is a unique x) such that $F = x^* Ax = y^* By$. Hence $x^* Ax$ and $y^* By$ must have the same range. In general, this property will not hold if the matrix S is singular. For example, if $S = 0$, the range of $y^* By$ contains only a single number 0. □

Definiteness of QFs - definitions

Definitions

Positive Definite QF: The quadratic form x^*Ax is said to be positive definite if it is positive (> 0) for every x except $x = 0$.
Positive Semi-Definite QF: The quadratic form x^*Ax is said to be positive semidefinite if it is non-negative (≥ 0) for every x, and there exist points $x \neq 0$ for which $x^*Ax = 0$.
Indefinite Forms: A quadratic form x^*Ax is said to be indefinite if the form is positive for some points x and negative for others.

Fact

Negative definite and semidefinite forms are defined by interchanging words "negative" and "positive" in the above definitions. If x^*Ax is positive definite (semidefinite), then $x^*(-A)x$ is negative definite (semidefinite).

A symmetric matrix A is often said to be positive definite, positive semidefinite, negative definite, etc., if the respective QF: x^*Ax is positive definite, positive semidefinite, negative definite, etc.

Another set of necessary and sufficient conditions for Definiteness:

A set of necessary and sufficient conditions for the form x^*Ax to be positive definite is

$$a_{11} > 0, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0, \begin{pmatrix} a_{11} & a_{12} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} > 0, \ldots, |\mathrm{A}|>0;$$

If these n minors of A are positive, x^*Ax is positive definite; and x^*Ax is positive definite only if these minors are positive.

Fact

(1) *x^*Ax is positive (negative) definite iff every eigenvalue of A is positive (negative).*
(2) *x^*Ax is positive (negative) semidefinite iff all eigenvalues of A are non-negative (nonpositive), and at least one of the eigenvalues vanishes.*
(3) *x^*Ax is indefinite iff A has both positive and negative eigenvalues.*

Examples

(1). $F = 3x_1^2 + 5x_2^2$, $F = 2x_1^2 + 3x_2^2 + x_3^2$, $F = x_1^2$ are positive definite forms in two, three, and one variable, respectively.
(2). $F = 4x_1^2 + x_2^2 - 4x_1x_2 + 3x_3^2 = (2x_1 - x_2)^2 + 3x_3^2$ is positive semidefinite since it is never negative and vanishes if $x_2 = 2x_1$, $x_3 = 0$.
(3) $F = -2x_1^2 - x_2^2$, $F = -x_1^2 - x_2^2$, $F = -x_1^2$ are negative definite forms in two, two, and one variable, respectively.
(4) $F = 4x_1^2 - 3x_2^2$ is indefinite since it is positive when $x_1 = 1$, $x_2 = 1$ and negative when $x_1 = 0$, $x_2{=}1$.

Note: *A positive (negative) definite form remains positive (negative) definite when expressed in terms of a new set of variables provided the transformation of the variables is nonsingular.*

Thus if x^*Ax is positive (negative) definite and S is nonsingular, then $(Sy)^*A(Sy) = y^*S^*ASy = y^*By$ $(x = Sy)$ is positive (negative) definite. Try to prove this statement given the previous theorem.

Diagonalisation of QFs

Theorem

By an orthogonal transformation of variables every QF x^*Ax may be reduced to a diagonal form.

Proof. Given a QF x^*Ax, consider a nonsingular transformation of variables $x = Qy$, the matrix Q has as its columns a set of **orthonormal** eigenvectors of A which span E^n. The matrix Q is therefore an orthogonal matrix, and the transformation of variables is called an orthogonal transformation. In the diagonal form, the coefficient of y_j^2 is the eigenvalue λ_j of A. In terms of the variables y, the F becomes

$$y^*Q^*AQy = y^*Dy, \qquad \text{and} \qquad D = ||\lambda_j\delta_{ij}||$$

is a diagonal matrix whose diagonal elements are the eigenvalues of A.

Thus $y^*Dy = \sum_{j=1}^{n} \lambda_j y_j^2$. Only the squares of the variables appear; there are no cross products $y_i y_j$. A QF containing only the squares of the variables is said to be in diagonal form. Furthermore, we say that the transformation of variables $x = Qy$ has diagonalized the QF x^*Ax.

Examples

Consider the QF: $F = 2x_1^2 + 2\sqrt{2}x_1x_2 + x_2^2 = x^*Ax$

the symmetric matrix A is then $\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$.

To diagonalise the form, we transform $x = Qy$, where columns of Q are - $u_1 = [-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}]^T$ and $u_2 = [\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}]^T$ are orthonormal e.v.s of A.

Thus the transformation variables are -

$$x_1 = \frac{1}{\sqrt{3}}(-y_1 + \sqrt{y_2}), \qquad y_1 = \frac{1}{\sqrt{3}}(-x_1 + \sqrt{2}x_2)$$
$$x_2 = \frac{1}{\sqrt{3}}(\sqrt{2}y_1 + y_2), \qquad y_2 = \frac{1}{\sqrt{3}}(\sqrt{2}x_1 + x_2).$$

Note that $Q^{-1} = Q^*$; hence it is very easy to find the inverse transformation for an orthogonal transformation of variables. Since the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 3$, the form F becomes $F = 3y_2^2$ under this transformation of variables. The form is therefore positive semidefinite. The point $y = [2, 0]$ causes the form to vanish. The x corresponding to this y is $x = [\frac{2}{\sqrt{3}}, 2 - 2\sqrt{\frac{2}{3}}]$, and of course, F vanishes for this value of x.