

Delhi School of Economics

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Introductory Math Econ Practice Problem Set - 1

INSTRUCTOR: SUGATA BAG

- (1) Fill in the blanks in the proofs on page 3.
(2) Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ -6x - 4y \end{pmatrix}$$

- (a) Find a basis for the kernel of F . (This boils down to solving a very simple system of linear equations.)
(b) Find a basis for the image of F .
(c) Draw the image of F .
(d) Draw the set of solutions of $F(x, y)^T = (0, 0)^T$.¹ On the same graph, draw the set of solutions of $F(x, y)^T = (3, -6)^T$. Notice anything?
(e) How many solutions are there to $F(x, y)^T = (1, 1)^T$? What does this have to do with the image of F ?
(f) Find the matrix for F with respect to the standard basis for \mathbb{R}^2 .
(3) Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y + 8z \\ y + 2z \\ x + y + 5z \end{pmatrix}$$

- (a) Find a basis for the kernel of F . (This boils down to solving a system of linear equations.) Draw the kernel of F in \mathbb{R}^3 .
(b) Find a basis for the image of F . Draw the image of F in \mathbb{R}^3 . (You will not be graded for neatness.)
(c) Plot the set of solutions to $F(x, y, z) = (2, 0, 1)^T$ in \mathbb{R}^3 .
(d) Find the matrix for F with respect to the standard basis for \mathbb{R}^3 .
(4) Prove that the kernel of a linear transformation is a vector subspace. (Your proof should start “Let V and W be vector spaces, and $F: V \rightarrow W$ a linear transformation.” The whole proof should be quite short.)
(5) Let V be a real vector space and U_1, U_2 linear subspaces of V . Then $U_1 \cup U_2$ is a linear subspace of V if and only if either $U_1 \subset U_2$ or $U_2 \subset U_1$. Either
• Prove this statement or
• Draw several pictures in \mathbb{R}^2 and/or \mathbb{R}^3 indicating why it’s true, and give a short explanation in words of why it’s true.
(6) Matrices for reflections in \mathbb{R}^2 .
(a) Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote reflection across the line $y = x$. Find the matrix for F with respect to the standard basis for \mathbb{R}^2 .

¹The notation $(x, y)^T$ is shorthand for $\begin{pmatrix} x \\ y \end{pmatrix}$.

- (b) Find a basis for \mathbb{R}^2 with respect to which the matrix for F is diagonal. (Hint: there is a vector v so that $F(v) = v$. There's another vector w so that $F(w) = -w$.)
- (c) Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote reflection across the line through the origin making an angle θ with the x axis. Find the matrix for F with respect to the standard basis for \mathbb{R}^2 . (This takes some drawing.)
- (d) Find a basis for \mathbb{R}^2 with respect to which the matrix for F is diagonal.
- (7) Matrices for some maps between other vector spaces:
- (a) Define a map $F: \mathcal{P}_{\leq 3} \rightarrow \mathcal{P}_{\leq 2}$ by $F(p(x)) = p'(x) + p(2) + 3xp''(x)$. Find the matrix for F with respect to the bases $[1, x, x^2, x^3]$ for $\mathcal{P}_{\leq 3}$ and $[1, x, x^2]$ for $\mathcal{P}_{\leq 2}$.
- (b) Recall that $\mathcal{C}^0(\mathbb{R})$ denotes the vector space of continuous functions $\mathbb{R} \rightarrow \mathbb{R}$. Let $V = \text{Span}\{\sin(x), \cos(x)\} \subset \mathcal{C}^0(\mathbb{R})$. Let $F: V \rightarrow V$ be defined by $F(f(x)) = f'(x)$. Find the matrix for F with respect to the basis $\mathcal{B} = [\sin(x), \cos(x)]$.
- (c) With notation as in part 7b, let $G: V \rightarrow V$ be the linear transformation $G(f(x)) = f''(x)$. Find the matrix for G with respect to the basis \mathcal{B} .

²Hom stands for *homomorphism*, which is another word for linear transformation.

We will use the following lemma,

Lemma 1. *Let V be an n -dimensional vector space. Then any set of linearly independent vectors in V can have at most n elements.*

The following is sometimes called the *basis extension theorem*.

Theorem 1. *Let V be a finite-dimensional vector space, and S a linearly independent subset of V . Then S is contained in a basis for V .*

Proof. Write $S = \{v_1, \dots, v_k\}$. If $\text{Span}(S) = V$ then S is a _____ for V and we're done. Otherwise, there is some vector $v \in V$ such that $v \notin \text{Span}(S)$. Let $S_2 = S \cup \{v\}$. We claim that S_2 is linearly _____. Suppose not. Then there are numbers $a_1, \dots, a_k, a \in \mathbb{F}$ such that _____ If $a \neq 0$ then we have $v =$ _____ So, v lies in the _____ of v_1, \dots, v_k , which is a contradiction.

So, $a = 0$. But then

$$a_1v_1 + \dots + a_kv_k = 0.$$

Since S is _____, all of the a_i must be zero.

Now, repeat this process with _____ in place of S . Either $\text{Span}(S_2) = V$, in which case S_2 is a _____ for V and we're done; or, we can find a still larger set S_3 which is still _____. Repeat. By Lemma 1, any set of linearly _____ vectors in V can have at most $\dim(V)$ elements, the process must eventually terminate. But the only way it terminates is if one of the S_n is a basis for V . \square

Corollary 2. *Let $F: V \rightarrow W$ be a linear transformation, with V finite-dimensional. Then*

$$\dim(\ker(F)) + \dim(\text{Im}(F)) = \dim(V).$$

Proof. Let $\{e_1, \dots, e_k\}$ be a basis for $\ker(F)$. By the basis extension theorem, we can find vectors f_1, \dots, f_l so that $\{e_1, \dots, e_k, f_1, \dots, f_l\}$ is a basis for V . We claim that $\{F(f_1), \dots, F(f_l)\}$ is a

basis for $\text{Im}(F)$. To see this, we must show that $\{F(f_1), \dots, F(f_l)\}$ _____ the image of F and are linearly _____.

First, we prove that $\{F(f_1), \dots, F(f_l)\}$ span $\text{Im}(F)$. Indeed, if $w \in \text{Im}(F)$ then $w = f(v)$ for some v in V . Since $\{e_1, \dots, e_k, f_1, \dots, f_l\}$ _____ V , there are numbers $a_1, \dots, a_k, b_1, \dots, b_l \in \mathbb{F}$ such that

$$v = a_1e_1 + \dots + a_ke_k + b_1f_1 + \dots + b_lf_l.$$

But then

$$\begin{aligned} w = F(v) &= \text{_____} \\ &= \text{_____} \\ &= b_1F(f_1) + \dots + b_lF(f_l) \end{aligned}$$

since $F(e_i) = 0$. So, w is in the span of $\{F(f_1), \dots, F(f_l)\}$.

Next, we prove that $F(f_1), \dots, F(f_l)$ are linearly _____. Suppose that

$$a_1F(f_1) + \dots + a_lF(f_l) = 0.$$

Then,

$$\text{_____}.$$

So, $a_1f_1 + \dots + a_lf_l$ is in the _____ of F . So, $a_1f_1 + \dots + a_lf_l = b_1e_1 + \dots + b_ke_k$ for some $b_1, \dots, b_k \in \mathbb{F}$, since e_1, \dots, e_k _____ $\ker(F)$. But then $(-b_1)e_1 + \dots + (-b_k)e_k + a_1f_1 + \dots + a_lf_l = 0$. So, since $\{e_1, \dots, f_l\}$ is _____, all a_i and b_j are zero. So, $F(f_1), \dots, F(f_l)$ are linearly _____.

Since _____ is a basis for $\text{Im}(F)$, it follows that the dimension of $\text{Im}(F)$ is l . Since _____ is a basis for $\ker(F)$, the dimension of the kernel of F is k . And, since $\{e_1, \dots, e_k, f_1, \dots, f_l\}$ is a basis for _____, the dimension of _____ is $k + l$. This proves the corollary. \square