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Author(s): Charles Blackorby, David Donaldson, John A. Weymark
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SOCIAL CHOICE WITH INTERPERSONAL UTILITY COMPARISONS: A DIAGRAMMATIC INTRODUCTION*

BY CHARLES BLACKORBY, DAVID DONALDSON AND JOHN A. WEYMARK[1]

1. INTRODUCTION

Arrow's [1951, 1963] social-choice model is the basis of almost all of the literature in social-choice theory. His social-welfare function is a mapping from the set of all possible individual orderings of social alternatives into the set of social orderings of them. If individual preferences are represented by utility functions, we would say that these utility functions are ordinally measurable and interpersonally noncomparable.

Arrow's impossibility result has proved extremely robust in this framework. Modifications of conditions on the social-welfare function and weakening of the basic rationality requirement (that the function produce an *ordering*) have made it clear that there is very little room for sensible social choices when interpersonal comparisons of utility are not allowed. In recent years, the noncomparability assumption has been dropped, and the possibilities for social-choice rules expanded as a consequence. However, interpersonal comparisons of utility are often rejected as impractical because they are thought to be difficult to make, or because "exact" numerical utility scales cannot be constructed. Both objections may be softened a good deal when it is discovered that many social-evaluation rules require only partial comparability. For example, in a society of two people, if one of them gains from a proposed move and the other loses, the utilitarian rule requires only that the gain can be compared to the loss.

Sen [1974, 1977c, 1984] has provided a taxonomy of different measurability and comparability assumptions, and we follow it in this paper. We discuss several of the major candidates: ordinal noncomparability, ordinal full comparability (where levels of individual ordinal utilities may be compared), cardinal full comparability (where individual cardinal utilities — the functions are unique up to a positive affine transformation — are fully comparable), and cardinal unit comparability (where the individual cardinal utility functions have arbitrary

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origins but are otherwise comparable). Diagrammatic proofs of the corresponding possibility theorems are also provided.[2] Our primary sources for these results are d'Aspremont and Gevers [1977], Hammond [1976], Roberts [1980a, 1980b], and Sen [1974, 1977c, 1984]. The proofs of many of the theorems in these articles involve some mathematical sophistication and are not easily accessible to the nonspecialist reader. However, by making use of what is known as the "welfarism theorem" it is possible to conduct the analysis in terms of a welfare ordering of vectors of utilities obtained by the members of society. This permits the use of diagrammatic arguments and we present simple geometric proofs of the main theorems in this literature for the case of two individuals; these results generalize to the n-person case. For example, we are able to provide an elementary diagrammatic proof of Arrow's theorem. We believe that our diagrammatic analysis sheds a good deal of light on the issues at hand as well as providing a nontechnical introduction to the literature.[3]

In Section 2, we present the model and show that under a set of assumptions, called "welfarism" by Sen, all social orderings associated with a given rule can be represented by a single ordering of the space of utility n-tuples. In Section 3, we introduce the taxonomy of measurability and comparability of utilities. We prove a version of Arrow's theorem in Section 4, a possibility result for ordinally measurable, noncomparable utilities. Section 5 moves to ordinal full comparability, and we find that if individuals are treated anonymously, some welfare *position* (such as the worst-off) must dictate. In Section 6, we discuss the lexicographic version of the maximin rule. We then proceed in Section 7 to cardinal measurability with full or partial comparability and utilitarianism.

2. SOCIAL-EVALUATION FUNCTIONALS AND WELFARISM

The social-choice problem we consider has as its objective the specification of a social ordering of alternative social states based on the utility functions of the individuals in society. We refer to these social orderings of alternatives as social evaluations and call the mapping from individuals' utility functions to the social ordering a social-evaluation functional.[4]

In this section, we discuss a number of properties that social-evaluation functionals commonly possess. We also present a key result on social-evaluation functionals, what Sen [1977a, 1977c, 1984] calls the welfarism theorem. It is this theorem which permits the use of our diagrammatic approach. The welfarism

[2] This list is by no means exhaustive. See Sen [1977c, 1984] for a more complete discussion of the results obtained in this literature.

[3] Deaton and Muellbauer [1980, Chapter 9] hint at the kind of analysis presented here.

[4] A functional is a function whose domain of definition is a set of functions. Social-evaluation functionals are often referred to as social-welfare functionals in the literature. Our choice of terminology reflects a desire to distinguish carefully between orderings of social alternatives (social evaluations) and orderings of utility n-tuples. Here social-welfare statements refer to rankings of utilities.

theorem states that three very simple and appealing conditions on the social-evaluation functional imply that all the social orderings of alternatives generated by a social-evaluation functional can be represented by a single ordering on the space of utility n-tuples. Thus, only the actual utilities received by the members of society count in forming social evaluations. This theorem implies that the social-evaluation functional cannot exhibit preferences for certain alternatives based on features that are independent of individual welfares. For example, an alternative can not be singled out for special treatment simply because it is the status quo, because it respects individuals' rights, or because it prohibits the consumption of alcohol.

The set of alternative social states is $X=\{x, y, z, w,\ldots\}$ and it has at least three members. The set of individuals comprising the society in question is $N=\{1,\ldots, n\}$ where n is a finite number greater than one. Person i ($i=1,\ldots, n$) has a utility function U_i: $X\to \boldsymbol{R}$; U_i is a real-valued mapping from the set of social states. Thus, $U_i(x)$ is the utility i gets in state x. An n-tuple of utility functions $U=(U_1,\ldots, U_n)$ is called a profile of utility functions, or a *profile* for short. Hence, U is a vectorvalued function $U: X\to \boldsymbol{R}^n$. The set of all possible profiles is $\mathscr{U}$.

The problem of social evaluation is to determine an ordering R of the social states X, where xRy is interpreted to mean that x is at least as good as y.[5] This ordering is, in general, different for different realizations of U (the social ordering depends on the individual utility functions). If $\mathscr{R}$ is the set of all orderings of X, and $\mathscr{D}\subseteq\mathscr{U}$ is the set of admissible profiles, then this phenomenon of dependence is represented by the *social-evaluation functional* F: $\mathscr{D}\to\mathscr{R}$. $F(U)$ is the ordering of X produced by the social-evaluation functional F when the utility profile is U, and we write $R_U=F(U)$ with corresponding indifference and strict preference relations I_U and P_U, respectively.

In this paper, we impose no admissibility restriction, so $\mathscr{D}=\mathscr{U}$. In some social-choice discussions *a priori* domain restrictions play an important role; this literature is surveyed in Sen [1977b, 1984]. The lack of a restriction on $\mathscr{D}$ is called *unrestricted domain*

Unrestricted Domain: $\mathscr{D} = \mathscr{U}$.

A second simple and appealing condition on F is that if everyone is indifferent between a pair of alternatives, society should be as well. This condition prevents, for example, the imposition of the preferences of an outsider, and is called *Pareto indifference.*

Pareto Indifference: For all $x, y\in X$ and for all $U\in\mathscr{D}$, if $U(x)=U(y)$, then xI_Uy.[6]

[5] An ordering is a complete, reflexive, and transitive binary relation. R is (a) complete if and only if for all $x, y\in X$, $x\neq y$, xRy or yRx, (b) reflexive if and only if for all $x\in X$, xRx, and (c) transitive if and only if for all $x, y, z\in X$, (xRy and yRz) implies xRz. For an ordering R, indifference is defined by $xIy\leftrightarrow(xRy$ and $yRx)$ while strict preference is defined by $xPy\leftrightarrow(xRy$ and not $yRx)$.

[6] $U(x)=U(y)$ means that $U_i(x)=U_i(y)$ for all $i\in N$.

A third condition, *binary independence of irrelevant alternatives*, requires the social ordering over any pair of alternatives to be independent of the utility information about other alternatives.

Binary Independence of Irrelevant Alternatives: For all $U', U'' \in \mathscr{D}$, if for any $x, y \in X$, $U'(x) = U''(x)$ and $U'(y) = U''(y)$, then $R_{U'}$ and $R_{U''}$ coincide on $\{x, y\}$ (i.e., $xR_{U'}y \leftrightarrow xR_{U''}y$ and $yR_{U'}x \leftrightarrow yR_{U''}x$).

These conditions taken together are called *welfarism* and have very strong implications for F. Welfarism requires that the social-evaluation functional F must ignore all non-utility features of the alternatives, such as their names, and concentrate on the vector of utilities associated with any social state.[7] This property is known as *strong neutrality*.

Strong Neutrality: For all $x, y, w, z \in X$ and $U', U'' \in \mathscr{D}$, if $U'(x) = U''(w)$ and $U'(y) = U''(z)$, then $xR_{U'}y \leftrightarrow wR_{U''}z$ and $yR_{U'}x \leftrightarrow zR_{U''}w$.

THEOREM 2.1. *If a social-evaluation functional F satisfies unrestricted domain, then F satisfies Pareto indifference and binary independence of irrelevant alternatives if and only if F satisfies strong neutrality.*[8]

In the statement of strong neutrality, the alternatives are not required to be distinct. Thus, setting $w = x$ and $z = y$, it is obvious that strong neutrality implies independence. Similarly, by setting $U' = U''$, $w = y$, and $z = x$, strong neutrality implies Pareto indifference. The reverse implication is, of course, more interesting. Because of the fundamental importance of this result, we present the proof for the case when $\{x, y, w, z\}$ are distinct.

Let $u = (u_1, \ldots, u_n)$ and $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$ where $u = U'(x) = U''(w)$ and $\bar{u} = U'(y) = U''(z)$. By unrestricted domain, we can find another profile U''' such that $u = U'''(x) = U'''(w)$ and $\bar{u} = U'''(y) = U'''(z)$. These profiles are depicted below.

	x	y	w	z
U'	u	$\bar{u}$		
U''			u	$\bar{u}$
U'''	u	$\bar{u}$	u	$\bar{u}$

Utility values for the blanks in the table are left unspecified as are the values for all other alternatives. We have

[7] The social-evaluation functional may pay attention to the names of the individuals when ranking alternatives.

[8] This theorem has been established by Sen [1977c, Theorem 6]. Variants have been de veloped by Guha [1972], Blau [1976], and d'Aspremont and Gevers [1977].

$$\begin{aligned} xR_{U'}y &\leftrightarrow xR_{U'''}y && \text{(by independence)} \\ &\leftrightarrow wR_{U'''}z && \text{(by repeated application of Pareto indifference)} \\ &\leftrightarrow wR_{U''}z && \text{(by independence).} \end{aligned}$$

Similarly, $yR_{U'}x \leftrightarrow zR_{U''}w$, yielding strong neutrality.

While each of the welfarism axioms in isolation seems reasonable, the fact that in combination they imply strong neutrality has generated a great deal of criticism. Critiques of welfarism and its variants may be found in Roberts [1980b] and Sen [1977a, 1977c, 1979a].

It is of particular importance to note that strong neutrality demands not only that non-utility information be disregarded within a single utility profile, but also across profiles as well. This has the rather remarkable implication that when F satisfies the welfarism axioms (unlimited domain, Pareto indifference, and binary independence), all of the orderings R_U can be represented by a single ordering R^* of $\boldsymbol{R}^n$, the space of utility n-tuples.

THEOREM 2.2. *If a social-evaluation functional F satisfies unlimited domain, then F satisfies Pareto indifference and binary independence if and only if there exists an ordering R^* of $\boldsymbol{R}^n$ such that $\forall x, y \in X$, $\forall U \in \mathscr{U}$, $xR_Uy \leftrightarrow \bar{u}R^*\bar{\bar{u}}$ where $\bar{u} = U(x)$ and $\bar{\bar{u}} = U(y)$*[9].

In fact the ordering R^* is unique and is constructed as follows. Consider two arbitrary points $\bar{u}$ and $\bar{\bar{u}} \in \boldsymbol{R}^n$. By unrestricted domain, there exists a pair of alternatives $x, y \in X$ and some $U \in \mathscr{U}$ such that $U(x) = \bar{u}$ and $U(y) = \bar{\bar{u}}$. We then define $\bar{u}R^*\bar{\bar{u}} \leftrightarrow xR_Uy$ and $\bar{\bar{u}}R^*\bar{u} \leftrightarrow yR_Ux$. The neutrality result, Theorem 2.1, guarantees that this procedure does not depend upon the choice of U, x, or y in the construction. To complete the sufficiency part of the proof, all that is needed is to check that this R^* is in fact an ordering, which we leave to the reader. We refer to R^* as a *social-welfare ordering*. If R^* is representable by a real-valued function, this function is called a *social-welfare function*.

In addition to the welfarism axioms, various other conditions are often imposed on the social-evaluation functional. When the functional F satisfies the welfarism axioms, the ordering R^* inherits these properties. It is therefore sufficient, and simplifies the discussion, to make these assumptions directly on R^*. D'Aspremont and Gevers [1977] rigourously derive these properties (with the exception of continuity) from their primitives on F.

In this paper, we make use the *strong Pareto* condition; a welfare gain by some without loss by others is a social improvement.

Strong Pareto: For all $\bar{u}, \bar{\bar{u}} \in \boldsymbol{R}^n$, if $\bar{u}_i \geq \bar{\bar{u}}_i$ for all $i \in N$, then $\bar{u}R^*\bar{\bar{u}}$; if, moreover, there exists $k \in N$ such that $\bar{u}_k > \bar{\bar{u}}_k$, then $\bar{u}P^*\bar{\bar{u}}$.

A weaker version of this principle, the *weak Pareto* condition, declares a gain

[9] Theorem 2.2 is due to d'Aspremont and Gevers [1977, Lemma 3] and Hammond [1979, Theorem 1]. Luce and Raiffa [1958] hint at such a result.

by everyone to be a social improvement, but is silent when some are indifferent.

Weak Pareto: For all $\bar{u}$, $\bar{\bar{u}} \in \boldsymbol{R}^n$, if $\bar{u}_i > \bar{\bar{u}}_i$ for all $i \in N$, then $\bar{u}P^*\bar{\bar{u}}$.

Strong neutrality makes the names of alternatives irrelevant to the social-decision process. *Anonymity* does the same for individuals' names.

Anonymity: For all $\bar{u}$, $\bar{\bar{u}} \in \boldsymbol{R}^n$, if $\bar{u}$ is a permutation of $\bar{\bar{u}}$, *then* $\bar{u}I^*\bar{\bar{u}}$.

The last condition we consider is *continuity*. If $\bar{u}P^*u$ and $uP^*\bar{\bar{u}}$, continuity requires any curve connecting $\bar{u}$ and $\bar{\bar{u}}$ to cross the indifference curve containing u; there are no sudden jumps from being better than u to being worse than u. In terms of the underlying social-evaluation functional F, continuity requires the social ordering R_U to vary continuously with variations in U. It is thus an inter-profile condition.

Continuity: For all $\bar{u} \in \boldsymbol{R}^n$, $\{u \in \boldsymbol{R}^n | uR^*\bar{u}\}$ is closed and $\{u \in \boldsymbol{R}^n | \bar{u}R^*u\}$ is closed.[10]

3. MEASURABILITY AND COMPARABILITY

Strong neutrality implies that all the relevant information for a social ordering of x and y is contained in the utility n-tuples $U(x)$ and $U(y)$. In Arrow's [1951, 1963] framework for social choice, the usable information is further restricted. He assumed (in effect) that the individual utility functions are ordinal and that they are noncomparable across individuals. Therefore, all usable information is contained in the profile of individual preference orderings $\{R_1, \ldots, R_n\}$ implicit in the profile U.[11] No interpersonal utility information is admissible.

In this section, we offer a general framework for considering alternative assumptions about the measurability and comparability of utilities. Our discussion is based upon the work of d'Aspremont and Gevers [1977], Roberts [1980b], and Sen [1974, 1977c, 1979b, 1984].

Measurability assumptions specify what transformations may be applied to an individual's utility function without altering the individually usable information. Comparability assumptions specify how much of this information may be used interpersonally. It is convenient to consider measurability and comparability conditions simultaneously. Different measurability and comparability assumptions are obtained by partitioning $\mathscr{D}$ into information sets and requiring all utility profiles in the same information set to be mapped by the social-evaluation functional F into the same ordering of X. For example, in Arrow's framework, if U' and $U'' \in \mathscr{D}$ and U_i' is an increasing transform of U_i'' for each $i \in N$, then $F(U') = F(U'')$ (they yield exactly the same ordering of X).

[10] A set is closed if it contains its boundary.

[11] A utility profile U can be used to construct a profile of individual preference orderings by defining person i's preference ordering R_i on X as $xR_iy \leftrightarrow U_i(x) \geq U_i(y)$, for all x, $y \in X$.

Formally, we suppose that $\mathscr{D}$ has been partitioned into a set of information sets $S=\{S_t | t \in T\}$ where S_t is an information set for each $t \in T$ and T indexes the elements of the partition. We require the social-evalution functional to be constant on each information set.

Information Invariance: For all U', $U'' \in \mathscr{D}$, if U', $U'' \in S_t$ for some $t \in T$, $F(U')=F(U'')$.

The information sets we shall consider can all be generated using rather simple rules. The basic idea is as follows. All of the information conditions we discuss have the property that person i ($i=1,\ldots, n$) has the same indifference curves (over X) for all utility profiles in a given information set. Thus the difference between any two profiles U' and U'' which are informationally equivalent lies only in the numbers attached to each person's indifference curves. Consequently, we implicitly have a vector of increasing functions $\phi=(\phi_1,\ldots, \phi_n)$ which tells us how to change the numbers assigned to any list of indifference curves, one for each individual, in the U' profile to obtain the corresponding list of numbers in the U'' profile. If we take another pair of profiles $\bar{U}$ and $\bar{\bar{U}}$ which are in the same information set as U' and U'' we can also find the vector of functions $\phi'=(\phi'_1,\ldots, \phi'_n)$ needed to obtain $\bar{\bar{U}}$ from $\bar{U}$. Continuing in this fashion we determine the class Φ of all such functions obtained from pairs of utility profiles in this information set. In the examples we consider, the class Φ is the same for every information set in S. Furthermore, if we choose an arbitrary $\hat{\phi} \in \Phi$ and an arbitrary $U \in \mathscr{D}$, the profile $\hat{U}$ obtained from U using $\hat{\phi}$ will be informationally equivalent to U.

Formally, we call a vector of increasing functions $\phi=(\phi_1,\ldots, \phi_n)$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ an *invariance transformation* For all $U' \in \mathscr{D}$, if $U''(x)=\phi(U'(x)) \equiv [\phi_1(U'_1(x)),\ldots, \phi_n(U'_n(x))]$ for all $x \in X$, then U', $U'' \in S_t$ for some $t \in T$. Thus, U' and U'' are informationally equivalent. Φ, the class of such information-preserving transformations,[12] is an alternative way of describing S.

If we consider a finer partition of $\mathscr{D}$, there will be fewer utility profiles in any information set. Consequently, when we construct the corresponding class of information transformations, this class will contain fewer members. Taking a finer partition of $\mathscr{D}$ (or, equivalently, shrinking the allowable invariance transformations) formally captures the idea that the usable information has increased. With a finer partition there are now utility profiles which are in different members of the partition, and hence can be informationally distinguished, but which were originally in the same information set. Thus there is an inverse relationship between the size of the class of invariance transformations and the amount of information available.

In the presence of the welfarism axioms and information invariance, any partition S of $\mathscr{U}$ will impose structure on the social-welfare ordering R^* of $\boldsymbol{R}^n$.

[12] For Φ to partition $\mathscr{D}$, it must satisfy reflexivity (the set of identity transformations is in Φ), symmetry (if $\phi \in \Phi$, so is its inverse), and transitivity (if ϕ and ϕ' are in Φ, so is $\phi \circ \phi'$). See Roberts [1980b] and d'Aspremont and Gevers [1977].

The nature of this structure can be seen by noting that an invariance transformation ϕ maps utility n-tuples into utility n-tuples. For example, suppose U'' is the utility profile obtained by applying the invariance transformation ϕ to U'. If $u'=U'(x)$ and $\bar{u}=U'(y)$, we have $u''=U''(x)=\phi(u')$ and $\bar{\bar{u}}=U''(y)=\phi(\bar{u})$. Since U' and U'' are informationally equivalent, $R_{U'}$ and $R_{U''}$ must rank x and y in the same fashion. Thus, the social-welfare ordering of u' and $\bar{u}$ must be identical with the social-welfare ordering of u'' and $\bar{\bar{u}}$. This can be stated formally as *information invariance for* R^*.

Information Invariance for R^*: For all $u', u'', \bar{u}, \bar{\bar{u}} \in \boldsymbol{R}^n$, if $u''=\phi(u')$ and $\bar{\bar{u}}=\phi(\bar{u})$ for some $\phi \in \Phi$, then $u'R^*\bar{u} \leftrightarrow u''R^*\bar{\bar{u}}$.

Information invariance for R^* says that if one pair of utility n-tuples is obtained from another pair of utility n-tuples by an invariance transformation, then the social rankings of the pairs are identical. Accordingly, if the class of invariance transformations is shrunk, there will be fewer such pairs that must be ranked identically on informational grounds. In other words, the greater the usable information, the fewer the restrictions imposed on the social-welfare ordering. Consequently, as usable utility information is increased, the number of social-welfare orderings compatible with a given list of properties will also increase.

These ideas can be made more concrete by considering specific examples. In the Arrow framework we have *ordinally measurable, noncomparable utilities.*

Ordinally Measurable, Noncomparable Utilities: $\phi \in \Phi$ if and only if ϕ_i is an increasing transformation for all $i \in N$.

Here U' and U'' are in the same information set if and only if there exist increasing functions $(\phi_1, \ldots, \phi_n)$ such that $U''(x)=[\phi_1(U_1'(x)), \ldots, \phi_n(U_n'(x))]$ for all $x \in X$. In terms of the social-welfare ordering R^*, an example will illustrate the principle involved. Let $n=2$ and suppose $\phi_1(t)=2t$ and $\phi_2(t)=t-2$. If $u'=(1, 1)$ and $\bar{u}=(-1, 5)$ these transformations map u' into $u''=(2, -1)$ and $\bar{u}$ into $\bar{\bar{u}}=(-2, 3)$ as shown in Figure 3.1. With ordinally measurable, noncomparable utilities, $(1, 1)R^*(-1, 5) \leftrightarrow (2, -1)R^*(-2, 3)$. Person 1 is better off in u' than $\bar{u}$, and better off in u'' than $\bar{\bar{u}}$. Person 2 is better off in $\bar{u}$ than u', and better off in $\bar{\bar{u}}$ than u''. Information invariance with ordinally measurable, noncomparable utilities makes this the only usable information and so R^* must rank u' and $\bar{u}$ the same way that it ranks u'' and $\bar{\bar{u}}$.

Each of the information partitionings of $\mathscr{U}$ which we consider can be defined in a similar fashion, i.e., by specifying a class of invariance transformations Φ. The terminology used for information partitions reflects both a measurability assumption and the degree to which the individually-usable information is available interpersonally. For each measurability assumption we speak of full, partial, or noncomparable utilities when all, part, or none (respectively) of the information that is available intrapersonally can be used interpersonally.

In the example we considered, the only information available intrapersonally

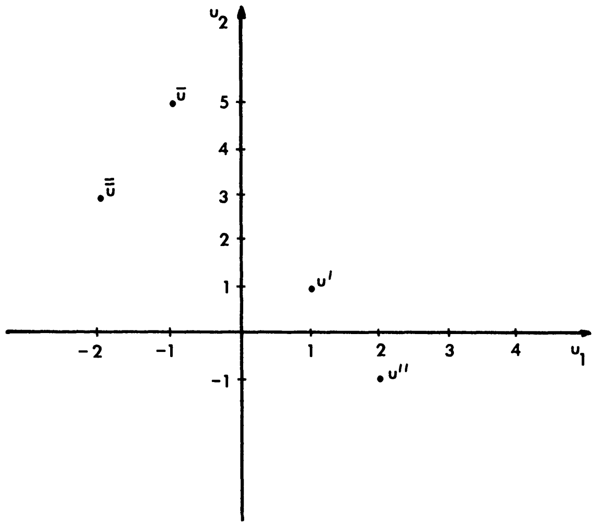

FIGURE 3.1

was that contained in the rankings $(R_1,\ldots, R_n)$ and none of this information was available for use interpersonally. With the same measurability assumption we can have full comparability.

Ordinally Measurable, Fully Comparable Utilities: $\phi \in \Phi$ if and only if $\phi_i = \phi_0$ for all $i \in N$ with ϕ_0 an increasing transformation.

In this case only a common transform is allowed. Consequently, this information restriction imposes fewer invariance requirements on the social-welfare ordering R^*. Comparisons of utility *levels* between people are permitted since $u_i \geq u_j \leftrightarrow \phi_0(u_i) \geq \phi_0(u_j)$.

If utilities are cardinally measurable, then we are restricted to increasing affine transformations.[13] We consider three comparability conditions.

Cardinally Measurable, Noncomparable Utilities: $\phi \in \Phi$ if and only if $\phi_i(t) = a_i + b_i t$ with $b_i > 0$ for all $i \in N$.

For this information restriction the affine transformations can be chosen independently for each person. As a consequence, while it is possible to compare utility gains and losses $[U_i(x) - U_i(y)]$ intrapersonally, it is not possible to do so interpersonally.

Cardinally Measurable, Unit-Comparable Utilities: $\phi \in \Phi$ if and only if $\phi_i(t) = a_i + bt$ with $b > 0$ for all $i \in N$.

[13] A function f: $\boldsymbol{R} \rightarrow \boldsymbol{R}$ is an increasing affine transformation if and only if $f(t) = a + bt$ with $b > 0$. Its graph is a straight line.

With cardinally measurable, unit-comparable utilities the affine transformations are restricted so that the terms scaling the units of measurement, the b's, are common to all individuals. Cardinal unit comparability permits the comparison of utility differences interpersonally, but does not allow the comparison of utility levels.

Cardinally Measurable, Fully Comparable Utilities: $\phi \in \Phi$ if and only if $\phi_i(t) = a + bt$ with $b > 0$ for all $i \in N$.

Moving to full comparability requires the use of identical transformations for each individual. Consequently, we may compare *both* utility levels and utility differences.

Utilities are measurable on a translation scale if and only if affine transformations which set the unit scaling term (b_i) equal to one are permitted. This measurability assumption enables one to calculate absolute differences in utility. If these changes are interpersonally comparable, the invariance condition is known as *translation-scale, full comparability*.

Translation-Scale Measurable, Fully Comparable Utilities: $\phi \in \Phi$ if and only if $\phi_i(t) = a + t$ for all $i \in N$.

As a final information partition,[14] we mention the possibility that all utility numbers have complete numerical significance. When this is the case each utility profile is in a different element of the partition S, so information invariance places no restrictions on R^*.

Perfectly Measurable, Fully Comparable Utilities: $\phi \in \Phi$ if and only if ϕ is the indentity mapping.

In some circumstances it is not necessary to specify exactly what information is available. This is modelled by considering collections of information partitions or collections of invariance-transformation classes. We shall have occasion to consider a situation where the ability to discriminate is at least as great as is possible with ordinally measurable, fully comparable utilities; this is known as *level-plus comparability*. Formally, let Φ_{OF} denote the class of invariance transformations corresponding to ordinally measurable fully comparable utilities.

Level-Plus Comparability: $\Phi \subseteq \Phi_{OF}$.

4. ARROW'S THEOREM

We are now in a position to put this apparatus to work. The first case we consider corresponds to the framework adopted by Arrow [1951, 1963]; utilities are

[14] There are, of course, many other information restrictions possible. Further examples are provided by Blackorby and Donaldson [1982], d'Aspremont and Gevers [1977], Gevers [1979], Roberts [1980b], and Sen [1977c, 1984], among others.

ordinally measurable and noncomparable. We strengthen the commonly used assumptions slightly, adding Pareto indifference to weak Pareto.[15] This enables us to use the social-welfare ordering R^*, since we also employ the other welfarism axioms, unrestricted domain and independence.

Arrow's theorem is usually stated as an impossibility result; his axioms are mutually inconsistent. Given that any subset of his axioms is consistent, Arrow's theorem can also be stated as a possibility theorem by negating one of his axioms. We adopt the latter approach.

The social-welfare ordering R^* is called a *dictatorship* if there is someone whose *strict* preferences are always replicated in the social ordering.[16]

Dictatorship: R^* is a dictatorship if and only if there is an individual $k \in N$ such that for all $\bar{u}$, $\bar{\bar{u}} \in \boldsymbol{R}^n$, if $\bar{u}_k > \bar{\bar{u}}_k$, then $\bar{u}P^*\bar{\bar{u}}$.

THEOREM 4.1. *If the social-evaluation functional F Fsatisfies the welfarism axioms, then the socil-welfare ordering R^* satisfies information invariance with ordinally measurable, noncomparable utilities and week Pareto if and only if it is a dictatorship.*

We now provide a diagrammatic proof of Theorem 4.1 for the two-person case.

We first order all utility 2-tuples in $\boldsymbol{R}^2$ with respect to the point $\bar{u}$ in Figure 4.1. Using $\bar{u}$ as a reference point, the plane has been divided into four regions; the boundaries of the regions are considered separately. By weak Pareto we know that all points in region I are preferred to $\bar{u}$ and $\bar{u}$ is preferred to all points in region III.

We now establish that all points inregion II (or region IV) must be ranked in the same way against $\bar{u}$. That is, for all u in region II, either $uP^*\bar{u}$, $uI^*\bar{u}$, or $\bar{u}P^*u$. All the points in region II make person one worse off than at $\bar{u}$ ($u_1 < \bar{u}_1$) and person two better off than at $\bar{u}$ ($u_2 > \bar{u}_2$). With ordinally measurable, noncomparable utilities, this is all the information that can be used. To put this somewhat more precisely, consider the points a and b and, for concreteness, suppose $aP^*\bar{u}$. Now consider an increasing monotone transformation of person one's utility scale which maps $\bar{u}_1$ back into itself and a_1 into b_1. This is possible with an increasing transformation since we have made sure that a_1 is mapped into a smaller number than $\bar{u}_1$, preserving person one's ranking. Similarly, we can choose an increasing transformation of person two's scale which maps $\bar{u}_2$ into itself and a_2 into b_2; this mapping perserves person two's ranking.[17] Since $aP^*\bar{u}$,

[15] Several different sets of assumptions are equivalent to Arrow's original set. We follow Sen [1970] in using unrestricted domain, weak Pareto, independence, and nondictatorship. Sen [1979a] has shown that unrestricted domain, independence, and weak Pareto imply a condition closely related to strong neutrality that he calls "strict-ranking welfarism."

[16] Of course, it is possible to define a dictatorship directly in terms of the social-evaluation functional F.

[17] Examples of the required mappings are given by $\phi_i(t) = \alpha_i t + \beta_i = \left[\frac{\bar{u}_i - b_i}{\bar{u}_i - a_i}\right] t + \left[\frac{b_i - a_i}{\bar{u}_i - a_i}\right] \bar{u}_i$.

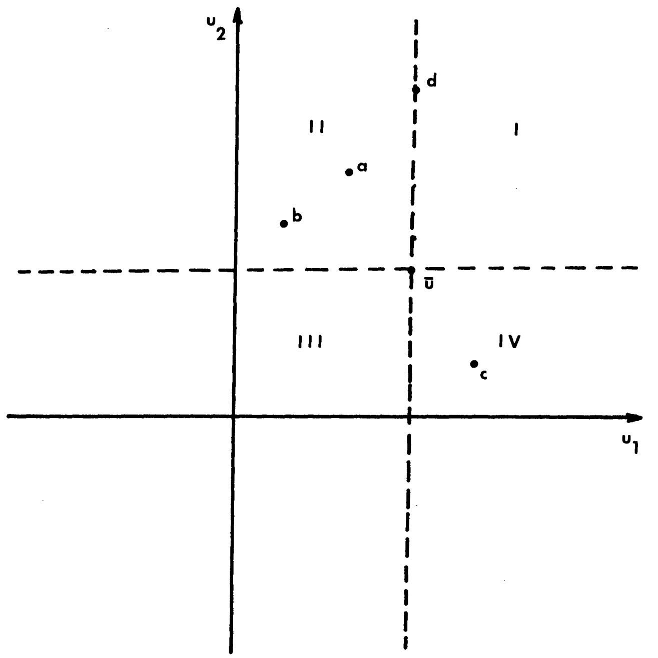

FIGURE 4.1

we must have $bP^*\bar{u}$ as well, by the reasoning outlined in the previous section. As a and b are arbitrary, we have now established that the whole of region II (region IV) is ranked in an identical fashion with respect to $\bar{u}$ (although not with respect to each other).

Because R^* is an ordering, there are three possible ways of ranking region II (region IV) with respect to $\bar{u}$: (i) indifferent, (ii) preferred, or (iii) worse. The first alternative is easily seen to lead to a contradiction. Indifference would imply not only that $aI^*\bar{u}$ and $bI^*\bar{u}$ but also aI^*b (by transitivity of R^*). But for the a and b shown in Figure 4.1, a Pareto-dominates b, a contradiction.

We now wish to establish that the ranking given to region II must be opposite from the ranking given to region IV. For concreteness, suppose region II is preferred to $\bar{u}$. Consider a transformation of person one's scale which shifts each point to the right by the value $\bar{u}_1 - a_1$ and a transformation of person two's scale which shifts each point down by the value $a_2 - \bar{u}_2$. These transformations map a into $\bar{u}$ and $\bar{u}$ into c. Since a is (by assumption) preferred to $\bar{u}$, this relationship must be preserved by the transformation; consequently $\bar{u}$ is preferred to c. But, using our earlier conclusion, this means region IV is worse than $\bar{u}$. Choosing region II to be preferred is arbitrary; if region II is worse than $\bar{u}$ this reasoning implies region IV is better.

Now it is simply a matter of tying together some loose threads. If two adjacent regions are both ranked the same way with respect to $\bar{u}$, so is their common boundary. For example, suppose region II is preferred to $\bar{u}$. Consider a point such as d on the boundary between regions I and II. For any choice of d there is always some point in region II which it Pareto-dominates (such as a). This implies dP^*a and together with $aP^*\bar{u}$, we obtain $dP^*\bar{u}$ by the transitivity of R^*.

With respect to $\bar{u}$, we have established that the social-welfare ordering must lead to one of the rankings shown in Figure 4.2. Note that we have not specified the rankings on the dotted lines.

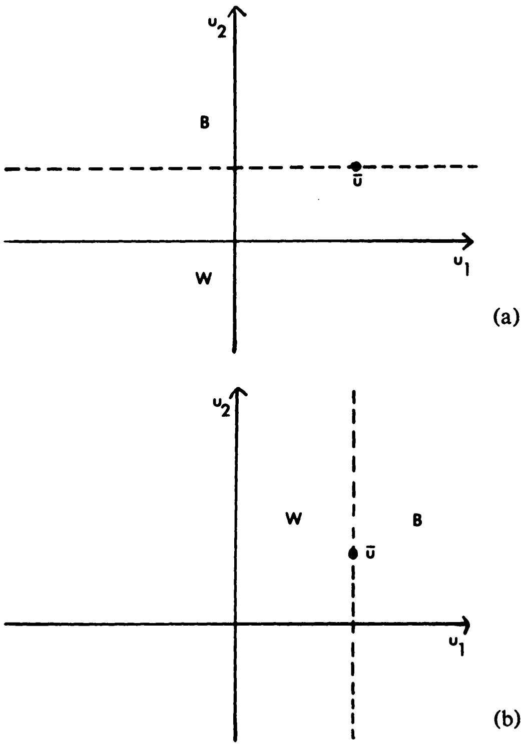

FIGURE 4.2

The only thing left to establish is that the choice of $\bar{u}$ as a reference point is irrelevant. That this is so follows immediately from the information assumption. Suppose one wishes to determine the rankings with respect to u'. Transform the utility scales by adding $u_1'-\bar{u}_1$ to person one's scale and $u_2'-\bar{u}_2$ to person two's scale. This simply shifts the pattern found in Figure 4.2. If the situation found in Figure 4.2 (a) prevails, person two is a dictator while if the situation in Figure 4.2 (b) prevails, person one is a dictator. It is easy to check that a dictatorship satisfies the invariance requirement and weak Pareto.[18]

[18] For n people, $n>2$, we can prove the theorem as follows. Consider a move away from

(*Continued on next page*)

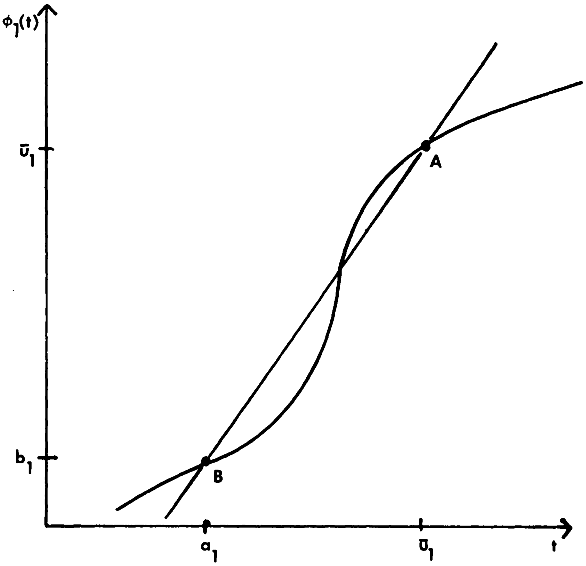

FIGURE 4.3

In establishing this theorem we have in fact used somewhat more demanding invariance properties than are needed. For example, in the first transformation used in the proof we wanted to find an increasing transformation which mapped $\bar{u}_1$ into $\bar{u}_1$ (point A in Figure 4.3) and a_1 into b_1 (point B in Figure 4.3). There are an infinity of such mappings, namely all the upward sloping lines through A and B. It is possible to obtain this result by using an increasing affine transformation, namely the straight line through A and B (see footnote 17). Note that, since there is a different transform for each person, adding more people does no harm to this argument. Thus Theorem 4.1 remains valid when the social ordering must remain invariant to independent affine transformations of the utility scales. This is stated formally as Corollary 4.1.

COROLLARY 4.1. *If the social-evaluation functional F satisfies the welfarism axioms, then the social-welfare ordering R^* satisfies information invariance with cardinally measurable, noncomparable utilities and weak Pareto if and only*

(*Continued*)
$\bar{u}$ where person k is made better off and all the rest worse off. Because of information invariance and the Pareto rule, if this move is ranked better than $\bar{u}$, then all moves from $\bar{u}$ where k is better off must be ranked as better than $\bar{u}$. By the argument in the text, this rule must be the same for all $\bar{u}$, and k is a dictator. Consequently, all such moves must be ranked as worse than the starting point if dictatorship is to be avoided. But this is not possible if R^* is to be transitive. To illustrate, let $n=3$ and $\bar{u}=(2, 2, 2)$. Then $u^1=(3, 1, 1)$ is worse than $\bar{u}$, $u^2=(2\frac{1}{2}, 2\frac{1}{2}, \frac{1}{2})$ is worse than u^1, and $\bar{u}=(2, 2, 2)$ is worse than u^2. Transitivity would require $\bar{u}$ to be worse than itself, an impossibility. Consequently, there must be a dictator.

if it is a dictatorship.[19]

Specifying the name of a dictator does not completely determine the social-evaluation function or the social-welfare ordering R^*. To get his own preference replicated as the social preference for any pair of alternatives, the dictator must exhibit a strict preference between them. If he is indifferent, the social ranking can be chosen arbitrarily (consistent with neutrality). By strengthening the assumptions, this choice can be removed.

One natural way to extend the axiom system is to adopt the strong Pareto assumption. Suppose person two is a dictator, as shown in Figure 4.2 (a). Strong Pareto now completely determines the ordering of points on the dotted line through $\bar{u}$; moving to the right (increasing u_1) is an improvement. Similarly, when person one is a dictator, moving upwards along the line through $\bar{u}$ in Figure 4.2 (b) is an improvement. In other words, whenever the dictator is indifferent, the other person's preferences come into play. In general, we obtain a lexicographic dictatorship; there is a ranking of the individuals with the first person being the dictator. If the dictator is indifferent, we move on to the second-ranked person's preferences. If both of the two top-ranked individuals are indifferent, we consider the third-ranked person, and so on.[20] Note that this social-welfare ordering is not continuous and cannot be represented by a social-welfare function.

Instead of adopting the strong Pareto principle, we could instead require R^* to satisfy continuity. In terms of Figure 4.2, this will imply that all points on the line through $\bar{u}$ are indifferent to each other. In this case, the social-welfare ordering is representable by a continuous social-welfare function. Adding continuity strengthens the dictator's powers. Not only does the social ordering reflect the dictator's preferences when he expresses a strict preference, but also indifference by the dictator results in social indifference as well. The social ordering coincides with the dictator's ordering.

Continuity and strong Pareto cannot be satisfied together by the social-welfare ordering R^* given ordinally measurable, noncomparable utilities. Similarly, for this information condition, weak Pareto is incompatable with any axiom that rules out dictatorships, including anonymity.

5. POSITIONAL DICTATORSHIPS

Arrow's result depends critically on the assumption that the welfares of different individuals cannot be compared. A very simple relaxation of this restriction on information is to allow levels of utility to be compared between people. That is, we want to be able to say whether person i in state x is better or worse off than

[19] See Sen [1970, Theorem 8*2].

[20] Curiously, we know of only one place where this result is actually stated in print; see Gevers [1979, Theorem 3]. Luce and Raiffa [1958] realize there is a lexicographic extension to Arrow's theorem, but do not state it precisely.

person j in state y. If these are purely ordinal comparisons, then we have ordinally measurable, fully comparable utilities. Thus, for any alternative, it is possible to rank the individuals (breaking ties arbitrarily) from best-off to worst-off, and this information may be used in determining the social ordering of X. If the social-evaluation functional F satisfies the welfarism axioms, then the social-welfare ordering R^* will be a *positional dictatorship* if R^* satisfies information invariance with this informational assumption, weak Pareto, and anonymity. The implications of dropping anonymity are discussed informally below.

To define a positional dictator it is necessary to develop some additional notation. For a vector of utilities u, let $r(u)$ be the person who is the rth best off. For example, if $u=(2, 7, 3)$ then $1(u)=2$ as person two receives the highest utility, $2(u)=3$, and $3(u)=1$.[21] For a positional dictatorship it is not some particular individual who becomes a dictator but rather this decision-making power is given to a particular rank or position. Who occupies this position depends on the vector of utilities.

Positional Dictatorship: R^* is a positional dictatorship if and only if there exists r, $1 \leq r \leq n$, such that for all $\bar{u}$, $\bar{\bar{u}} \in \boldsymbol{R}^n$, if $\bar{u}_{r(\bar{u})} > \bar{\bar{u}}_{r(\bar{\bar{u}})}$, then $\bar{u} P^* \bar{\bar{u}}$.

A well-known example of a positional dictatorship is the maximin rule. With maximin, social welfare is identified with the welfare of the worst-off individual, $n(u)$, and we have $\bar{u} R^* \bar{\bar{u}}$ if and only if $\bar{u}_{n(\bar{u})} \geq \bar{\bar{u}}_{n(\bar{\bar{u}})}$.[22] Maximin implies that the nth position is a (positional) dictator but the converse is not true since positional dictatorships do not require social indifference whenever the positional dictator is indifferent.

THEOREM 5.1. *If the social-evaluation functional F satisfies the welfarism axioms, then the social-welfare ordering R^* satisfies information invariance with ordinally measurable, fully comparable utilities, weak Pareto, and anonymity if and only if it is a positional dictatorship.*[23]

In Figure 5.1, $\bar{u}$ is taken as a reference point. Because of anonymity, $\bar{\bar{u}} = (\bar{u}_2, \bar{u}_1)$ is indifferent to $\bar{u}$. By weak Pareto, regions I and II are preferred to $\bar{u}$ and regions II and III are preferred to $\bar{\bar{u}}$. By transitivity, this means III is preferred to $\bar{u}$ as well. Similarly, regions V, VI, and VII are ranked as worse than $\bar{u}$ (and $\bar{\bar{u}}$, of course).

We now wish to establish that all points in region VIII (and by anonymity region IX) have the same ranking vis-a-vis $\bar{u}$. Utility vectors u in region VIII are distinguished by five inequalities: (i) Person one is better off than person two

[21] Ties may be broken arbitrarily. For $u=(2, 2, 7)$, $2(u)=2$ and $3(u)=1$, or $2(u)=1$ and $3(u)=2$.

[22] This differs somewhat from the way maximin is employed by Rawls [1971]. He defines positions in terms of an index of primary goods rather than utilities.

[23] Versions of this theorem have been established by Gevers [1979, Theorem 4] and Roberts [1980a, Theorem 4].

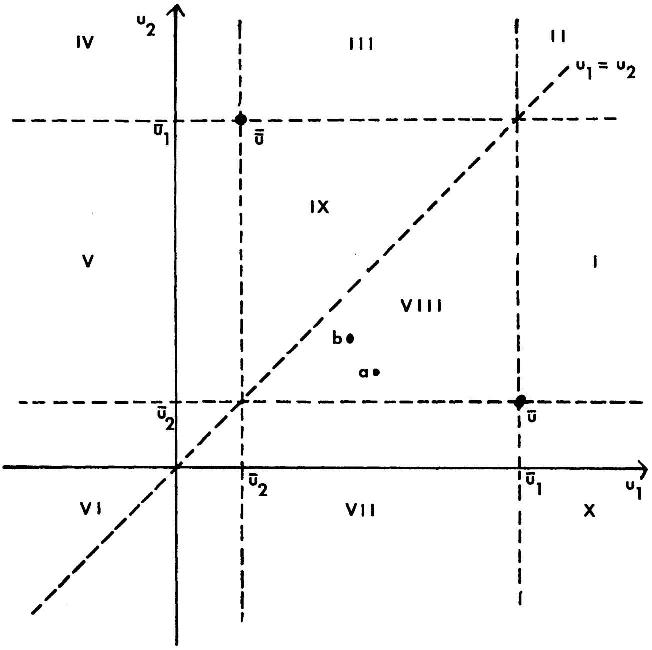

FIGURE 5.1

$(u_1 > u_2)$, (ii) person one is worse off in u than in $\bar{u}$ $(u_1 < \bar{u}_1)$, (iii) person two is better off in u than in $\bar{u}$ $(u_2 > \bar{u}_2)$, (iv) person one is better off in u than person two in $\bar{u}$ $(u_1 > \bar{u}_2)$, and (v) person two is worse off in u than person one in $\bar{u}$ $(u_2 < \bar{u}_1)$. These five inequalities together with the fact that $\bar{u}_1 > \bar{u}_2$, are all usable information. Since levels of utility are all that can be compared, however, these inequalities exhaust the information. To see this in terms of invariance transformations, consider points a and b. Since they are both in region VIII, we have $\bar{u}_1 > a_1 > a_2 > \bar{u}_2$ and $\bar{u}_1 > b_1 > b_2 > \bar{u}_2$. We can find a single increasing transform (Figure 5.2), mapping $\bar{u}$ into itself, and a into b. Therefore, the ranking of a against $\bar{u}$ must be the same as the ranking of b against $\bar{u}$.[24]

Since R^* is an ordering we must have (a) $uP^*\bar{u}$, (b) $\bar{u}P^*u$, or (c) $uI^*\bar{u}$ for all u in region VIII. By the argument of Theorem 4.1, the last case, indifference, is eliminated.

By anonymity, every point in region IX is indifferent to a point in region VIII and hence, by transitivity, points in IX must be ranked the same way with respect to $\bar{u}$ as points in VIII.

An argument similar to that of Theorem 4.1 demonstrates that all points in region X (and by anonymity, region IV) are ranked identically with respect to $\bar{u}$,

[24] The fact that R^* is a binary relation implies binary independence (the ranking of a and $\bar{u}$ depends only on information about a and $\bar{u}$). Thus information about a and b $(a_1 > b_1 > b_2 > a_2)$ is irrelevant to the problem of ranking a and $\bar{u}$.

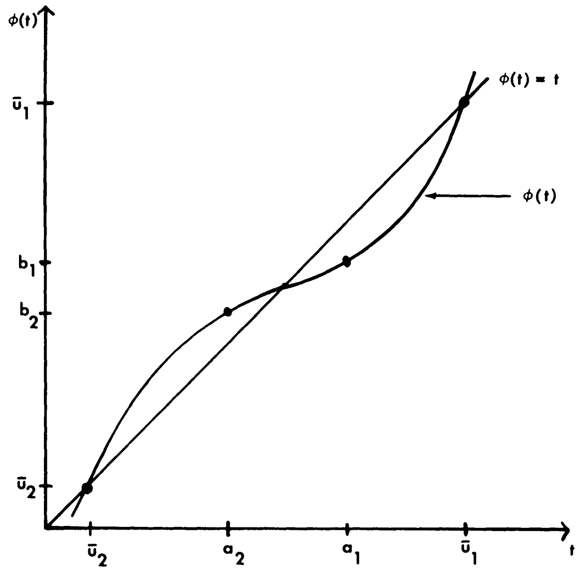

FIGURE 5.2

and in the opposite way to points in regions VIII and IX. Applying weak Pareto and transitivity, as we did in the proof of Theorem 4.1, if two adjacent regions have the same ranking with respect to $\bar{u}$, then so does their common boundary.

It follows that by specifying the ranking in region VIII, the social-welfare ordering must be one of the two shown in Figure 5.3. In Figure 5.3 (a), the lower rank is given dictatorial power, while in Figure 5.3 (b), the higher rank dictates. Points on the dotted lines through $\bar{u}$ are ranked in any arbitrary symmetric fashion.

Whichever position dictates for a particular $\bar{u}$ must dictate for all possible choices of $\bar{u}$. To see this, imagine that the lower position dictated for $\bar{u}$ and the higher position for $\hat{u}$. Transitivity and weak Pareto could not be compatible with this since the dotted lines through $\bar{u}$ and $\hat{u}$ would cross.

It is easy to check that a positional dictatorship satisfies the invariance requirement, weak Pareto, and anonymity.[25]

Adding continuity to our list of assumptions turns the dotted lines into indifference curves. Then Figure 5.3 (a) would depict maximin while Figure 5.3

[25] The proofs provided by Gevers [1979] and Roberts [1980a] for the general case with $n>2$ are extremely complex. An alternative method of proof can be constructed using our remarks in footnote 18 on the n-person proof of Arrow's theorem. Anonymity is used to first restrict attention to utility n-tuples having the same rank order. References to dictators are replaced by references to positional dictators. In the step which involves showing transitivity is violated it is possible to construct the utility n-tuples in such a way that strict rank-orderings are preserved. Ties in rankings must be handled separately, a fact which unfortunately leads to a general proof of comparable complexity to those of Gevers and Roberts.

(b) would depict maximax.

By analogy with the discussion of Arrow's theorem one might think that using strong Pareto in place of continuity results in lexicographic positional dictatorships, "leximin" in Figure 5.3 (a) and "leximax" in Figure 5.3 (b). Unfortunately, this is not the case for $n > 2$ as can be seen from an example constructed by Gevers [1979, p. 79]. If the dictating position has the same utility in u and $\bar{u}$, then the information about that utility level relative to those of the other individuals is usable; in the Arrow framework it is not.

One way to circumvent this problem is to simply adopt an axiom which prevents

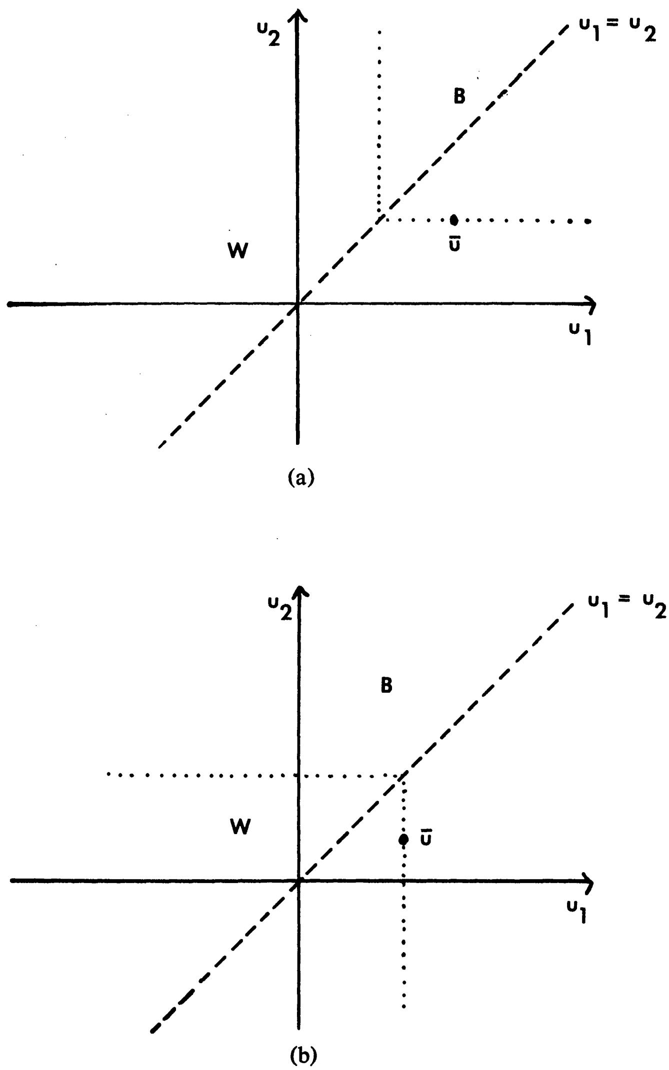

FIGURE 5.3

this information from being used, as in d'Aspremont and Gevers [1977]. This assumption requires that whenever some individuals are indifferent between two alternatives, the ordering of the alternatives is independent of the utility levels that the indifferent individuals enjoy. This is a separability requirement, and it allows Theorem 5.1 to be applied to the remaining positions whenever the dictating position is indifferent. Thus, a sequence of positional dictators is found and they "rule" lexicographically. If the sequence of positions begins at the worst off, then turns to the second worst off, and so on, the rule is "leximin" or lexicographic maximin.

Dropping the anonymity requirement opens up the possibility of other social-welfare orderings. The orderings need not be symmetric in this case. Let us call all utility n-tuples which have the same rank-ordering a *rank-ordered set.* When $n=2$ there are only two rank-ordered sets, $Z_1=\{u\in \boldsymbol{R}^2 | u_1 \geq u_2\}$ and $Z_2=\{u\in \boldsymbol{R}^2 | u_2 \geq u_1\}$. Within rank-ordered sets our previous reasoning still applies; within any rank-ordered set there is a dictator. However, without symmetry the position assigned the role of dictator can depend on the set being considered. With $n=2$ this opens up only two new possibilities, namely the two dictatorial rules (Figure 4.2). With three people, the rule could, for example, take the form that person one is a dictator in the rank-ordered set where he is best off, otherwise person two is the dictator.

We thus see that when anonymity is dropped, the set of social-welfare orderings obtained contains the dictatorial rules found in Theorem 4.1. This is to be expected. The Arrow dictators were found by requiring that social choices be invariant when each person's utility function is subjected to a monotone transformation. If invariance to a common increasing transform is all that is required, dictatorships satisfy this weaker invariance requirement.

6. LEXIMIN

In Section 4, the assumptions used implied the existence of a dictator, but were not sufficiently strong to determine which person would be the dictator. Similarly, in Section 5 we obtained positional dictatorships, but which position is to be dictatorial is left unspecified. In the latter setting, the assumptions are thus compatible with both maximax and maximin. Clearly, some additional axiom is needed to add ethical content.

Such an ethical assumption was introduced by Hammond [1976]. For this axiom to make sense, it must be possible to make interpersonal comparisons of utility levels. However, given this possibility, we shall see that the precise information available does not affect the class of social-welfare orderings consistent with our axiom system. In other words, any information in addition to that obtained when preferences are ordinally measurable and fully comparable is simply ignored. In the terminology of Section 3, we are considering level-plus comparability.

The equity axiom we consider is applicable to what has been called *two-person* situations. A two-person situation occurs if, when comparing two social states, there are only two individuals who are not indifferent between the alternatives. Hammond's equity axiom states that in two-person situations, if one of the concerned individuals is worse off than the other concerned individual in each of the two states considered, this person's preferences should determine the social preference.

Hammond Equity: For all i, j in N, and for all $\bar{u}$, $\bar{\bar{u}}$ in $\boldsymbol{R}^n$, if $\bar{u}_k = \bar{\bar{u}}_k$, for all $k \neq i, j$, and $\bar{\bar{u}}_i > \bar{u}_i > \bar{u}_j > \bar{\bar{u}}_j$, then $\bar{u} P^* \bar{\bar{u}}$.

As has been remarked earlier, it is necessary to be able to make interpersonal comparisons of utility levels to apply the Hammond equity principle. A social-welfare ordering R^* satisfies information invariance with level-plus comparability, Hammond equity, strong Pareto, and anonymity if and only if R^* is the lexicographic maximin rule (leximin). Before stating the theorem precisely, it is useful to present a formal definition of leximin.

Leximin: R^* is the leximin rule if and only if for all $\bar{u}$, $\bar{\bar{u}} \in \boldsymbol{R}^n$, $\bar{u} P^* \bar{\bar{u}}$ if and only if $\exists k \in N$ such that $\bar{u}_{k(\bar{u})} > \bar{\bar{u}}_{k(\bar{\bar{u}})}$ and $\bar{u}_{j(\bar{u})} = \bar{\bar{u}}_{j(\bar{\bar{u}})}$ for all $j > k$.

THEOREM 6.1. *If the social-evaluation functional F satisfies the welfarism axioms, then the social-welfare ordering R^* satisfies information invariance with level-plus comparability, Hammond equity, strong Pareto, and anonymity if and only if R^* is the lexicographic maximin rule (leximin).*

The proof of Theorem 6.1 in the two-person case is presented with the aid of Figure 6.1.[26] In Figure 6.1, $\bar{u}$ is taken as a reference point.

Before proceeding, it is useful to recall that transitivity and the Pareto principle will imply that if any pair of the eight regions share a common boundary, and both regions are ranked the same way with respect to $\bar{u}$, then so is their boundary.

The strong Pareto principle immediately yields both regions I and II and their boundaries as better than $\bar{u}$ and both regions VI and VII and their boundaries as worse. Consider an arbitrary point a in region IV. In both a and $\bar{u}$, person one is better off than person two with person one preferring $\bar{u}$, while person two prefers a. Hammond's equity principle applies, so each point in region IV is preferred to $\bar{u}$. Similarly, in ranking region VIII with $\bar{u}$, the Hammond equity principle is applicable; in this case, $\bar{u}$ is better than all points in region VIII. Note that we have not appealed to invariance transformations in ranking these last two regions. Indeed, if the actual information available was, say, cardinally measurable and fully comparable utilities, information invariance considered in isolation would permit certain elements in IV to be ranked differently with respect to $\bar{u}$.

Anonymity yields region III better than $\bar{u}$, region V worse, $\bar{\bar{u}} = (\bar{u}_2, \bar{u}_1)$ indifferent, points on the dotted line below $\bar{\bar{u}}$ worse, and points above $\bar{\bar{u}}$ better than $\bar{u}$. This is the two-person lexicographic maximin rule, which is depicted in

[26] Theorem 6.1 was established by Hammond [1976, Theorem 7.2].

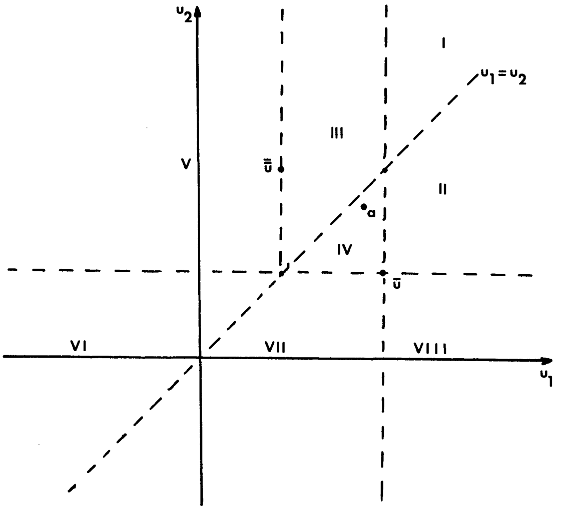

FIGURE 6.1

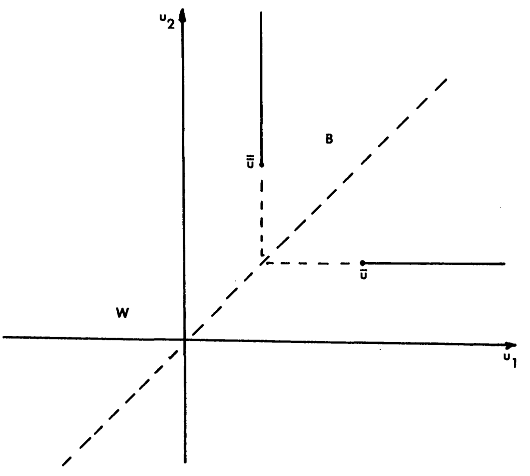

FIGURE 6.2

Figure 6.2.

The reverse implication, that leximin implies the assumptions of the theorem, is easily checked.[27]

[27] In the n-person case, the result can be proved for any two-person subgroup, holding the utilities of all other people constant. Then it is possible to show that the worst-off person must be a positional dictator, and that R^* is the n-person leximin rule. See Sen [1977c, Theorem 8]. Extensions of the idea of establishing results which hold in general when they are valid for two-person situations may be found in Hammond [1979].

If, in addition to the assumptions of Theorem 6.1 we require continuity, then an impossibility results. This impossibility remains even when strong Pareto is relaxed to weak and anonymity is dropped, as long as the number of people is greater than two. If there are only two people, these axioms yield the maximin rule.

Note that the Hammond equity axiom contains a good deal of separability. Specifically, it makes the social ordering for any pair of people separable from the rest. It is this feature of the axiom that allows generalization of the two-person result to n-person situations. This separability drives the above impossibility result as well.[28]

7. UTILITARIANISM

In Section 5 we considered the possibility of making interpersonal comparisons when individual utilities are ordinally measurable. In this section, we consider interpersonal comparisons with cardinal utility functions. With ordinally measurable, fully comparable utilities, it is possible to compare utility levels of different individuals. With cardinally measurable unit-comparable utilities, it is not possible to compare utility levels, but it is possible to compare utility gains and losses. In the presence of the welfarism axioms, a social-welfare ordering satisfies information invariance with this informational assumption, weak Pareto, and anonymity if and only if it is the utilitarian rule; alternatives are ranked by the sum of utilities across individuals. Before presenting this result, it is useful to consider the rules obtained without anonymity. To keep matters simple, it is convenient to consider only continuous social-welfare orderings.

Theorem 7.1 isolates a class of decision rules which we call *generalized utilitarianism* since the weights given to different individuals need not be the same.

Generalized Utilitarianism: R^* is a generalized utilitarian ordering if and only if there exists $\alpha \in \boldsymbol{R}^n_+$, $\alpha_i > 0$ for some i, such that $\bar{u}R^*\bar{\bar{u}}$ if and only if $\sum_i \alpha_i \bar{u}_i \geq \sum_i \alpha_i \bar{\bar{u}}_i$.

The utilitarian rule is the special case obtained by setting all components of α to be equal.[29]

THEOREM 7.1. *If the social-evaluation functional F satisfies the welfarism axioms, then the social-welfare ordering R^* satisfies information invariance with cardinally measurable, unit-comparable utilities, weak Pareto, and continuity if and only if it is a generalized utilitarian rule.*

[28] Additional theorems related to the results considered in this section may be found in d'Aspremont and Gevers [1977], Deschamps and Gevers [1978], Hammond [1976, 1979], Sen [1976, 1977c, 1984], and Strasnick [1976a, 1976b].

[29] It is possible to normalize α so that $\sum_i \alpha_i = n$.

It is easy to see that if R^* is a generalized utilitarian ordering, then it satisfies the information invariance condition, weak Pareto, and continuity. We need, therefore, to show that the axioms are sufficient for R^* to be a generalized utilitarian rule.

For our two-person proof, let the origin $0_2=(0, 0)$ be the reference point. By weak Pareto, all points in the interior of the first quadrant are preferred to 0_2, and 0_2 is preferred to all points in the interior of the third quadrant. Continuity then implies that there must be a point $\tilde{u}$ in the fourth quadrant (or perhaps on a boundary) with $\tilde{u} \neq 0_2$ indifferent to 0_2. This is illustrated in Figure 7.1.

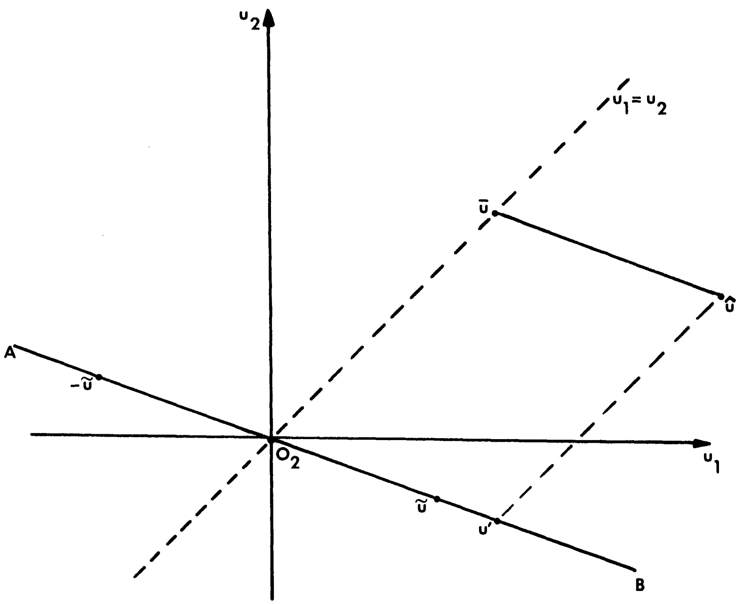

FIGURE 7.1

For any $\lambda > 0$, by the information assumption, $\lambda\tilde{u}I^*0_2$. Therefore, all points on the ray 0_2B are indifferent.

Note that, so far, we have used only transformations that are common to all individuals. In other words, the result so far obtained is compatible with cardinally measurable, fully comparable utilities. To extend indifference to the rest of the line AB, it is necessary to use transformations that change the origins of different people independently. Subtract $\tilde{u}$ from 0_2 and $\tilde{u}$. This maps 0_2 into $-\tilde{u}$ and $\tilde{u}$ into 0_2. The information invariance condition requires $-\tilde{u}I^*0_2$ and, by the above argument, all points on the ray 0_2A are indifferent. Hence, by transitivity, all points on the line AB are indifferent to each other. The line AB may be written as $\sum_i \alpha_i u_i = 0$.

Now consider the point $\hat{u}$ and find the point $\bar{u}$ on the ray of equality such that

$\sum_i \alpha_i \hat{u}_i = \sum_i \alpha_i \bar{u}_i$. Let t be the common value of $\bar{u}_i$, so $\bar{u}=(t, t)$. Let $u'=\hat{u}-(t, t)$ and note that $0_2=\bar{u}-(t, t)$, as shown in Figure 7.1. By the information assumption $\bar{u}I^*\hat{u}$ since $0_2 I^* u'$. Hence, the invariance axiom requires that indifference prevails among all points where $\sum_i \alpha_i u_i$ is a constant. Weak Pareto requires that $\alpha_i \geq 0$ for all i and $\alpha_j > 0$ for some j, and that $\sum \alpha_i u_i > \sum \alpha_i \mathring{u}_i$ implies $uP^*\mathring{u}$. This establishes the result for the case of two people.[30]

By adding anonymity to our list of axioms, we obtain an axiomatization of utilitarianism. In fact, it is now possible to drop an explicit reference to continuity, as it is implied by the other axioms.

COROLLARY 7.1. *If the social-evaluation functional F satisfies the welfarism axioms, then the social-welfare ordering R^* satisfies information invariance with cardinally measurable, unit-comparable utilities, weak Pareto, and anonymity if and only if R^* is the utilitarian ordering.*[31]

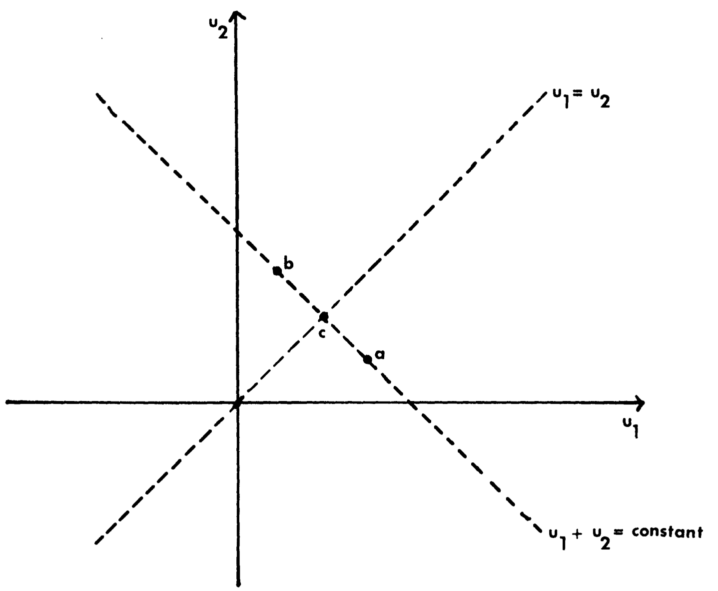

FIGURE 7.2

To show that it is not necessary to assume continuity explicitly we present a simple direct proof of Corollary 7.1. Let c be a point on the ray of equality and

[30] The n-person proof is very similar. Weak Pareto and continuity guarantee that there exist $n-1$ non-zero linearly independent vectors in $\boldsymbol{R}^n$ indifferent to 0_n. These vectors play the role of $\bar{u}$ in our two-person diagrammatic proof.

[31] The utilitarian ordering is a generalized utilitarian rule with $\alpha_i=\alpha_j$ for all $i, j \in N$. Corollary 7.1 was established by d'Aspremont and Gevers [1977, Theorem 3].

let a be any other point yielding the same total utility. This is depicted in Figure 7.2. By anonymity, aI^*b where b is a permutation of a. Suppose cP^*a. By adding $c-a$ to both a and c we map a into c and c into b. Since this is an allowable information transformation and cP^*a, we have bP^*c and, by transitivity, bP^*a, a contradiction. Similarly, aP^*c leads to a contradiction. Hence, aI^*c and the line through a and b is a social indifference curve. Weak Pareto then implies that the social welfare ordering is utilitarianism.

Returning to the larger class of generalized utilitarian rules, it is worth remarking that continuous dictatorships are members of this class. Whenever there is some $k \in N$ such that $\alpha_k > 0$ and $\alpha_i = 0$ for $i \neq k$, there is a dictator; in fact, the social ordering coincides with the dictator's ordering, even in the case of indifference. Of course, this is to be expected since dictatorships are invariant to any monotone transformations, including the ones allowed in this information framework.

The consequences of dropping continuity from the axioms used in Theorem 7.1 are easy to state, although this modification would necessitate major changes in our method of proof. First, observe that all the generalized utilitarian rules are still possible. All social-welfare orderings obtained from this relaxation of continuity have the property that there exists a generalized utilitarian rule such that the new rule and its generalized utilitarian counterpart coincide whenever the generalized utilitarian rule exhibits strict preference. Precisely the same relationship exists between the Arrow dictators and continuous dictatorships.

We shall not consider cardinally measurable, fully comparable utilities in detail, but a few remarks are in order. This particular informational basis has been particularly popular, since in this framework it is possible to compare both utility levels and utility gains and losses. There is thus sufficient information to implement either maximin or utilitarianism.

One way to restrict the set of social choice rules is to adopt a separability axiom on R^*. As the importance of this kind of assumption can not be captured in two-dimensional diagrams, it is not appropriate to develop these theorems here. Separability assumptions have been employed with cardinally measurable, fully comparable utilities to establish variants of utilitarian or generalized utilitarian rules (by Blackorby and Donaldson [1982], Blackorby, Donaldson, and Weymark [1980], d'Aspremont and Gevers [1977], Deschamps and Gevers [1977, 1978], Gevers [1979], Harsanyi [1955], Maskin [1978], and Roberts [1980b]).[32]

Even without separability, it is easy to see that information invariance with cardinally measurable, fully comparable utilities places a great deal of structure on the continuous social-welfare ordering R^*. The argument used in the proof of Theorem 7.1 implies that with this information assumption R^* must be linear in the rank-ordered sets $Z_1 = \{u | u_1 \geq u_2\}$ and $Z_2 = \{u | u_2 \geq u_1\}$ when $n=2$. When n is greater than two, this linearity is preserved in the sets of utility n-tuples

[32] See also the surveys in Sen [1977b, 1977c, 1984]. These references also contain results employing information assumptions which differ somewhat from cardinally measurable, fully comparable utilities.

possessing the same rank order.[33] A family of such social-welfare orderings is provided by the *single-parameter Gini family* (S-Ginis) studied by Donaldson and Weymark [1980]. For S-Ginis, $\bar{u}R^*\bar{\bar{u}}$ if and only if

$$\sum_r [r^\delta-(r-1)^\delta]\bar{u}_{r(\bar{u})} \geq \sum_r [r^\delta-(r-1)^\delta]\bar{\bar{u}}_{r(\bar{\bar{u}})}$$

where δ is a parameter greater than or equal to one. These social-welfare functions satisfy anonymity since they are positional rules. (Recall that $r(u)$ is the rth best off person in u.) When $\delta=1$, the rule is utilitarian and when δ approaches infinity it approaches maximin. When $\delta=2$, we obtain the social-welfare function underlying the Gini inequality index.

8. CONCLUDING REMARKS

It is commonplace in welfare economics to use social-welfare functions which are strictly quasiconcave in utilities. None of the information assumptions considered so far permits this kind of curvature for social indifference curves. Excluding level-plus comparability, of all the information assumptions we have discussed, cardinally measurable, fully comparable utilities contain the most usable information. As was noted in the previous section, even with this information condition, social-welfare functions must be linear in rank-ordered sets. Consequently, curvature of social indifference curves requires that more information be available.

Some strictly quasiconcave social-welfare functions are compatible with translation-scale, fully comparable utilities. As with cardinally measurable, fully comparable utilities, this information condition permits interpersonal comparisons of utility levels and of utility differences. Furthermore, the comparison of utility differences is now on an absolute scale. It is this extra information which allows for curvature in the social indifference curves.[34] Of course, with perfectly measurable, fully comparable utilities, any continuous social-welfare function is compatible with the welfarism axioms and continuity.

The preceding discussion has presupposed that the social-welfare function (or ordering) is derived from a social-evaluation functional satisfying unrestricted domain. Samuelson [1967] has argued that Bergson [1938]-Samuelson [1947] social-welfare functions are, in our terminology, obtained from a social-evaluation functional whose domain is restricted to the single utility profile (more precisely, single information set) that society actually has. Analogues to the multiprofile version of Arrow's theorem presented above have been developed in this single

[33] See Roberts [1980b] for a discussion of this point.

[34] In some circumstances, curvature of the social indifference curves is also possible with ratio-scale, fully comparable utilities, an invariance condition which permits interpersonal comparisons of percentage changes in utility. For a detailed discussion of ratio-scale and translation-scale measurability, see Blackorby and Donaldson [1982] and Roberts [1980b].

profile setting.[35] This is done by assuming strong neutrality directly and by adding an axiom concerning the diversity of tastes found in society. Together, these conditions can be shown to yield an ordering in utility space, permitting the methods of this paper to be employed for this restricted domain.

We have required that the social-evaluation functional yield *orderings* of the alternatives. One way to weaken this requirement is to replace transitivity by quasitransitivity (transitivity of strict preference), acyclicity (no finite strict preference cycles), or to drop these rationality requirements altogether. Transitivity of social indifference was used in an essential fashion in our strong neutrality theorem. However, variants of this theorem (which employ the weak Pareto principle) are available (Sen [1979a], for example) which would permit the use of our diagrams to rank pairs of utility n-tuples that do not involve indifferent individuals. Thus our diagrams can be used to illustrate the theorems on oligarchies (Gibbard [1969]) and collegiums (Brown [1975])[36] as well as May's [1952] axiomatization of majority rule.

The welfarism-neutrality results of Section 2 apply if the range of the social-evaluation functional is broadened to quasi-orderings (reflexive and transitive but not necessarily complete binary relations) of the social alternatives. Social decisions are made in this case with a quasi-ordering of utility n-tuples. Consequently it is also possible to present the work of Blackorby and Donaldson [1977], Suppes [1966], and Weymark [1984] in this framework.

In bargaining models, binary independence of irrelevant alternatives is violated because of the presence of a threat point (status quo). However, bargaining theories do satisfy a modified form of independence and recently Roberts [1980b] has used the welfarism framework to study bargaining under different information assumptions. He is able to show, for example, how to obtain Nash's [1950] bargaining solution as the social-choice procedure when there are cardinally measurable noncomparable utilities.

We hope that the results we have considered in detail in the previous sections and the extensions mentioned in these concluding remarks will have convinced the reader that a very substantial part of the social-choice literature is amenable to a simple diagrammatic analysis. In addition, we hope that the intuition provided by this geometric approach will lead to a deeper understanding of the theorems we have considered and, perhaps, serve as a guide to the development of future results.

University of British Columbia, Canada.

[35] The first results of this sort were due to Kemp and Ng [1976] and Parks [1976]. See Sen [1984] for a survey.

[36] Sen [1984] surveys this literature.

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