Answer any <u>five</u>, each question carries 14 points

1. Consider an economy with two goods and two agents. Agents 1 has an endowment of 0.6 units of good $X$ and 0.6 units of good $Y$. Whereas agent 2 has an endowment of 0.4 units of both the goods. Utility functions are as follows, $u_1(x, y) = x + 2y$ and $u_2(x, y) = 2x + y$.

   a) Model this situation as a bargaining problem, that is, find the utility possibility set and the disagreement point. [5]

   b) Find the Nash bargaining outcome and the Kalai-Smorodinsky bargaining outcome. [4]

   c) Model this as a co-operative game and find the core allocations. [5]

2. Consider the following class of bargaining problems, $\Sigma = \{(S, \mathbf{0}) \mid S \subset \mathcal{R}^2_+\}$ where $S$ is compact, convex and comprehensive. A bargaining solution is a function $F : \Sigma \to S$. Consider the following properties.
   **Property** $A$: Let $\mathbf{p} = \left(\frac{1}{F_1(S,\mathbf{0})}, \frac{1}{F_2(S,\mathbf{0})}\right)$. $F$ satisfies property A if $p \cdot F(S, \mathbf{0}) \geq p \cdot \mathbf{v}$ for all $\mathbf{v} \in S$.
   **Property** $B$: Suppose that $S \subset T$. $F$ satisfies Property B if $F(T, \mathbf{0}) \geq F(S, \mathbf{0})$.

   a) Why is property $A$ intuitively appealing? [3]

   b) Show that the Nash bargaining rule satisfies property $A$. [3]

   c) Why is property $B$ intuitively appealing? [2]

   d) Show that the Nash bargaining rule fails to satisfy Property $B$. [3]

   e) Show that the above properties are incompatible, that is, there is no $F$ which satisfies both. [3]

3. Consider a Transferable utility game $(N, v)$. A 'subgame' on $S \subset N$ can be constructed as follows, $w(T) = v(T)$ for all $T \subseteq S$. That is $(S, w)$ is the restriction of $v$ on $S$. An allocation $\Phi$ is population monotonic if for all possible subgames $(S, w)$, we have $\Phi_k(N, v) \geq \Phi_k(S, w)$ for all $k \in S$.

   a) Does population monotonicity have normative appeal? Discuss. [4]

   b) Show that if $(N, v)$ is a convex game then the Shapley value is population monotonic. [5]

   c) Egalitarian allocation divides the worth of the grand coalition equally among all agents. Provide an example to show that egalitarian allocation is not population monotonic even when $(N, v)$ is a convex game. [5]

4. A public facility costs $c > 0$ and brings a benefit $b_i \geq 0$ to each of its users $i = 1, \ldots, n$. Consider the following cost sharing method (called head tax): Find $\lambda$ such that $\sum_{i=1}^{n} \min\{\lambda, b_i\} = c$ . Then agent $i$ pays $x_i = \min\{\lambda, b_i\}$.

   a) Show that this allocation is well defined. That is, given a society $(b_1, \ldots, b_n, c)$, the above cost sharing method has a unique outcome. [2]

   b) One can construct a TU game $(N, v)$ as follows, $v(S) = \max(\sum_{i \in S} b_i - c, 0)$. Show that head tax is a core allocation. [3]

   c) Show that the head tax of all agents increase with an increase in $c$ (benefits remaining unchanged). [3]

   d) Show that the head tax of agent $i$ weakly increases with an increase in $b_i$ (other benefits and cost remaining unchanged). [3]

   e) Compute the head tax, when $b_1 = 0.5, b_2 = 0.2, b_3 = 0.9$ and $c = 1$. [3]

5. Consider a bilateral trade setting. Agent 1 is a seller of an indivisible object and has a valuation $\theta_s$, while Agent 2 is a buyer with valuation $\theta_b$. Valuations are private information and independently drawn from a uniform distribution on $[0, 1]$. Suppose $y_i(\theta)$ denote the probability that agent $i$ receives the object given valuations $\theta = (\theta_s, \theta_b)$. Agents have quasilinear utility functions, given by $[y_i(\theta)\theta_i + t_i(\theta)]$, where $t_i(\theta)$ denotes the transfer.

   a) Consider the following trade rule $y_b(\theta) = 1$ if $\theta_b \geq \theta_s$ and 0 otherwise. $t_s(\theta) = \frac{1}{2}(\theta_b + \theta_s)y_b(\theta)$ and $t_b(\theta) = -\frac{1}{2}(\theta_b + \theta_s)y_b(\theta)$. Is this rule ex-post efficient and individually rational? [2]

   b) Suppose that the seller truthfully reveals her type. Will the buyer find it worthwhile to reveal his type? [4]

   c) Define an expected externality mechanism and compute the transfers? [4]

   d) Verify that truth telling is a Bayesian Nash equilibrium in the expected externality mechanism. [4]

6. In a sealed bid first price auction, the highest bidder receives the good and pays the seller the amount of his bid. Suppose there are two bidders whose valuations are independently drawn from a uniform distribution on $[0, 1]$.

   a) Find an equilibrium bidding strategy. [5]

   b) Compute the expected revenue from this auction. [5]

   c) Rewrite first price auction as a mechanism and check that it meets the conditions of the revenue equivalence theorem. [4]

7. Let $G^N$ be the set of all graphs on $N$. Given a graph $g \in G^N$ and value function $v$ (a mapping from $G^N$ to $\mathcal{R}$), a network game $(N, w)$ can be constructed as follows: $w^g(S) = v(g_S)$, where $S \subseteq N$ and $g_S = \{(ij) \in g \mid i, j \in S\}$. The Shapley value of $w^g$ is an allocation at $g$ and is denoted by $\Psi(g)$.

   a) Consider a three person society, where $v$ is defined as follows. $v(g) = 0$ if $g$ has no link, $v(g) = \frac{14}{13}$ if $g$ has two links and $v(g) = 1$ for all other graphs. Compute $\Psi(g)$ for all $g \in G^N$. [5]

   b) Which of the above graphs are pairwise stable under $\Psi$? [4]

   c) In general, prove that $\Psi$ satisfies the equal bargaining power property, that is, $\Psi_i(g + ij) - \Psi_i(g) = \Psi_j(g + ij) - \Psi_j(g)$. [5]