Problem set 1, Game Theory 2 , 2013-14

1. Let A be the set of alternatives and N be the set of citizens. Strict preference of citizen $i \in N$ over A is denoted by P_i.
(a) Majority rule: $a \in A$ is socially as good as $b \in A \Leftrightarrow |\{i \in N \mid aP_ib\}| \geq |\{i \in N \mid bP_ia\}|$
(b) Approval voting rule: Fix an integer, $1 \leq k \leq |A|$. Take $a \in A$ and $i \in N$, $s(a,i) = 1$ if i ranks a among top k alternatives, otherwise $s(a,i) = 0$. $a \in A$ socially as good as $b \in A \Leftrightarrow \sum_{i \in N} s(a,i) \geq \sum_{i \in N} s(b,i)$
(c) Reverse dictatorship: Fix $k \in N$. $a \in A$ is socially preferred to $b \in A \Leftrightarrow bP_ka$
Check whether the above social rankings satisfy transitivity, IIA and Praeto.

2. $A = \{x, y, z\}$ and $N = \{1, 2\}$. Strict preference of citizen $i \in N$ over A is denoted by P_i. A social aggregation rule f is defined as follows, $f(P_1^*, P_2^*) = P_1^*$, where $P_1^* : xP_1^*yP_1^*z$, $P_2^* : yP_2^*xP_2^*z$ and $f(P_1, P_2) = P_2$ for all other P_1, P_2. Does this aggregation rule satisfy Pareto and IIA?

3. Consider the Arrow domain.
(i) Show that if there are only two alternatives, then there are non-dictatorial rules which satisfy IIA and Pareto.
(ii) Consider the following voting problem. There are three candidates $\{L, C, R\}$ and n voters. Voters are of the following types: left wing ($L \succ C \succ R$), right wing ($R \succ C \succ L$) and centrists (either $C \succ L \succ R$ or $C \succ R \succ L$). Find a social ranking in this setting which satisfies IIA and Pareto. Does it violate Arrow's impossibility theorem?

4. A social preference R is 'Acyclic' if the preference has at least one maximal element in every subset of A, that is for all $A' \subseteq A$, $\{x \in A' \mid xRy, \ \forall y \in A'\} \neq \emptyset$.
(a) Show that Acyclicity is weaker condition than transitivity.
(b) Consider the following situation: $A = \{x, y, z\}$, $N = \{1, 2\}$ and strict preference of citizen $i \in N$ over A is denoted by P_i. An aggregation rule called 'Veto rule' is defined as follows. Citizen 1 is the dictator with one qualification that citizen 2 can veto the possibility that x be socially preferred to y. In other words, social preference coincide with citizen 1's preference except the situation when xP_1y but yP_2x, in which case xIy, that is x is socially indifferent to y. Show that the veto rule satisfies Acyclicity, IIA and Pareto but fails transitivity. Does this violate Arrow's impossibility theorem.

5. Check whether the following social welfare rankings satisfy 'Invariance under cardinal unit comparable utilities', 'Invariance under ordinal comparable utilities', 'Anonimity' and 'Hammond equity'. Suppose there are only two agents in a society. Let $x_{(k)}$ denote the $k-th$ highest utility under policy x, where $1 \leq k \leq 2$.
Ranking1: x is as good as $y \Leftrightarrow x_{(k)} \leq y_{(k)}$.
Ranking2: x is as good as $y \Leftrightarrow \sum_k \omega_k x_{(k)} \geq \sum_k \omega_k y_{(k)}$, $\omega_k > 0$ for $k = 1, 2$.

6. A family of social welfare rankings called 'generalized Gini rankings' is defined as follows. Let $x_{(k)}$ denote the $k-th$ highest utility under policy x, where $1 \leq k \leq n$.
x is as good as $y \Leftrightarrow \frac{1}{n^\delta}\sum_{k=1}^n [k^\delta - (k-1)^\delta]x_{(k)} \geq \frac{1}{n^\delta}\sum_{k=1}^n [k^\delta - (k-1)^\delta]y_{(k)}$
(i) Show that 'generalized Gini rankings' satisfy 'Invariance under cardinal full comparable utilities' and 'Anonymity'.
(ii) Show that $\delta = 1$ is the same as the utilitarian rule.
(iii) Show that 'generalized Gini rankings' converge to 'Rawlsian ranking' as $\delta \rightarrow \infty$.

7. Consider a profile of individual utility function $(u_1, u_2, \ldots, u_n)$. Consider another profile $(u'_1, u'_2, \ldots, u'_n)$, where $u'_i = u_j$ and $u'_j = u_i$, $u'_k = u_k$ for all $k \neq i, j$. A social welfare ranking satisfies Anonimity* if social raking remains the same under profiles $(u_1, u_2, \ldots, u_n)$ and $(u'_1, u'_2, \ldots, u'_n)$. Show that under 'Welfarism', Anonimity is equivalent to Anonimity*.

8. Either prove or provide counterexample:
(a) Invariance under ordinal non-comparability is strictly stronger than Invariance under cardinal unit comparability.
(b) Invariance under ordinal comparability is strictly stronger than Invariance under cardinal unit comparability.
(c) IIA is strictly weaker than Welfarism (strong-neutrality).

9. Let $\widehat{R}$ be a ranking of policies which satisfies Welfarism, that is policies are ranked just by comparing utility vectors. Moreover, assume that $u_i(x) > 0$ for all $i \in N$ and for all $x \in A$. Let us construct a ranking R^* from $\widehat{R}$ as follows,
$(e^{u_1(x)}, \ldots, e^{u_n(x)}) R^* (e^{u_1(y)}, \ldots, e^{u_n(y)}) \Leftrightarrow (u_1(x), \ldots, u_n(x)) \widehat{R} (u_1(x), \ldots, u_n(y))$
(a) Show that if $\widehat{R}$ satisfies Weak Pareto, Continuity and Anonimity then R^* also satisfies these axioms.
(b) Let us define two invariance properties,
Invariance under non-comparable unit shift:
$(u_1(x), \ldots, u_n(x)) R (u_1(y), \ldots, u_n(y)) \Leftrightarrow (a_1 + u_1(x), \ldots, a_n + u_n(x)) R (a_1 +$

$u_1(y), \ldots, a_n + u_n(y))$
Invariance under non-comparable scale shift:
$(u_1(x), \ldots, u_n(x))R(u_1(y), \ldots, u_n(y)) \Leftrightarrow (b_1u_1(x), \ldots, b_nu_n(x))R(b_1u_1(y), \ldots, b_nu_n(y))$
Show that: $\widehat{R}$ satisfies Invariance under non-comparable unit shift $\Leftrightarrow$ R^* satisfies Invariance under non-comparable scale shift.
(c) Show that there is only one aggregation rule which satisfies Invariance under non-comparable scale shift, Weak Pareto, Continuity and Anonimity. Identify this rule.

10. Suppose that a society consists of two groups, Red and Blue, of equal population. The society has a per capita education budget of $\frac{1}{2}$. A typical education policy is a division of this budget between Reds and Blues, where x_R and x_B denote per capita education expenditures on Reds and Blues respectively. For each group t, given any policy, earning (w_t) is a strictly increasing function of years of schooling. Given a policy (x_R, x_B), w_R is uniform random variable in the range $[x_R, 2]$ and w_B is uniform random variable in the range $[0.5 + x_B, 2]$.
(a) Find and interpret the equal opportunity policy.
(b) Is there any efficiency loss under the equal opportunity policy?