# Topic 10: Hypothesis Testing

**Rohini Somanathan**

**Course 003, 2018**

# The Problem of Hypothesis Testing

A statistical hypothesis is an assertion or conjecture about the probability distribution of one or more random variables.

A test of a statistical hypothesis is a rule or procedure for deciding whether to reject that assertion.

Suppose we have a sample $x = (x_1, \dots, x_n)$ from a density $f$. We have two hypotheses about $f$. On the basis of our sample, one of the hypotheses is accepted and the other is rejected.

The two hypotheses have different status:

- the null hypothesis, $H_0$, is the hypothesis under test. It is the conservative hypothesis, not to be rejected unless the evidence is clear
- the alternative hypothesis $H_1$ specifies the kind of departure from the null hypothesis that is of interest to us

A hypothesis is simple if it completely specifies the probability distribution, else it is composite.

Examples:

- Income $\sim$ log-normally with known variance but unknown mean. $H_0 : \mu \geq 8,000$ rupees per month, $H_1 : \mu < 8,000$
- We would like to know whether parents are more likely to have boys than girls. The probability of a boy child $\sim$ Bernoulli $(p)$. $H_0 : p = \frac{1}{2}$ and $H_1 : p > \frac{1}{2}$

# Statistical tests

**Before deciding whether or not to accept $H_0$, we observe a random sample. Denote by $S$, the set of all possible sample outcomes.**

**A test procedure partitions $S$ into two subsets, the acceptance region with values that lead us to accept $H_0$ and the critical region $R$ , which has values which lead its rejection.**

**These sets are usually defined in terms of values taken by a test statistic ( the same mean, the sample variance or functions of these). The critical values of a test statistic are the bounds of $R$.**

**When arriving at a decision based on a sample and a test, we may make two types of errors:**

- **$H_0$ may be rejected when it is true- a Type I error**
- **$H_0$ may be accepted when it is false- a Type II error**

**We use $\alpha$ and $\beta$ to denote these errors.**

**The power function is useful in computing these errors and summarizes the properties of a test. We will define these functions.**

**We also identify the set of hypothesis testing problems for which there is an optimal test and characterize these tests.**

# The power function

The power function of a test is the probability of rejecting $H_0$ as a function of the parameter $\theta \in \Omega$. If we are using a test statistic $T$

$$\pi(\theta) = Pr(T \in R) \text{ for } \theta \in \Omega$$

Since the power function of a test specifies the probability of rejecting $H_0$ as a function of the real parameter value, we can evaluate our test by asking how often it leads to mistakes.

What is the power function of an ideal test ? Think of examples when such a test exists.

It is common to specify an upper bound $\alpha_0$ on $\pi(\theta)$ for every value $\theta \in \Omega_0$. This bound $\alpha_0$ is the level of significance of the test.

The size of a test, $\alpha$ is the maximum probability, among all values of $\theta \in \Omega_0$ of making an incorrect decision:

$$\alpha = \sup_{\theta_0 \in \Omega_0} \pi(\theta)$$

Given a level of significance $\alpha_0$, only tests for which $\alpha \leq \alpha_0$ are admissible.

# Example 1: Normal distribution

$X \sim N(\mu, 100)$ and we are interested in testing $H_0 : \mu = 80$ against $H_1 : \mu > 80$.

Let $\bar{x}$ denote the mean of a sample $n = 25$ from this distribution and suppose we use the critical region $R = \{(x_1, x_2, \dots x_{25}) : \bar{x} > 83\}$.

The power function is

$$\pi(\mu) = P(\bar{X} > 83) = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > \frac{83 - \mu}{2}\right) = 1 - \Phi\left(\frac{83 - \mu}{2}\right)$$

The size of this test is the probability of Type 1 error: $\alpha = 1 - \Phi(\frac{3}{2}) = .067 = \pi(80)$

What are the values of $\pi(83)$, and $\pi(86)$?

$\pi(80)$ is given above, $\pi(83) = 0.5$, $\pi(86) = 1 - \Phi(-\frac{3}{2}) = \Phi(\frac{3}{2}) = .933$

stata: display normal(1.5)

We can sketch the graph of the power function using the command

stata: twoway function 1-normal((83-x)/2), range (70 90)

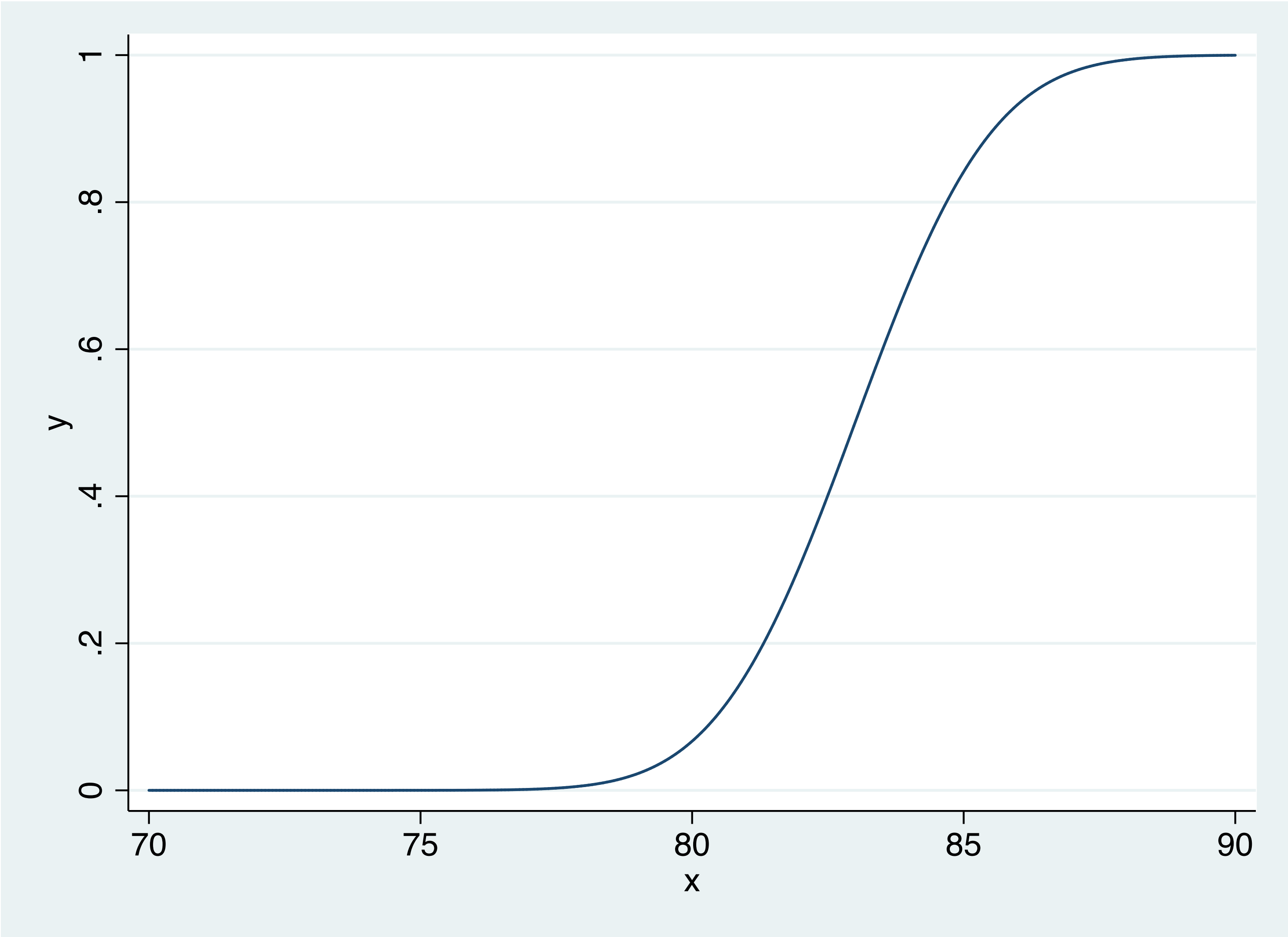
What is the p-value corresponding to $\bar{x} = 83.41$?

This is the smallest level of significance, $\alpha_0$ at which a given hypothesis would be rejected based on the observed outcome of X?

In this case, $Pr(\bar{X} \geq 83.41) = 1 - \Phi(\frac{3.41}{2}) = .044$.

Can you find a test for which $\alpha_0 = .05$?

# The power graph..example 1



# Example 2 : Uniform distribution

A random sample is taken from a uniform distribution on $[0, \theta]$ and we would like to test

$$H_0 : 3 \leq \theta \leq 4 \qquad \text{against} \qquad H_1 : \theta < 3 \text{ or } \theta > 4$$

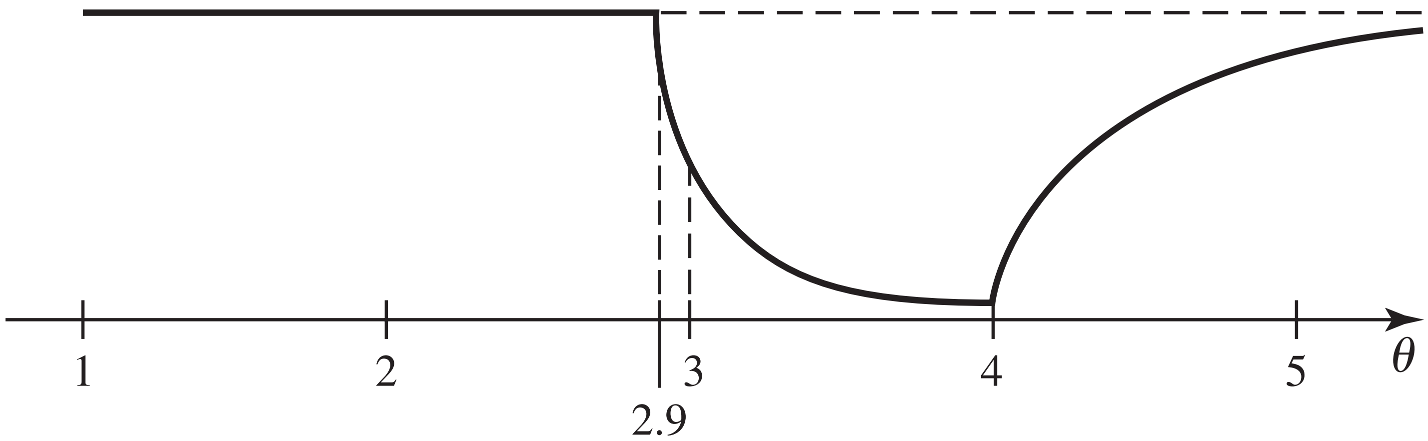
Our test procedure uses the M.L.E. of $\theta$, $Y_n = \max(X_1, \dots, X_n)$ and rejects the null hypothesis whenever $Y_n$ lies outside $[2.9, 4]$. What might be the rationale for this type of test?

The power function for this test is given by

$$\pi(\theta) = Pr(Y_n < 2.9|\theta) + Pr(Y_n > 4|\theta)$$

- What is the power of the test if $\theta < 2.9$?
- When $\theta$ takes values between 2.9 and 4, the probability that any sample value is less than 2.9 is given by $\frac{2.9}{\theta}$ and therefore $Pr(Y_n < 2.9|\theta) = (\frac{2.9}{\theta})^n$ and $Pr(Y_n > 4|\theta) = 0$. Therefore the power function $\pi(\theta) = (\frac{2.9}{\theta})^n$
- When $\theta > 4$, $\pi(\theta) = (\frac{2.9}{\theta})^n + [1 - (\frac{4}{\theta})^n]$

# The power graph..example 2



# Example 3: Binomial distribution

**The probability of a defective bolts in a shipment is given by a Bernoulli random variable.**

**We have the following null and alternative hypotheses: $H_0 : p \leq .02$, $H_1 : p > .02$.**

**Our sample consists of 200 bolts and our test statistic is number of defective items $X$ in the sample. We want to find a test for which $\alpha_0 = .05$.**

**Let us now think of how our test statistic $X$ behaves for different values of $p$. We want to find $X$ such that $\alpha_0 \leq 0.05$**

**Since $X \sim Bin(n, p)$, the probability of $X$ being greater than any given $x$ is increasing in $p$, we can focus on $p = .02$. If $Pr(X > x) \leq .05$ for this $p$, it will be true for all smaller values of $p$.**

**It turns out that for $p = .02$, the probability that the number of defective items is greater than 7 is .049 ( display 1-binomial(200,7,.02)).**

**$R = \{x : x > 7\}$ is therefore the test we choose and its size is 0.049.**

**In general, for discrete distributions, the size will typically be strictly smaller than $\alpha_0$.**

**The size of tests which reject for more than 4, 5 and 6 defective pieces are .37. .21 and .11 respectively.**

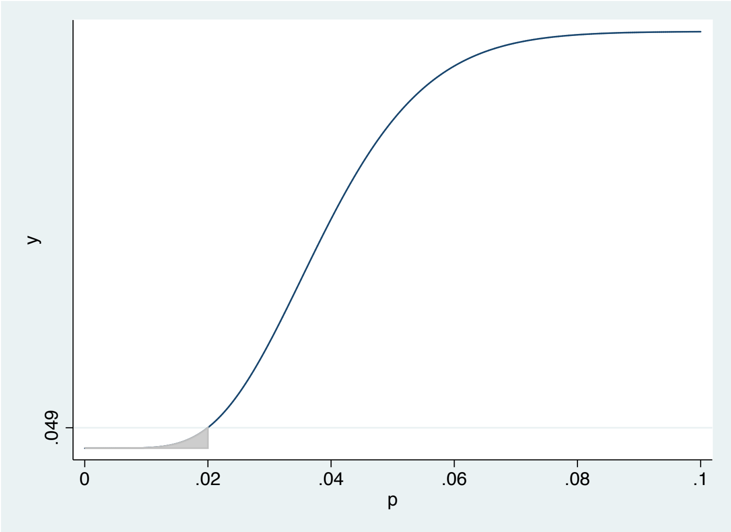
 

# Example 3: power function

**Let's graph the power function of this test:**

```
twoway function y=1-binomial(200,7,x), range(0 .1) xtitle(p)||function y=1-binomial(200,7,x),
range(0 .02) color(gs12) recast(area) legend(off) ylabel(.049)
```

**(all on one line)**




**Can you mark off the two types of errors for different values of $p$?**

**What happens to this power function as we increase or decrease the critical region? (say $R = \{x : x > 6\}$ or $R = \{x : x > 8\}$.**

# Testing simple hypotheses

We have so far focussed on understanding the properties of given tests. What does an optimal test look like and when does such a test exist?

Suppose that $\Omega_0$ and $\Omega_1$ each contain a single element:

$$H_0 : \theta = \theta_0 \text{ and } H_1 : \theta = \theta_1$$

Denote by $f_i(x)$ the joint density function or p.f. of the observations in our sample under $H_i$:

$$f_i(x) = f(x_1|\theta_i)f(x_2|\theta_i)\dots f(x_n|\theta_i)$$

For a test $\delta$, type I and type II errors are $\alpha(\delta)$ and $\beta(\delta)$ respectively:

$$\alpha(\delta) = Pr(\text{ Rejecting } H_0|\theta = \theta_0) \quad \text{and} \quad \beta(\delta) = Pr(\text{ Not Rejecting } H_0|\theta = \theta_1)$$

By always accepting $H_0$, we achieve $\alpha(\delta) = 0$ but then $\beta(\delta) = 1$. The converse is true if we always reject $H_0$.

It turns out that we can find an optimal test which minimizes any linear combination of $\alpha(\delta)$ and $\beta(\delta)$.

# Optimal tests for simple hypotheses

**Theorem (Minimizing the linear combination $a\alpha(\delta) + b\beta(\delta)$):**

*Let $\delta^*$ denote a test procedure such that the hypothesis $H_0$ is accepted if $af_0(x) > bf_1(x)$ and $H_1$ is accepted if $af_0(x) < bf_1(x)$. Either $H_0$ or $H_1$ may be accepted if $af_0(x) = bf_1(x)$. Then for any other test procedure $\delta$,*

$$a\alpha(\delta^*) + b\beta(\delta^*) \leq a\alpha(\delta) + b\beta(\delta)$$

**So we reject whenever the likelihood ratio $\frac{f_1(x)}{f_0(x)} > \frac{a}{b}$. If we are minimizing the sum of errors, we would reject whenever $\frac{f_1(x)}{f_0(x)} > 1$.**

*Proof.* (for discrete distributions)

$$a\alpha(\delta)+b\beta(\delta) = a\sum_{x\in R} f_0(x)+b\sum_{x\in R^c} f_1(x) = a\sum_{x\in R} f_0(x)+b\left[1-\sum_{x\in R} f_1(x)\right] = b+\sum_{x\in R}[af_0(x)-bf_1(x)]$$

The desired function $a\alpha(\delta) + b\beta(\delta)$ will be minimized if the critical region includes only those points for which $af_0(x) - bf_1(x) < 0$. We therefore reject when the likelihood ratio exceeds $\frac{a}{b}$.

□

# Minimizing $\beta(\delta)$, given $\alpha_0$

Suppose we fix a level of significance $\alpha_0$ and want a test procedure that minimizes $\beta(\delta)$? The optimal test in this case is obtained by modifying the previous result.

We know that the optimal test which minimizes $\alpha(\delta) + k\beta(\delta)$ involves rejecting the null hypothesis whenever

$$\frac{f_1(x)}{f_0(x)} > \frac{1}{k}$$

.

If we choose k such that $Pr(\frac{f_1(x)}{f_0(x)} > \frac{1}{k}) = .05$ under $H_0$, then the previous result tells us we are minimizing $\beta$. Hence the following result.

**The Neyman-Pearson Lemma :** Let $\delta^*$ denote a test procedure such that, for some constant k, the hypothesis $H_0$ is accepted if $f_0(x) > kf_1(x)$ and $H_1$ is accepted if $f_0(x) < kf_1(x)$. Either $H_0$ or $H_1$ may be accepted if $f_0(x) = kf_1(x)$. If $\delta$ is any other test procedure such that $\alpha(\delta) \leq \alpha(\delta^*)$, then it follows that $\beta(\delta) \geq \beta(\delta^*)$. Furthermore if $\alpha(\delta) < \alpha(\delta^*)$ then $\beta(\delta) > \beta(\delta^*)$

Let us now take an example to show how we find k such that $\alpha(\delta^*) = .05$. This procedure will then have the minimum possible value of $\beta(\delta)$.

# Neyman Pearson Lemma..example 1

**Let $X_1 \dots X_n$ be a normal random sample, $X_i \sim N(\mu, 1)$.**

$$H_0 : \theta = 0 \text{ and } H_1 : \theta = 1$$

**We want a test procedure $\delta$ for which $\beta$ is minimized given $\alpha_0 = .05$.**

$$f_0(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\sum x_i^2} \text{ and } f_1(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\sum (x_i - 1)^2} \text{ so } \frac{f_1(x)}{f_0(x)} = e^{n(\bar{x}_n - \frac{1}{2})} > k$$

**by the NP Lemma. This condition can be re-written in terms of our sample mean $\bar{x}_n$:**

$$\bar{x}_n > k' = \frac{1}{2} + \frac{1}{n} \log k$$

**How do we find $k'$? $Pr(\bar{X}_n > k' | \theta = 0) = Pr(Z > k'\sqrt{n})$. For $\alpha_0 = .05$, we have $k'\sqrt{n} = 1.645$ or $k' = \frac{1.645}{\sqrt{n}}$**

**Under this procedure, Type II error is**

$$\beta(\delta^*) = Pr(\bar{X}_n < \frac{1.645}{\sqrt{n}} | \theta = 1) = Pr(Z < 1.645 - \sqrt{n})$$

**For $n = 9$, $\beta(\delta^*) = 0.0877$ (display normal(1.645-3))**

**If instead, we want to minimize $2\alpha(\delta) + \beta(\delta)$, we choose $k' = \frac{1}{2} + \frac{1}{n} \log 2$, our optimal procedure rejects $H_0$ when $\bar{x}_n > 0.577$. In this case, $\alpha(\delta_0) = 0.0417$ (display 1-normal( (.577)*3)) and $\beta(\delta_0) = 0.1022$ (display normal( (.577-1)*3)) and the minimized value of $2\alpha(\delta) + \beta(\delta)$ is 0.186**

# Neyman Pearson Lemma..example 2

Let $X_1 \dots X_n$ be a sample from a Bernoulli distribution

$$H_0 : p = 0.2 \text{ and } H_1 : p = 0.4$$

How do we find a test procedure $\delta$ which limits us to an $\alpha_0$ and minimizes $\beta$. let $y$ denote values taken by $Y = \sum X_i$

$$f_0(x) = (0.2)^y (0.8)^{n-y} \text{ and } f_1(x) = (0.4)^y (0.6)^{n-y}$$

$$\frac{f_1(x)}{f_0(x)} = \left(\frac{3}{4}\right)^n \left(\frac{8}{3}\right)^y$$

The lemma tells us to use a procedure which rejects $H_0$ when the likelihood ratio is greater than a constant $k$. This condition can be re-written in terms of our sample mean $y > \frac{\log k + n \log \frac{4}{3}}{\log \frac{8}{3}} = k'$.

Now we would like to find $k'$ such that $Pr(Y > k'|p = 0.2) = .05$.

We may not however be able to do since $Y$ is discrete. If $n = 10$, we find that $Pr(Y > 3|p = 0.2) = .121$ and $Pr(Y > 4|p = 0.2) = .038$, (display 1-binomial(10,4,.2)) so we can decide to set one of these probabilities as the values of $\alpha(\delta)$ and use the corresponding values for $k'$.

Can you calculate $\beta(\delta)$ if $\delta$ rejects for $y > 4$?