LINEAR ALGEBRA

Chapter 1

LINEAR EQUATIONS

1.1. Introduction

EXAMPLE 1.1.1. Try to draw the two lines

$$\begin{aligned} 3x + 2y &= 5, \\ 6x + 4y &= 5. \end{aligned}$$

It is easy to see that the two lines are parallel and do not intersect, so that this system of two linear equations has no solution.

EXAMPLE 1.1.2. Try to draw the two lines

$$\begin{aligned} 3x + 2y &= 5, \\ x + y &= 2. \end{aligned}$$

It is easy to see that the two lines are not parallel and intersect at the point $(1, 1)$, so that this system of two linear equations has exactly one solution.

EXAMPLE 1.1.3. Try to draw the two lines

$$\begin{aligned} 3x + 2y &= 5, \\ 6x + 4y &= 10. \end{aligned}$$

It is easy to see that the two lines overlap completely, so that this system of two linear equations has infinitely many solutions.

In these three examples, we have shown that a system of two linear equations on the plane $\mathbb{R}^2$ may have no solution, one solution or infinitely many solutions. A natural question to ask is whether there can be any other conclusion. Well, we can see geometrically that two lines cannot intersect at more than one point without overlapping completely. Hence there can be no other conclusion.

In general, we shall study a system of m linear equations of the form

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n &= \ b_1, \\ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n &= \ b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned} \tag{1}$$

with n variables $x_1, x_2, \ldots, x_n$. Here we may not be so lucky as to be able to see geometrically what is going on. We therefore need to study the problem from a more algebraic viewpoint. In this chapter, we shall confine ourselves to the simpler aspects of the problem. In Chapter 6, we shall study the problem again from the viewpoint of vector spaces.

If we omit reference to the variables, then system (1) can be represented by the array

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array} \right) \tag{2}$$

of all the coefficients. This is known as the augmented matrix of the system. Here the first row of the array represents the first linear equation, and so on.

We also write $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

represent the coefficients and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

represents the variables.

EXAMPLE 1.1.4. The array

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array} \right) \tag{3}$$

represents the system of 3 linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{4}$$

with 5 variables x_1, x_2, x_3, x_4, x_5. We can also write

$$\begin{pmatrix} 1 & 3 & 1 & 5 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 2 & 4 & 0 & 7 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

1.2. Elementary Row Operations

Let us continue with Example 1.1.4.

EXAMPLE 1.2.1. Consider the array (3). Let us interchange the first and second rows to obtain

$$\left(\begin{array}{ccccc|c} 0 & 1 & 1 & 2 & 1 & 4 \\ 1 & 3 & 1 & 5 & 1 & 5 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array}\right).$$

Then this represents the system of equations

$$\begin{aligned} x_2 + x_3 + 2x_4 + x_5 &= 4, \\ x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{5}$$

essentially the same as the system (4), the only difference being that the first and second equations have been interchanged. Any solution of the system (4) is a solution of the system (5), and vice versa.

EXAMPLE 1.2.2. Consider the array (3). Let us add 2 times the second row to the first row to obtain

$$\left(\begin{array}{ccccc|c} 1 & 5 & 3 & 9 & 3 & 13 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array}\right).$$

Then this represents the system of equations

$$\begin{aligned} x_1 + 5x_2 + 3x_3 + 9x_4 + 3x_5 &= 13, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{6}$$

essentially the same as the system (4), the only difference being that we have added 2 times the second equation to the first equation. Any solution of the system (4) is a solution of the system (6), and vice versa.

EXAMPLE 1.2.3. Consider the array (3). Let us multiply the second row by 2 to obtain

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 2 & 2 & 4 & 2 & 8 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array}\right).$$

Then this represents the system of equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ 2x_2 + 2x_3 + 4x_4 + 2x_5 &= 8, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{7}$$

essentially the same as the system (4), the only difference being that the second equation has been multiplied through by 2. Any solution of the system (4) is a solution of the system (7), and vice versa.

In the general situation, it is not difficult to see the following.

PROPOSITION 1A. (ELEMENTARY ROW OPERATIONS) *Consider the array* (2) *corresponding to the system* (1).
(a) Interchanging the i-th and j-th rows of (2) *corresponds to interchanging the i-th and j-th equations in* (1).
(b) Adding a multiple of the i-th row of (2) *to the j-th row corresponds to adding the same multiple of the i-th equation in* (1) *to the j-th equation.*
(c) Multiplying the i-th row of (2) *by a non-zero constant corresponds to multiplying the i-th equation in* (1) *by the same non-zero constant.*
In all three cases, the collection of solutions to the system (1) *remains unchanged.*

Let us investigate how we may use elementary row operations to help us solve a system of linear equations. As a first step, let us continue again with Example 1.1.4.

EXAMPLE 1.2.4. Consider again the system of linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{8}$$

represented by the array

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array}\right). \tag{9}$$

Let us now perform elementary row operations on the array (9). At this point, do not worry if you do not understand why we are taking the following steps. Adding -2 times the first row of (9) to the third row, we obtain

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & -2 & -2 & -3 & -1 & -7 \end{array}\right).$$

From here, we add 2 times the second row to the third row to obtain

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right). \tag{10}$$

Next, we add -3 times the second row to the first row to obtain

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & -1 & -2 & -7 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

Next, we add the third row to the first row to obtain

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -1 & -6 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

Finally, we add -2 times the third to row to the second row to obtain

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -1 & -6 \\ 0 & 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right). \tag{11}$$

We remark here that the array (10) is said to be in row echelon form, while the array (11) is said to be in reduced row echelon form – precise definitions will follow in Sections 1.5–1.6. Let us see how we may solve the system (8) by using the arrays (10) or (11). First consider (10). Note that this represents the system

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ x_4 + x_5 &= 1. \end{aligned} \tag{12}$$

First of all, take the third equation

$$x_4 + x_5 = 1.$$

If we let $x_5 = t$, then $x_4 = 1 - t$. Substituting these into the second equation, we obtain (you must do the calculation here)

$$x_2 + x_3 = 2 + t.$$

If we let $x_3 = s$, then $x_2 = 2 + t - s$. Substituting all these into the first equation, we obtain (you must do the calculation here)

$$x_1 = -6 + t + 2s.$$

Hence

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (-6 + t + 2s, 2 + t - s, s, 1 - t, t)$$

is a solution of the system (12) for every $s, t \in \mathbb{R}$. In view of Proposition 1A, these are also precisely the solutions of the system (8). Alternatively, consider (11) instead. Note that this represents the system

$$\begin{aligned} x_1 \quad - 2x_3 \quad - x_5 &= -6, \\ x_2 + \ x_3 \quad - x_5 &= \ \ 2, \\ x_4 + x_5 &= \ \ 1. \end{aligned} \tag{13}$$

First of all, take the third equation

$$x_4 + x_5 = 1.$$

If we let $x_5 = t$, then $x_4 = 1 - t$. Substituting these into the second equation, we obtain (you must do the calculation here)

$$x_2 + x_3 = 2 + t.$$

If we let $x_3 = s$, then $x_2 = 2 + t - s$. Substituting all these into the first equation, we obtain (you must do the calculation here)

$$x_1 = -6 + t + 2s.$$

Hence

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (-6 + t + 2s, 2 + t - s, s, 1 - t, t)$$

is a solution of the system (13) for every $s, t \in \mathbb{R}$. In view of Proposition 1A, these are also precisely the solutions of the system (8). However, if you have done the calculations as suggested, you will notice that the calculation is easier for the system (13) than for the system (12). This is clearly a case of the array (11) in reduced row echelon form having more 0's than the array (10) in row echelon form, so that the system (13) has fewer non-zero coefficients than the system (12).

1.3. Row Echelon Form

DEFINITION. A rectangular array of numbers is said to be in row echelon form if the following conditions are satisfied:

(1) The left-most non-zero entry of any non-zero row has value 1. These are called the pivot entries.
(2) All zero rows are grouped together at the bottom of the array.
(3) The pivot entry of a non-zero row occurring lower in the array is to the right of the pivot entry of a non-zero row occurring higher in the array.

Next, we investigate how we may reduce a given array to row echelon form. We shall illustrate the ideas by working on an example.

EXAMPLE 1.3.1. Consider the array

$$\begin{pmatrix} 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 & 1 & 2 \end{pmatrix}.$$

Step 1: Locate the left-most non-zero column and cover all columns to the left of this column (in our illustration here, $\times$ denotes an entry that has been covered). We now have

$$\begin{pmatrix} \times & 0 & 5 & 0 & 15 & 5 \\ \times & 2 & 4 & 7 & 1 & 3 \\ \times & 1 & 2 & 3 & 0 & 1 \\ \times & 1 & 2 & 4 & 1 & 2 \end{pmatrix}.$$

Step 2: Consider the part of the array that remains uncovered. By interchanging rows if necessary, ensure that the top-left entry is non-zero. So let us interchange rows 1 and 4 to obtain

$$\begin{pmatrix} \times & 1 & 2 & 4 & 1 & 2 \\ \times & 2 & 4 & 7 & 1 & 3 \\ \times & 1 & 2 & 3 & 0 & 1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Step 3: If the top entry on the left-most uncovered column is a, then we multiply the top uncovered row by $1/a$ to ensure that this entry becomes 1. So let us divide row 1 by 1 to obtain

$$\begin{pmatrix} \times & 1 & 2 & 4 & 1 & 2 \\ \times & 2 & 4 & 7 & 1 & 3 \\ \times & 1 & 2 & 3 & 0 & 1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}!$$

Step 4: We now try to make all entries below the top entry on the left-most uncovered column zero. This can be achieved by adding suitable multiples of row 1 to the other rows. So let us add -2 times row 1 to row 2 to obtain

$$\begin{pmatrix} \times & 1 & 2 & 4 & 1 & 2 \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 1 & 2 & 3 & 0 & 1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Then let us add -1 times row 1 to row 3 to obtain

$$\begin{pmatrix} \times & 1 & 2 & 4 & 1 & 2 \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Step 5: Now cover the top row. We then obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Step 6: Repeat Steps 1–5 on the uncovered array, and as many times as necessary so that eventually the whole array gets covered. So let us continue. Following Step 1, we locate the left-most non-zero column and cover all columns to the left of this column. We now have

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Following Step 2, we interchanging rows if necessary to ensure that the top-left entry is non-zero. So let us interchange rows 1 and 3 (here we do not count any covered rows) to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & 5 & 0 & 15 & 5 \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 3, we multiply the top row by a suitable number to ensure that the top entry on the left-most uncovered column becomes 1. So let us multiply row 1 by $1/5$ to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & 1 & 0 & 3 & 1 \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 4, we do nothing! Following Step 5, we cover the top row. We then obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 1, we locate the left-most non-zero column and cover all columns to the left of this column. We now have

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & -1 & -1 & -1 \\ \times & \times & \times & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 2, we do nothing! Following Step 3, we multiply the top row by a suitable number to ensure that the top entry on the left-most uncovered column becomes 1. So let us multiply row 1 by -1 to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & 1 & 1 & 1 \\ \times & \times & \times & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 4, we now try to make all entries below the top entry on the left-most uncovered column zero. So let us add row 1 to row 2 to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & 1 & 1 & 1 \\ \times & \times & \times & 0 & 0 & 0 \end{pmatrix}.$$

Following Step 5, we cover the top row. We then obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & 0 & 0 & 0 \end{pmatrix}.$$

Following Step 1, we locate the left-most non-zero column and cover all columns to the left of this column. We now have

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step ∞. Uncover everything! We then have

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in row echelon form.

In practice, we do not actually cover any entries of the array, so let us repeat here the same argument without covering anything – the reader is advised to compare this with the earlier discussion. We start with the array

$$\begin{pmatrix} 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 & 1 & 2 \end{pmatrix}.$$

Interchanging rows 1 and 4, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Adding -2 times row 1 to row 2, and adding -1 times row 1 to row 3, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Interchanging rows 2 and 4, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Multiplying row 1 by 1/5, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Multiplying row 3 by -1, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Adding row 3 to row 4, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in row echelon form.

Remarks. (1) As already observed earlier, we do not actually physically cover rows or columns. In any practical situation, we simply copy these entries without changes.

(2) The steps indicated the the first part of the last example are for guidance only. In practice, we do not have to follow the steps above religiously, and what we do is to a great extent dictated by good common sense. For instance, suppose that we are faced with the array

$$\begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 2 & 0 & 2 \end{pmatrix}.$$

If we follow the steps religiously, then we shall multiply row 1 by 1/2. However, note that this will introduce fractions to some entries of the array, and any subsequent calculation will become rather messy. Instead, let us multiply row 1 by 3 to obtain

$$\begin{pmatrix} 6 & 9 & 6 & 3 \\ 3 & 2 & 0 & 2 \end{pmatrix}.$$

Then let us multiply row 2 by 2 to obtain

$$\begin{pmatrix} 6 & 9 & 6 & 3 \\ 6 & 4 & 0 & 4 \end{pmatrix}.$$

Adding -1 times row 1 to row 2, we obtain

$$\begin{pmatrix} 6 & 9 & 6 & 3 \\ 0 & -5 & -6 & 1 \end{pmatrix}.$$

In this way, we have avoided the introduction of fractions until later in the process. In general, if we start with an array with integer entries, then it is possible to delay the introduction of fractions by omitting Step 3 until the very end.

EXAMPLE 1.3.2. Consider the array

$$\begin{pmatrix} 2 & 1 & 3 & 2 & 5 \\ 1 & 3 & 2 & 4 & 1 \\ 3 & 2 & 0 & 0 & 2 \end{pmatrix}.$$

Try following the steps indicated in the first part of the previous example religiously and try to see how complicated the calculations get. On the other hand, we can modify the steps with some common sense. First of all, we interchange rows 1 and 2 to obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 2 & 0 & 0 & 2 \end{pmatrix}.$$

The reason for taking this step is to put an entry 1 at the top left without introducing fractions anywhere. When we next add multiples of row 1 to the other rows to make 0's below this 1, we do not introduce fractions either. Now adding -2 times row 1 to row 2, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & -5 & -1 & -6 & 3 \\ 3 & 2 & 0 & 0 & 2 \end{pmatrix}.$$

Adding -3 times row 1 to row 3, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & -5 & -1 & -6 & 3 \\ 0 & -7 & -6 & -12 & -1 \end{pmatrix}.$$

Next, multiplying row 2 by -7, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 35 & 7 & 42 & -21 \\ 0 & -7 & -6 & -12 & -1 \end{pmatrix}.$$

Multiplying row 3 by -5, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 35 & 7 & 42 & -21 \\ 0 & 35 & 30 & 60 & 5 \end{pmatrix}.$$

Note that here we are essentially covering up row 1. Also, we have multiplied rows 2 and 3 by suitable multiples to that their leading non-zero entries are the same, in preparation for taking the next step without introducing fractions. Now adding -1 times row 2 to row 3, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 35 & 7 & 42 & -21 \\ 0 & 0 & 23 & 18 & 26 \end{pmatrix}.$$

Here, the array is almost in row echelon form, except that the leading non-zero entries in rows 2 and 3 are not equal to 1. However, we can always multiply row 2 by $1/35$ and row 3 by $1/23$ if we want to obtain the row echelon form

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 1 & 1/5 & 6/5 & -3/5 \\ 0 & 0 & 1 & 18/23 & 26/23 \end{pmatrix}.$$

If this differs from the answer you got when you followed the steps indicated in the previous example religiously, do not worry. row echelon forms are not unique!

1.4. Reduced Row Echelon Form

Definition. A rectangular array of numbers is said to be in reduced row echelon form if the following conditions are satisfied:
(1) The left-most non-zero entry of any non-zero row has value 1. These are called the pivot entries.
(2) All zero rows are grouped together at the bottom of the array.
(3) The pivot entry of a non-zero row occurring lower in the array is to the right of the pivot entry of a non-zero row occurring higher in the array.
(4) Each column containing a pivot entry has 0's everywhere else in the column.

We now investigate how we may reduce a given array to reduced row echelon form. Here, we basically take an extra step to convert an array from row echelon form to reduced row echelon form. We shall illustrate the ideas by continuing on an earlier example.

Example 1.4.1. Consider again the array

$$\begin{pmatrix} 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 & 1 & 2 \end{pmatrix}.$$

We have already shown in Example 1.3.1 that this array can be reduced to row echelon form

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 1: Cover all zero rows at the bottom of the array. We now have

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step 2: We now try to make all the entries above the pivot entry on the bottom row zero (here again we do not count any covered rows). This can be achieved by adding suitable multiples of the bottom row to the other rows. So let us add -4 times row 3 to row 1 to obtain

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step 3: Now cover the bottom row. We then obtain

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step 4: Repeat Steps 2–3 on the uncovered array, and as many times as necessary so that eventually the whole array gets covered. So let us continue. Following Step 2, we add -2 times row 2 to row 1 to obtain

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Following Step 3, we cover row 2 to obtain

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -9 & -4 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Following Step 2, we do nothing! Following Step 3, we cover row 1 to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step ∞. Uncover everything! We then have

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in reduced row echelon form.
Again, in practice, we do not actually cover any entries of the array, so let us repeat here the same argument without covering anything – the reader is advised to compare this with the earlier discussion. We start with the row echelon form

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Adding -4 times row 3 to row 1, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Adding -2 times row 2 to row 1, we obtain

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in reduced row echelon form.

1.5. Solving a System of Linear Equations

Let us first summarize what we have done so far. We study a system (1) of m linear equations in n variables $x_1, \ldots, x_n$. If we omit reference to the variables, then the system (1) can be represented by the array (2), with m rows and $n+1$ columns. We next reduce the array (2) to row echelon form or reduced row echelon form by elementary row operations.

By Proposition 1A, the system of linear equations represented by the array in row echelon form or reduced row echelon form has the same solution set as the system (1). It follows that to solve the system

(1), it remains to solve the system represented by the array in row echelon form or reduced row echelon form. We now describe a simple way to obtain all solutions of this system.

Definition. Any column of an array (2) in row echelon form or reduced row echelon form containing a pivot entry is called a pivot column.

First of all, let us eliminate the situation when the system has no solutions. Suppose that the array (2) has been reduced to row echelon form, and that this contains a row of the form

$$\underbrace{0 \quad \ldots \quad 0}_{n} \quad 1$$

corresponding to the last column of the array being a pivot column. This row represents the equation

$$0x_1 + \ldots + 0x_n = 1;$$

clearly the system cannot have any solution.

Definition. Suppose that the array (2) in row echelon form or reduced row echelon form satisfies the condition that its last column is not a pivot column. Then any variable x_i corresponding to a pivot column is called a pivot variable. All other variables are called free variables.

Example 1.5.1. Consider the array

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

representing the system

$$\begin{aligned} x_2 \qquad\qquad - 9x_5 &= -4, \\ x_3 \qquad + 3x_5 &= 1, \\ x_4 + x_5 &= 1. \end{aligned}$$

Note that the zero row in the array represents an equation which is trivial! Here the last column of the array is not a pivot column. Now columns $2, 3, 4$ are the pivot columns, so that x_2, x_3, x_4 are the pivot variables and x_1, x_5 are the free variables.

To solve the system, we allow the free variables to take any values we choose, and then solve for the pivot variables in terms of the values of these free variables.

Example 1.5.2. Consider the system of 4 linear equations

$$\begin{aligned} 5x_3 \qquad\quad + 15x_5 &= 5, \\ 2x_2 + 4x_3 + 7x_4 + x_5 &= 3, \\ x_2 + 2x_3 + 3x_4 \qquad &= 1, \\ x_2 + 2x_3 + 4x_4 + x_5 &= 2, \end{aligned} \tag{14}$$

in the 5 variables x_1, x_2, x_3, x_4, x_5. If we omit reference to the variables, then the system can be represented by the array

$$\left(\begin{array}{ccccc|c} 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 & 1 & 2 \end{array}\right). \tag{15}$$

As in Example 1.3.1, we can reduce the array (15) to row echelon form

$$\left(\begin{array}{ccccc|c} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{16}$$

representing the system

$$\begin{aligned} x_2 + 2x_3 + 4x_4 + x_5 &= 2, \\ x_3 + 3x_5 &= 1, \\ x_4 + x_5 &= 1. \end{aligned} \tag{17}$$

Alternatively, as in Example 1.4.1, we can reduce the array (15) to reduced row echelon form

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{18}$$

representing the system

$$\begin{aligned} x_2 - 9x_5 &= -4, \\ x_3 + 3x_5 &= 1, \\ x_4 + x_5 &= 1. \end{aligned} \tag{19}$$

By Proposition 1A, the three systems (14), (17) and (19) have exactly the same solution set. Now, we observe from (16) or (18) that columns $2, 3, 4$ are the pivot columns, so that x_2, x_3, x_4 are the pivot variables and x_1, x_5 are the free variables. If we assign values $x_1 = s$ and $x_5 = t$, then we have, from (17) (harder) or (19) (easier), that

$$(x_1, x_2, x_3, x_4, x_5) = (s, 9t - 4, -3t + 1, -t + 1, t). \tag{20}$$

It follows that (20) is a solution of the system (14) for every $s, t \in \mathbb{R}$.

Example 1.5.3. Let us return to Example 1.2.4, and consider again the system (8) of 3 linear equations in the 5 variables x_1, x_2, x_3, x_4, x_5. If we omit reference to the variables, then the system can be represented by the array (9). We can reduce the array (9) to row echelon form (10), representing the system (12). Alternatively, we can reduce the array (9) to reduced row echelon form (11), representing the system (13). By Proposition 1A, the three systems (8), (12) and (13) have exactly the same solution set. Now, we observe from (10) or (11) that columns $1, 2, 4$ are the pivot columns, so that x_1, x_2, x_4 are the pivot variables and x_3, x_5 are the free variables. If we assign values $x_3 = s$ and $x_5 = t$, then we have, from (12) (harder) or (13) (easier), that

$$(x_1, x_2, x_3, x_4, x_5) = (-6 + t + 2s, 2 + t - s, s, 1 - t, t). \tag{21}$$

It follows that (21) is a solution of the system (8) for every $s, t \in \mathbb{R}$.

Example 1.5.4. In this example, we do not bother even to reduce the matrix to row echelon form. Consider the system of 3 linear equations

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + 2x_4 &= 5, \\ x_1 + 3x_2 + 2x_3 + 4x_4 &= 1, \\ 3x_1 + 2x_2 &= 2, \end{aligned} \tag{22}$$

in the 4 variables x_1, x_2, x_3, x_4. If we omit reference to the variables, then the system can be represented by the array

$$\left(\begin{array}{cccc|c} 2 & 1 & 3 & 2 & 5 \\ 1 & 3 & 2 & 4 & 1 \\ 3 & 2 & 0 & 0 & 2 \end{array}\right). \tag{23}$$

As in Example 1.3.2, we can reduce the array (23) to the form

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 1 \\ 0 & 35 & 7 & 42 & -21 \\ 0 & 0 & 23 & 18 & 26 \end{array}\right), \tag{24}$$

representing the system

$$\begin{aligned} x_1 + \ 3x_2 + \ 2x_3 + \ 4x_4 &= \quad 1, \\ 35x_2 + \ 7x_3 + 42x_4 &= -21, \\ 23x_3 + 18x_4 &= \quad 26. \end{aligned} \tag{25}$$

Note that the array (24) is almost in row echelon form, except that the pivot entries are not 1. By Proposition 1A, the two systems (22) and (25) have exactly the same solution set. Now, we observe from (24) that columns $1, 2, 3$ are the pivot columns, so that x_1, x_2, x_3 are the pivot variables and x_4 is the free variable. If we assign values $x_4 = s$, then we have, from (25), that

$$(x_1, x_2, x_3, x_4) = \left(\frac{16}{23}s + \frac{28}{23}, -\frac{24}{23}s - \frac{19}{23}, -\frac{18}{23}s + \frac{26}{23}, s\right). \tag{26}$$

It follows that (26) is a solution of the system (22) for every $s \in \mathbb{R}$.

1.6. Homogeneous Systems

Consider a homogeneous system of m linear equations of the form

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n &= 0, \\ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0, \end{aligned} \tag{27}$$

with n variables $x_1, x_2, \ldots, x_n$. If we omit reference to the variables, then system (27) can be represented by the array

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 \end{array}\right) \tag{28}$$

of all the coefficients.

Note that the system (27) always has a solution, namely the trivial solution

$$x_1 = x_2 = \ldots = x_n = 0.$$

Indeed, if we reduce the array (28) to row echelon form or reduced row echelon form, then it is not difficult to see that the last column is a zero column and so cannot be a pivot column.

On the other hand, if the system (27) has a non-trivial solution, then we can multiply this solution by any non-zero real number different from 1 to obtain another non-trivial solution. We have therefore proved the following simple result.

PROPOSITION 1B. *The homogeneous system* (27) *either has the trivial solution as its only solution or has infinitely many solutions.*

The purpose of this section is to discuss the following stronger result.

PROPOSITION 1C. *Suppose that the system* (27) *has more variables than equations; in other words, suppose that* $n > m$. *Then there are infinitely many solutions.*

To see this, let us consider the array (28) representing the system (27). Note that (28) has m rows, corresponding to the number of equations. Also (28) has $n + 1$ columns, where n is the number of variables. However, the column of (28) on the extreme right is a zero column, corresponding to the fact that the system is homogeneous. Furthermore, this column remains a zero column if we perform elementary row operations on the array (28). If we now reduce (28) to row echelon form by elementary row operations, then there are at most m pivot columns, since there are only m equations in (27) and m rows in (28). It follows that if we exclude the zero column on the extreme right, then the remaining n columns cannot all be pivot columns. Hence at least one of the variables is a free variable. By assigning this free variable arbitrary real values, we end up with infinitely many solutions for the system (27).

1.9. Application to Economics

In this section, we describe a simple exchange model due to the economist Leontief. An economy is divided into sectors. We know the total output for each sector as well as how outputs are exchanged among the sectors. The value of the total output of a given sector is known as the price of the output.

Leontief has shown that there exist equilibrium prices that can be assigned to the total outpu sectors in such a way that the income for each sector is exactly the same as its expenses.

EXAMPLE 1.9.1. An economy consists of three sectors A, B, C which purchase from each other according to the table below:

	proportion of output from sector A	B	C
purchased by sector A	0.2	0.6	0.1
purchased by sector B	0.4	0.1	0.5
purchased by sector C	0.4	0.3	0.4

Let p_A, p_B, p_C denote respectively the value of the total output of sectors A, B, C. For the expense to match the value for each sector, we must have

$$\begin{aligned} 0.2p_A + 0.6p_B + 0.1p_C &= p_A, \\ 0.4p_A + 0.1p_B + 0.5p_C &= p_B, \\ 0.4p_A + 0.3p_B + 0.4p_C &= p_C, \end{aligned}$$

leading to the homogeneous linear equations

$$\begin{aligned} 0.8p_A - 0.6p_B - 0.1p_C &= 0, \\ 0.4p_A - 0.9p_B + 0.5p_C &= 0, \\ 0.4p_A + 0.3p_B - 0.6p_C &= 0, \end{aligned}$$

giving rise to the augmented matrix

$$\left(\begin{array}{ccc|c} 0.8 & -0.6 & -0.1 & 0 \\ 0.4 & -0.9 & 0.5 & 0 \\ 0.4 & 0.3 & -0.6 & 0 \end{array}\right), \qquad \text{or simply} \qquad \left(\begin{array}{ccc|c} 8 & -6 & -1 & 0 \\ 4 & -9 & 5 & 0 \\ 4 & 3 & -6 & 0 \end{array}\right).$$

This can be reduced by elementary row operations to

$$\left(\begin{array}{ccc|c} 16 & 0 & -13 & 0 \\ 0 & 12 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

leading to the solution $(p_A, p_B, p_C) = t(\frac{13}{16}, \frac{11}{12}, 1)$ if we assign the free variable $p_C = t$, or to the solution $(p_A, p_B, p_C) = t(39, 44, 48)$ if we assign the free variable $p_C = 48t$, where t is a real parameter. For the latter, the choice $t = 10^6$ gives rise to the prices of 39, 44 and 48 million for the three sectors A, B, C respectively.

Problems for Chapter 1

1. Consider the system of linear equations

$$\begin{aligned} 2x_1 + 5x_2 + 8x_3 &= 2, \\ x_1 + 2x_2 + 3x_3 &= 4, \\ 3x_1 + 4x_2 + 4x_3 &= 1. \end{aligned}$$

   a) Write down the augmented matrix for this system.
   b) Reduce the augmented matrix by elementary row operations to row echelon form.
   c) Use your answer in part (b) to solve the system of linear equations.

2. Consider the system of linear equations

$$\begin{aligned} 4x_1 + 5x_2 + 8x_3 &= 0, \\ x_1 \qquad\quad + 3x_3 &= 6, \\ 3x_1 + 4x_2 + 6x_3 &= 9. \end{aligned}$$

   a) Write down the augmented matrix for this system.
   b) Reduce the augmented matrix by elementary row operations to row echelon form.
   c) Use your answer in part (b) to solve the system of linear equations.

3. Consider the system of linear equations

$$\begin{aligned} x_1 - x_2 - 7x_3 + 7x_4 &= 5, \\ -x_1 + x_2 + 8x_3 - 5x_4 &= -7, \\ 3x_1 - 2x_2 - 17x_3 + 13x_4 &= 14, \\ 2x_1 - x_2 - 11x_3 + 8x_4 &= 7. \end{aligned}$$

   a) Write down the augmented matrix for this system.
   b) Reduce the augmented matrix by elementary row operations to row echelon form.
   c) Use your answer in part (b) to solve the system of linear equations.

4. Solve the system of linear equations

$$\begin{aligned} x + 3y - 2z &= 4, \\ 2x + 7y + 2z &= 10. \end{aligned}$$

5. For each of the augmented matrices below, reduce the matrix to row echelon or reduced row echelon form, and solve the system of linear equations represented by the matrix:

a) $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 3 & 2 & -1 & 3 & 6 \\ 4 & 3 & 1 & 4 & 11 \\ 2 & 1 & -3 & 2 & 1 \end{array}\right)$ b) $\left(\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 1 \\ 2 & -5 & -3 & 12 & 2 \\ 7 & 1 & 8 & 5 & 7 \end{array}\right)$

6. Reduce each of the following arrays by elementary row operations to reduced row echelon form:

a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 11 & 21 & 31 & 41 & 51 \\ 2 & 12 & 22 & 32 & 42 & 52 \\ 3 & 13 & 23 & 33 & 43 & 53 \end{pmatrix}$

7. Consider a system of linear equations in five variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ and expressed in matrix form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is written as a column matrix. Suppose that the augmented matrix $(A|\mathbf{b})$ can be reduced by elementary row operations to the row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 3 & 2 & 0 & 6 & 4 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

a) Which are the pivot variables and which are the free variables?
b) Determine all the solutions of the system of linear equations.

8. Consider a system of linear equations in five variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ and expressed in matrix form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is written as a column matrix. Suppose that the augmented matrix $(A|\mathbf{b})$ can be reduced by elementary row operations to the row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 1 & 5 \\ 0 & 1 & 3 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

a) Which are the pivot variables and which are the free variables?
b) Determine all the solutions of the system of linear equations.

9. Consider the system of linear equations

$$\begin{aligned} x_1 + \lambda x_2 - x_3 &= 1, \\ 2x_1 + x_2 + 2x_3 &= 5\lambda + 1, \\ x_1 - x_2 + 3x_3 &= 4\lambda + 2, \\ x_1 - 2\lambda x_2 + 7x_3 &= 10\lambda - 1. \end{aligned}$$

a) Reduce its associated augmented matrix to row echelon form.
[HINT: After one or two steps, we will find the calculations extremely unpleasant, particularly since we do not know whether λ is zero or non-zero. Try rewriting the system of equations as a system in the variables x_1, x_3, x_2, so that columns 2 and 3 of the augmented matrix are now swapped.]
b) Find a value of λ for which the system is soluble.
c) Solve the system.