LINEAR ALGEBRA

Chapter 6

VECTOR SPACES ASSOCIATED WITH MATRICES

6.1. Introduction

Consider an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \tag{1}$$

with entries in $\mathbb{R}$. Then the rows of A can be described as vectors in $\mathbb{R}^n$ as

$$\mathbf{r}_1 = (a_{11}, \ldots . a_{1n}), \qquad \ldots, \qquad \mathbf{r}_m = (a_{m1}, \ldots . a_{mn}), \tag{2}$$

while the columns of A can be described as vectors in $\mathbb{R}^m$ as

$$\mathbf{c}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \qquad \ldots, \qquad \mathbf{c}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}. \tag{3}$$

For simplicity, we sometimes write

$$A = \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{pmatrix} \qquad \text{and} \qquad A = (\mathbf{c}_1 \quad \ldots \quad \mathbf{c}_n).$$

We also consider the system of homogeneous equations $A\mathbf{x} = \mathbf{0}$.

In this chapter, we shall be concerned with three vector spaces that arise from the matrix A.

DEFINITION. Suppose that A is an $m \times n$ matrix of the form (1), with entries in $\mathbb{R}$.

(RS) The subspace $\text{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ of $\mathbb{R}^n$, where $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are given by (2) and are the rows of the matrix A, is called the row space of A.

(CS) The subspace $\text{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ of $\mathbb{R}^m$, where $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are given by (3) and are the columns of the matrix A, is called the column space of A.

(NS) The solution space of the system of homogeneous linear equations $A\mathbf{x} = \mathbf{0}$ is called the nullspace of A.

REMARKS. (1) To see that $\text{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ is a subspace of of $\mathbb{R}^n$ and that $\text{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ is a subspace of of $\mathbb{R}^m$, recall Proposition 5C.

(2) To see that the nullspace of A is a subspace of $\mathbb{R}^n$, recall Example 5.2.5.

6.2. Row Spaces

Our aim in this section is to find a basis for the row space of a given matrix A with entries in $\mathbb{R}$. This task is made considerably easier by the following result.

PROPOSITION 6A. *Suppose that the matrix B can be obtained from the matrix A by elementary row operations. Then the row space of B is identical to the row space of A.*

PROOF. Clearly the rows of B are linear combinations of the rows of A, so that any linear combination of the rows of B is a linear combination of the rows of A. Hence the row space of B is contained in the row space of A. On the other hand, the rows of A are linear combinations of the rows of B, so a similar argument shows that the row space of A is contained in the row space of B. ◯

To find a basis for the row space of A, we can now reduce A to row echelon form, and consider the non-zero rows that result from this reduction. It is easily seen that these non-zero rows are linearly independent.

EXAMPLE 6.2.1. Let

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{r}_1 &= (1, 3, -5, 1, 5), \\ \mathbf{r}_2 &= (1, 4, -7, 3, -2), \\ \mathbf{r}_3 &= (1, 5, -9, 5, -9), \\ \mathbf{r}_4 &= (0, 3, -6, 2, -1). \end{aligned}$$

Also the matrix A can be reduced to row echelon form as

$$\begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that

$$\mathbf{v}_1 = (1, 3, -5, 1, 5), \qquad \mathbf{v}_2 = (0, 1, -2, 2, -7), \qquad \mathbf{v}_3 = (0, 0, 0, 1, -5)$$

form a basis for the row space of A.

REMARK. Naturally, it is not necessary that the first non-zero entry of a basis element has to be 1.

6.3. Column Spaces

Our aim in this section is to find a basis for the column space of a given matrix A with entries in $\mathbb{R}$. Naturally, we can consider the transpose A^t of A, and use the technique in Section 6.2 to find a basis for the row space of A^t. This basis naturally gives rise to a basis for the column space of A.

EXAMPLE 6.3.1. Let

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix}.$$

Then

$$A^t = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 4 & 5 & 3 \\ -5 & -7 & -9 & -6 \\ 1 & 3 & 5 & 2 \\ 5 & -2 & -9 & -1 \end{pmatrix}.$$

The matrix A^t can be reduced to row echelon form as

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that

$$\mathbf{w}_1 = (1, 1, 1, 0), \qquad \mathbf{w}_2 = (0, 1, 2, 2), \qquad \mathbf{w}_3 = (0, 0, 0, 1)$$

form a basis for the row space of A^t, and so a basis for the column space of A.

Alternatively, we may pursue the following argument, which shows that elementary row operations do not affect the linear dependence relations among the columns of a matrix.

PROPOSITION 6B. *Suppose that the matrix B can be obtained from the matrix A by elementary row operations. Then any collection of columns of A are linearly independent if and only if the corresponding collection of columns of B are linearly independent.*

PROOF. Let A^* be a matrix made up of a collection of columns of A, and let B^* be the matrix made up of the corresponding collection of columns of B. Consider the two systems of homogeneous linear equations

$$A^*\mathbf{x} = \mathbf{0} \qquad \text{and} \qquad B^*\mathbf{x} = \mathbf{0}.$$

Since B^* can be obtained from the matrix A^* by elementary row operations, the two systems have the same solution set. On the other hand, the columns of A^* are linearly independent precisely when the system $A^*\mathbf{x} = \mathbf{0}$ has only the trivial solution, precisely when the system $B^*\mathbf{x} = \mathbf{0}$ has only the trivial solution, precsiely when the columns of B^* are linearly independent. ◯

To find a basis for the column space of A, we can now reduce A to row echelon form, and consider the pivot columns that result from this reduction. It is easily seen that these pivot columns are linearly independent, and that any non-pivot column is a linear combination of the pivot columns.

Example 6.3.2. Let

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix}.$$

Then A can be reduced to row echelon form as

$$\begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that the pivot columns of A are the first, second and fourth columns. Hence

$$\mathbf{u}_1 = (1,1,1,0), \qquad \mathbf{u}_2 = (3,4,5,3), \qquad \mathbf{u}_3 = (1,3,5,2)$$

form a basis for the column space of A.

6.4. Rank of a Matrix

For the matrix

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix},$$

we have shown that the row space has dimension 3, and so does the column space. In fact, we have the following important result.

PROPOSITION 6C. *For any matrix A with entries in $\mathbb{R}$, the dimension of the row space is the same as the dimension of the column space.*

Proof. For any matrix A, we can reduce A to row echelon form. Then the dimension of the row space of A is equal to the number of non-zero rows in the row echelon form. On the other hand, the dimension of the column space of A is equal to the number of pivot columns in the row echelon form. However, the number of non-zero rows in the row echelon form is the same as the number of pivot columns. ◯

Definition. The rank of a matrix A, denoted by $\text{rank}(A)$, is equal to the common value of the dimension of its row space and the dimension of its column space.

Example 6.4.1. The matrix

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix}$$

has rank 3.

6.5. Nullspaces

EXAMPLE 6.5.1. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix}.$$

We showed in Example 5.5.11 that the space of solutions of $A\mathbf{x} = \mathbf{0}$ has dimension 2. In other words, the nullspace of A has dimension 2. Note that in this particular case, the dimension of the nullspace of A and the dimension of the column space of A have a sum of 5, the number of columns of A.

Recall now that the nullspace of A is a subspace of $\mathbb{R}^n$, where n is the number of columns of the matrix A.

PROPOSITION 6D. *For any matrix A with entries in $\mathbb{R}$, the sum of the dimension of its column space and the dimension of its nullspace is equal to the number of columns of A.*

SKETCH OF PROOF. We consider the system of homogeneous linear equations $A\mathbf{x} = \mathbf{0}$, and reduce A to row echelon form. The number of leading variables is now equal to the dimension of the row space of A, and so equal to the dimension of the column space of A. On the other hand, the number of free variables is equal to the dimension of the space of solutions, which is the nullspace. Note now that the total number of variables is equal to the number of columns of A. $\bigcirc$

REMARK. Proposition 6D is sometimes known as the Rank-nullity theorem, where the nullity of a matrix is the dimension of its nullspace.

We conclude this section by stating the following result for square matrices.

PROPOSITION 6E. *Suppose that A is an $n \times n$ matrix with entries in $\mathbb{R}$. Then the following statements are equivalent:*
(a) A can be reduced to I_n by elementary row operations.
(b) A is invertible.
(c) $\det A \neq 0$.
(d) The system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(e) The system $A\mathbf{x} = \mathbf{b}$ is soluble for every $\mathbf{b} \in \mathbb{R}^n$.
(f) The rows of A are linearly independent.
(g) The columns of A are linearly independent.
(h) A has rank n.

6.6. Solution of Non-Homogeneous Systems

Consider now a non-homogeneous system of equations

$$A\mathbf{x} = \mathbf{b}, \tag{4}$$

where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \tag{5}$$

with entries in $\mathbb{R}$.

Our aim here is to determine whether a given system (4) has a solution without making any attempt to actually find any solution.

Note first of all that

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \ldots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

It follows that $A\mathbf{x}$ can be described by

$$A\mathbf{x} = x_1\mathbf{c}_1 + \ldots + x_n\mathbf{c}_n,$$

where $\mathbf{c}, \ldots, \mathbf{c}_n$ are defined by (3) and are the columns of A. In other words, $A\mathbf{x}$ is a linear combination of the columns of A. It follows that if the system (4) has a solution, then $\mathbf{b}$ must be a linear combination of the columns of A. This means that $\mathbf{b}$ must belong to the column space of A, so that the two matrices

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \qquad \text{and} \qquad (A|\mathbf{b}) = \begin{pmatrix} a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \ldots & a_{mn} & b_m \end{pmatrix} \tag{6}$$

must have the same (column) rank.

On the other hand, if the two matrices A and $(A|\mathbf{b})$ have the same rank, then $\mathbf{b}$ must be a linear combination of the columns of A, so that

$$\mathbf{b} = x_1\mathbf{c}_1 + \ldots + x_n\mathbf{c}_n$$

for some $x_1, \ldots, x_n \in \mathbb{R}$. This gives a solution of the system (4).

We have just proved the following result.

PROPOSITION 6F. *For any matrix A with entries in $\mathbb{R}$, the non-homogeneous system of equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if the matrices A and $(A|\mathbf{b})$ have the same rank. Here $(A|\mathbf{b})$ is defined by* (5) *and* (6).

EXAMPLE 6.6.1. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$$

We have already shown that $\text{rank}(A) = 3$. Now

$$(A|\mathbf{b}) = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 & 1 \\ 1 & 4 & -7 & 3 & -2 & 2 \\ 1 & 5 & -9 & 5 & -9 & 3 \\ 0 & 3 & -6 & 2 & -1 & 3 \end{pmatrix}$$

can be reduced to row echelon form as

$$\begin{pmatrix} 1 & 3 & -5 & 1 & 5 & 1 \\ 0 & 1 & -2 & 2 & -7 & 1 \\ 0 & 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so that $\text{rank}(A|\mathbf{b}) = 3$. It follows that the system has a solution.

EXAMPLE 6.6.2. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}.$$

We have already shown that $\text{rank}(A) = 3$. Now

$$(A|\mathbf{b}) = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 & 1 \\ 1 & 4 & -7 & 3 & -2 & 2 \\ 1 & 5 & -9 & 5 & -9 & 4 \\ 0 & 3 & -6 & 2 & -1 & 3 \end{pmatrix}$$

can be reduced to row echelon form as

$$\begin{pmatrix} 1 & 3 & -5 & 1 & 5 & 1 \\ 0 & 1 & -2 & 2 & -7 & 1 \\ 0 & 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so that $\text{rank}(A|\mathbf{b}) = 4$. It follows that the system has no solution.

REMARK. The matrix $(A|\mathbf{b})$ is sometimes known as the augmented matrix.

We conclude this chapter by describing the set of all solutions of a non-homogeneous system of equations.

PROPOSITION 6G. *Suppose that A is a matrix with entries in $\mathbb{R}$. Suppose further that $\mathbf{x}_0$ is a solution of the non-homogeneous system of equations $A\mathbf{x} = \mathbf{b}$, and that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis for the nullspace of A. Then every solution of the system $A\mathbf{x} = \mathbf{b}$ can be written in the form*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r, \quad \textit{where } c_1, \ldots, c_r \in \mathbb{R}. \tag{7}$$

On the other hand, every vector of the form (7) is a solution to the system $A\mathbf{x} = \mathbf{b}$.

PROOF. Let $\mathbf{x}$ be any solution of the system $A\mathbf{x} = \mathbf{b}$. Since $A\mathbf{x}_0 = \mathbf{b}$, it follows that $A(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}$. Hence there exist $c_1, \ldots, c_r \in \mathbb{R}$ such that

$$\mathbf{x} - \mathbf{x}_0 = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r,$$

giving (7). On the other hand, it follows from (7) that

$$A\mathbf{x} = A(\mathbf{x}_0 + c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r) = A\mathbf{x}_0 + c_1A\mathbf{v}_1 + \ldots + c_rA\mathbf{v}_r = \mathbf{b} + \mathbf{0} + \ldots + \mathbf{0} = \mathbf{b}.$$

Hence every vector of the form (7) is a solution to the system $A\mathbf{x} = \mathbf{b}$. ◯

EXAMPLE 6.6.3. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$$

We have already shown in Example 5.5.11 that $\mathbf{v}_1 = (1, -2, -1, 0, 0)$ and $\mathbf{v}_2 = (1, 3, 0, -5, -1)$ form a basis for the nullspace of A. On the other hand, $\mathbf{x}_0 = (-4, 0, 1, 5, 1)$ is a solution of the non-homogeneous system. It follows that the solutions of the non-homogeneous system are given by

$$\mathbf{x} = (-4, 0, 1, 5, 1) + c_1(1, -2, -1, 0, 0) + c_2(1, 3, 0, -5, -1) \quad \text{where } c_1, c_2 \in \mathbb{R}.$$

EXAMPLE 6.6.4. Consider the non-homogeneous system $x - 2y + z = 2$ in $\mathbb{R}^3$. Note that this system has only one equation. The corresponding homogeneous system is given by $x - 2y + z = 0$, and this represents a plane through the origin. It is easily seen that $(1, 1, 1)$ and $(2, 1, 0)$ form a basis for the solution space of $x - 2y + z = 0$. On the other hand, note that $(1, 0, 1)$ is a solution of $x - 2y + z = 2$. It follows that the solutions of $x - 2y + z = 2$ are of the form

$$(x, y, z) = (1, 0, 1) + c_1(1, 1, 1) + c_2(2, 1, 0), \quad \text{where } c_1, c_2 \in \mathbb{R}.$$

Try to draw a picture for this problem.

Problems for Chapter 6

1. For each of the following matrices, find a basis for the row space and a basis for the column space by first reducing the matrix to row echelon form:

a) $\begin{pmatrix} 5 & 9 & 3 \\ 3 & -5 & -6 \\ 1 & 5 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 4 & -1 & 5 \\ 1 & 2 & 3 & -1 & 3 \\ 1 & 2 & 0 & -4 & -3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & -8 \\ 4 & -3 & -7 \\ 1 & 12 & -3 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 5 \\ 3 & 4 & 11 & 2 \end{pmatrix}$

2. For each of the following matrices, determine whether the non-homogeneous system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution:

a) $A = \begin{pmatrix} 5 & 9 & 3 \\ 3 & -5 & -6 \\ 1 & 5 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 2 & 4 & -1 & 5 \\ 1 & 2 & 3 & -1 & 3 \\ 1 & 2 & 0 & -4 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & -8 \\ 4 & -3 & -7 \\ 1 & 12 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 5 \\ 3 & 4 & 11 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$