M.A. ECONOMICS SUMMER SEMESTER
COURSE 002. INTRODUCTORY MATHEMATICAL ECONOMICS
Final Examination

Maximum Marks: 70. Time Allowed: $2\frac{1}{2}$ hours.

Instructions. There are 3 parts to the question paper. Answer each part in a ***separate*** *booklet. Answer 2 of the 3 questions each in* **Parts I and II**. The single question in **Part III** is mandatory. Marks corresponding to each question are indicated at the end of the question.

PART I

(1) $P_n(F)$ is the vector space of all polynomials of degree $\leq n$ and with coefficients in F.
(a) Show that the map $T : P_3(F) \to P_4(F)$ defined by $T(p(x)) = (x+1)p(x)$ is a linear map.
(b) Describe $\ker T$ and $rangeT$ for the map$T$ above.
(c) (unrelated to (a) and (b)). Let $b_1, ..., b_n$ be positive real numbers. Check that the form $\langle z, w_i \rangle = b_1 z_1 \overline{w}_1 + \cdots + b_n z_n \overline{w}_n$ defines an inner product on F^n, where $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$.

$(5, 4, 4)$

(2)(a) Find a basis for the subspace $U = \{(x, y, z, w) \in R^4 | x = y + z, y = x + w, z + w = 0\} \subseteq R^4$. Justify your answer.
(b) What is the dimension of the subspace $U = \{(x_1, x_2, ..., x_n) \in F^n | x_1 + 2x_2 + \cdots + nx_n = 0\} \subseteq F^n$? Justify your answer by applying the rank-nullity theorem.
(c) In R^4, let $U = span(1, 1, 0, 0), (1, 1, 1, 2)$. Find $u \in U$ such that $||u - (1, 2, 3, 4)||$ is as small as possible.Use Gram-Schmidt method.

$(4.5, 4.5, 4)$

(3) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
(a) Find a change of basis matrix C[which is a matrix of e-vectors] such that $C^{-1}AC$ is diagonal. What is this diagonal matrix?
(b) Compute A^{100}.
(c) The matrix $\begin{pmatrix} -2 & 11 \\ 4 & 2 \end{pmatrix}$represents a linear transformation $T : R^2 \to R^2$ with respect to the basis $\{v_1, v_2\}$ where $v_1 = (3, 1)$ and $v_2 = (0, 2)$. Find the matrix of T with respect to the basis $\{w_1, w_2\}$ where $w_1 = (1, 1)$ and $w_2 = (-1, 1)$. $(6, 3, 4)$

PART II

(4) Consider the following social welfare maximization problem in the setting of an economy with 2 consumers (1 and 2) and 2 goods (X and Y). Consumer 1's utility function is $U_1(x_1, y_1) = 6 + 0.4\ln(x_1) + 0.6\ln(x_2)$. Consumer 2's utility function is $U_2(x_2, y_2) = 8 + \ln(x_2) + \ln(y_2)$. The economy is endowed with 15 units of good X and 20 units of good Y. Social welfare, as a function of the 2 utility functions $U_1(x_1, y_1)$ and $U_2(x_2, y_2)$ is a weighted sum with weight 2 on the lesser of $U_1(x_1, y_1)$ and $U_2(x_2, y_2)$, and weight 1 on the greater of the two. Find an allocation (x_1, y_1, x_2, y_2) that maximizes this social welfare function subject to the constraints $x_1 + x_2 \leq 15$, $y_1 + y_2 \leq 20$. *It is helpful to do this question, including dealing with the constraints, as simply as possible, to save time.*

(15)

(5) The manager of a firm wishes to maximize the firm's revenue, subject to profits being at least as great as m. Specifically, the problem is to choose an output level y and advertising a to maximize revenue $R(y, a)$ subject to the constraints

$a \geq 0$ (1),
$\Pi(y, a) \equiv R(y, a) - C(y) - a \geq m$ (2), and
$y \geq 0$ (3).

The revenue function $R(y, a)$ is C^1, and $\partial R/\partial a > 0$ everywhere. The cost $C(y)$ of producing y units of output is a C^1 function, and $C'(y) > 0$ everywhere.

Assume that (y^*, a^*) is an optimal solution, with $y^* > 0$, and that the constraint qualification holds.

(a) Associate multipliers $\lambda_1, \lambda_2, \lambda_3$ with the 3 constraints above respectively, set up the Lagrangean and write down the Kuhn-Tucker first order conditions.

(b) Using the Kuhn-Tucker conditions, show that $\Pi(y^*, a^*) = m$, that is, constraint (2) binds at the optimum.

(c) Using the Kuhn-Tucker conditions, show that at the optimum, $\partial R/\partial y > 0$ at the optimum.

(d) Again using the Kuhn-Tucker conditions, show that $\partial \Pi/\partial y < 0$ at the optimum.

(5, 3, 3, 4)

(6) (a) Does the system of equations

$xz^3 + y^2v^4 = 2$ (1)
$xz + yvz^2 = 2$ (2)

define v and z as C^1 functions of x and y around the solution point $(v, z, x, y) = (1, 1, 1, 1)$? If it does, evaluate $\partial v/\partial x$ and $\partial v/\partial y$ at this point.

(b) Let $S_1 = \{(x, y) \in R^2 | xy = 1\}$, $S_2 = \{(x, y) \in R^2 | xy = -1\}$. Let $S_1 + S_2 = \{(z_1, z_2) | z_1 = x_1 + x_2, z_2 = y_1 + y_2,$ for some $(x_1, y_1) \in S_1, (x_2, y_2) \in S_2\}$. Show that $S_1 + S_2$ is not closed.

$(8, 7)$

PART III

(7) A firm must deliver an amount Y of a certain good at time T. The good can be produced continuously over the time interval $[0, T]$, and the produced quantities are accumulated as inventory. Let $q(t)$ denote the flow production and $y(t)$ the stock of inventory at time t. Obviously, $\dot{y}(t) = q(t)$. Both production and holding of inventory are costly, and the flow cost at date t is given by $a [q(t)]^2 + by(t)$, where $a, b > 0$ are constants. The firm wants to minimize its total cost subject to meeting its delivery target, i.e., the inventory at date T must be equal to Y. The firm starts with no inventory at date 0. There is no discounting.

(a) Formulate the above problem as an optimal control problem. Derive the first-order necessary conditions for an optimal path using Pontryagin's maximum principle.

(b) Solve for the optimal production path $q(t)$ and the optimal inventory accumulation path $y(t)$.

(c) Suppose the value of b increases from b_1 to b_2. Let the corresponding optimal production paths be $q_1(t)$ and $q_2(t)$. Show that there is some cutoff date $\widehat{t}$ such that $q_1(t) \geq q_2(t)$ for $t \leq \widehat{t}$ and $q_1(t) < q_2(t)$ for $t > \widehat{t}$.

$(5, 6, 3)$