LINEAR ALGEBRA

Chapter 2

MATRICES

2.1. Introduction

A rectangular array of numbers of the form

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \tag{1}$$

is called an $m \times n$ matrix, with m rows and n columns. We count rows from the top and columns from the left. Hence

$$\begin{pmatrix} a_{i1} & \dots & a_{in} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

represent respectively the i-th row and the j-th column of the matrix (1), and a_{ij} represents the entry in the matrix (1) on the i-th row and j-th column.

EXAMPLE 2.1.1. Consider the 3×4 matrix

$$\begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}.$$

Here

$$\begin{pmatrix} 3 & 1 & 5 & 2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$$

represent respectively the 2-nd row and the 3-rd column of the matrix, and 5 represents the entry in the matrix on the 2-nd row and 3-rd column.

We now consider the question of arithmetic involving matrices. First of all, let us study the problem of addition. A reasonable theory can be derived from the following definition.

DEFINITION. Suppose that the two matrices

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$$

both have m rows and n columns. Then we write

$$A + B = \begin{pmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

and call this the sum of the two matrices A and B.

EXAMPLE 2.1.2. Suppose that

$$A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 2 & -2 & 7 \\ 0 & 2 & 4 & -1 \\ -2 & 1 & 3 & 3 \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} 2+1 & 4+2 & 3-2 & -1+7 \\ 3+0 & 1+2 & 5+4 & 2-1 \\ -1-2 & 0+1 & 7+3 & 6+3 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 1 & 6 \\ 3 & 3 & 9 & 1 \\ -3 & 1 & 10 & 9 \end{pmatrix}.$$

EXAMPLE 2.1.3. We do not have a definition for "adding" the matrices

$$\begin{pmatrix} 2 & 4 & 3 & -1 \\ -1 & 0 & 7 & 6 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 & 4 & 3 \\ 3 & 1 & 5 \\ -1 & 0 & 7 \end{pmatrix}.$$

PROPOSITION 2A. (MATRIX ADDITION) *Suppose that A, B, C are $m \times n$ matrices. Suppose further that O represents the $m \times n$ matrix with all entries zero. Then*
(a) $A + B = B + A$;
(b) $A + (B + C) = (A + B) + C$;
(c) $A + O = A$; and
(d) there is an $m \times n$ matrix A' such that $A + A' = O$.

PROOF. Parts (a)–(c) are easy consequences of ordinary addition, as matrix addition is simply entry-wise addition. For part (d), we can consider the matrix A' obtained from A by multiplying each entry of A by -1. ◯

The theory of multiplication is rather more complicated, and includes multiplication of a matrix by a scalar as well as multiplication of two matrices.

We first study the simpler case of multiplication by scalars.

Definition. Suppose that the matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$$

has m rows and n columns, and that $c \in \mathbb{R}$. Then we write

$$cA = \begin{pmatrix} ca_{11} & \ldots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \ldots & ca_{mn} \end{pmatrix}$$

and call this the product of the matrix A by the scalar c.

Example 2.1.4. Suppose that

$$A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix}.$$

Then

$$2A = \begin{pmatrix} 4 & 8 & 6 & -2 \\ 6 & 2 & 10 & 4 \\ -2 & 0 & 14 & 12 \end{pmatrix}.$$

PROPOSITION 2B. (MULTIPLICATION BY SCALAR) *Suppose that A, B are $m \times n$ matrices, and that $c, d \in \mathbb{R}$. Suppose further that O represents the $m \times n$ matrix with all entries zero. Then*
(a) $c(A + B) = cA + cB$;
(b) $(c + d)A = cA + dA$;
(c) $0A = O$; *and*
(d) $c(dA) = (cd)A$.

Proof. These are all easy consequences of ordinary multiplication, as multiplication by scalar c is simply entry-wise multiplication by the number c. ◯

The question of multiplication of two matrices is rather more complicated. To motivate this, let us consider the representation of a system of linear equations

$$\begin{aligned} a_{11}x_1 + \ldots + \ a_{1n}x_n &= \ b_1, \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m, \end{aligned} \tag{2}$$

in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \tag{3}$$

represent the coefficients and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{4}$$

represents the variables. This can be written in full matrix notation by

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

Can you work out the meaning of this representation?

Now let us define matrix multiplication more formally.

DEFINITION. Suppose that

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & \dots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{np} \end{pmatrix}$$

are respectively an $m \times n$ matrix and an $n \times p$ matrix. Then the matrix product AB is given by the $m \times p$ matrix

$$AB = \begin{pmatrix} q_{11} & \dots & q_{1p} \\ \vdots & & \vdots \\ q_{m1} & \dots & q_{mp} \end{pmatrix},$$

where for every $i = 1, \dots, m$ and $j = 1, \dots, p$, we have

$$q_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + \dots + a_{in} b_{nj}.$$

REMARK. Note first of all that the number of columns of the first matrix must be equal to the number of rows of the second matrix. On the other hand, for a simple way to work out q_{ij}, the entry in the i-th row and j-th column of AB, we observe that the i-th row of A and the j-th column of B are respectively

$$\begin{pmatrix} a_{i1} & \dots & a_{in} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}.$$

We now multiply the corresponding entries – from a_{i1} with b_{1j}, and so on, until a_{in} with b_{nj} – and then add these products to obtain q_{ij}.

EXAMPLE 2.1.5. Consider the matrices

$$A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix}.$$

Note that A is a 3×4 matrix and B is a 4×2 matrix, so that the product AB is a 3×2 matrix. Let us calculate the product

$$AB = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{pmatrix}.$$

Consider first of all q_{11}. To calculate this, we need the 1-st row of A and the 1-st column of B, so let us cover up all unnecessary information, so that

$$\begin{pmatrix} 2 & 4 & 3 & -1 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} q_{11} & \times \\ \times & \times \\ \times & \times \end{pmatrix}.$$

From the definition, we have

$$q_{11} = 2 \cdot 1 + 4 \cdot 2 + 3 \cdot 0 + (-1) \cdot 3 = 2 + 8 + 0 - 3 = 7.$$

Consider next q_{12}. To calculate this, we need the 1-st row of A and the 2-nd column of B, so let us cover up all unnecessary information, so that

$$\begin{pmatrix} 2 & 4 & 3 & -1 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \begin{pmatrix} \times & 4 \\ \times & 3 \\ \times & -2 \\ \times & 1 \end{pmatrix} = \begin{pmatrix} \times & q_{12} \\ \times & \times \\ \times & \times \end{pmatrix}.$$

From the definition, we have

$$q_{12} = 2 \cdot 4 + 4 \cdot 3 + 3 \cdot (-2) + (-1) \cdot 1 = 8 + 12 - 6 - 1 = 13.$$

Consider next q_{21}. To calculate this, we need the 2-nd row of A and the 1-st column of B, so let us cover up all unnecessary information, so that

$$\begin{pmatrix} \times & \times & \times & \times \\ 3 & 1 & 5 & 2 \\ \times & \times & \times & \times \end{pmatrix} \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} \times & \times \\ q_{21} & \times \\ \times & \times \end{pmatrix}.$$

From the definition, we have

$$q_{21} = 3 \cdot 1 + 1 \cdot 2 + 5 \cdot 0 + 2 \cdot 3 = 3 + 2 + 0 + 6 = 11.$$

Consider next q_{22}. To calculate this, we need the 2-nd row of A and the 2-nd column of B, so let us cover up all unnecessary information, so that

$$\begin{pmatrix} \times & \times & \times & \times \\ 3 & 1 & 5 & 2 \\ \times & \times & \times & \times \end{pmatrix} \begin{pmatrix} \times & 4 \\ \times & 3 \\ \times & -2 \\ \times & 1 \end{pmatrix} = \begin{pmatrix} \times & \times \\ \times & q_{22} \\ \times & \times \end{pmatrix}.$$

From the definition, we have

$$q_{22} = 3 \cdot 4 + 1 \cdot 3 + 5 \cdot (-2) + 2 \cdot 1 = 12 + 3 - 10 + 2 = 7.$$

Consider next q_{31}. To calculate this, we need the 3-rd row of A and the 1-st column of B, so let us cover up all unnecessary information, so that

$$\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ -1 & 0 & 7 & 6 \end{pmatrix} \begin{pmatrix} 1 & \times \\ 2 & \times \\ 0 & \times \\ 3 & \times \end{pmatrix} = \begin{pmatrix} \times & \times \\ \times & \times \\ q_{31} & \times \end{pmatrix}.$$

From the definition, we have

$$q_{31} = (-1) \cdot 1 + 0 \cdot 2 + 7 \cdot 0 + 6 \cdot 3 = -1 + 0 + 0 + 18 = 17.$$

Consider finally q_{32}. To calculate this, we need the 3-rd row of A and the 2-nd column of B, so let us cover up all unnecessary information, so that

$$\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ -1 & 0 & 7 & 6 \end{pmatrix} \begin{pmatrix} \times & 4 \\ \times & 3 \\ \times & -2 \\ \times & 1 \end{pmatrix} = \begin{pmatrix} \times & \times \\ \times & \times \\ \times & q_{32} \end{pmatrix}.$$

From the definition, we have

$$q_{32} = (-1) \cdot 4 + 0 \cdot 3 + 7 \cdot (-2) + 6 \cdot 1 = -4 + 0 + -14 + 6 = -12.$$

We therefore conclude that

$$AB = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 13 \\ 11 & 7 \\ 17 & -12 \end{pmatrix}.$$

EXAMPLE 2.1.6. Consider again the matrices

$$A = \begin{pmatrix} 2 & 4 & 3 & -1 \\ 3 & 1 & 5 & 2 \\ -1 & 0 & 7 & 6 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & -2 \\ 3 & 1 \end{pmatrix}.$$

Note that B is a 4×2 matrix and A is a 3×4 matrix, so that we do not have a definition for the "product" BA.

We leave the proofs of the following results as exercises for the interested reader.

PROPOSITION 2C. (ASSOCIATIVE LAW) *Suppose that A is an $m \times n$ matrix, B is an $n \times p$ matrix and C is an $p \times r$ matrix. Then $A(BC) = (AB)C$.*

PROPOSITION 2D. (DISTRIBUTIVE LAWS)
(a) Suppose that A is an $m \times n$ matrix and B and C are $n \times p$ matrices. Then $A(B + C) = AB + AC$.
(b) Suppose that A and B are $m \times n$ matrices and C is an $n \times p$ matrix. Then $(A + B)C = AC + BC$.

PROPOSITION 2E. *Suppose that A is an $m \times n$ matrix, B is an $n \times p$ matrix, and that $c \in \mathbb{R}$. Then $c(AB) = (cA)B = A(cB)$.*

2.2. Systems of Linear Equations

Note that the system (2) of linear equations can be written in matrix form as

$$A\mathbf{x} = \mathbf{b},$$

where the matrices A, $\mathbf{x}$ and $\mathbf{b}$ are given by (3) and (4). In this section, we shall establish the following important result.

PROPOSITION 2F. *Every system of linear equations of the form (2) has either no solution, one solution or infinitely many solutions.*

PROOF. Clearly the system (2) has either no solution, exactly one solution, or more than one solution. It remains to show that if the system (2) has two distinct solutions, then it must have infinitely many solutions. Suppose that $\mathbf{x} = \mathbf{u}$ and $\mathbf{x} = \mathbf{v}$ represent two distinct solutions. Then

$$A\mathbf{u} = \mathbf{b} \qquad \text{and} \qquad A\mathbf{v} = \mathbf{b},$$

so that

$$A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

where $\mathbf{0}$ is the zero $m \times 1$ matrix. It now follows that for every $c \in \mathbb{R}$, we have

$$A(\mathbf{u} + c(\mathbf{u} - \mathbf{v})) = A\mathbf{u} + A(c(\mathbf{u} - \mathbf{v})) = A\mathbf{u} + c(A(\mathbf{u} - \mathbf{v})) = \mathbf{b} + c\mathbf{0} = \mathbf{b},$$

so that $\mathbf{x} = \mathbf{u} + c(\mathbf{u} - \mathbf{v})$ is a solution for every $c \in \mathbb{R}$. Clearly we have infinitely many solutions. ○

2.3. Inversion of Matrices

For the remainder of this chapter, we shall deal with square matrices, those where the number of rows equals the number of columns.

DEFINITION. The $n \times n$ matrix

$$I_n = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix},$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

is called the identity matrix of order n.

REMARK. Note that

$$I_1 = (\,1\,) \qquad \text{and} \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following result is relatively easy to check. It shows that the identity matrix I_n acts as the identity for multiplication of $n \times n$ matrices.

PROPOSITION 2G. *For every $n \times n$ matrix A, we have $AI_n = I_nA = A$.*

This raises the following question: Given an $n \times n$ matrix A, is it possible to find another $n \times n$ matrix B such that $AB = BA = I_n$?

We shall postpone the full answer to this question until the next chapter. In Section 2.5, however, we shall be content with finding such a matrix B if it exists. In Section 2.6, we shall relate the existence of such a matrix B to some properties of the matrix A.

Definition. An $n \times n$ matrix A is said to be invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, we say that B is the inverse of A and write $B = A^{-1}$.

PROPOSITION 2H. *Suppose that A is an invertible $n \times n$ matrix. Then its inverse A^{-1} is unique.*

Proof. Suppose that B satisfies the requirements for being the inverse of A. Then $AB = BA = I_n$. It follows that

$$A^{-1} = A^{-1}I_n = A^{-1}(AB) = (A^{-1}A)B = I_nB = B.$$

Hence the inverse A^{-1} is unique. ◯

PROPOSITION 2J. *Suppose that A and B are invertible $n \times n$ matrices. Then $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof. In view of the uniqueness of inverse, it is sufficient to show that $B^{-1}A^{-1}$ satisfies the requirements for being the inverse of AB. Note that

$$(AB)(B^{-1}A^{-1}) = A(B(B^{-1}A^{-1})) = A((BB^{-1})A^{-1}) = A(I_nA^{-1}) = AA^{-1} = I_n$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}(AB)) = B^{-1}((A^{-1}A)B) = B^{-1}(I_nB) = B^{-1}B = I_n$$

as required. ◯

PROPOSITION 2K. *Suppose that A is an invertible $n \times n$ matrix. Then $(A^{-1})^{-1} = A$.*

Proof. Note that both $(A^{-1})^{-1}$ and A satisfy the requirements for being the inverse of A^{-1}. Equality follows from the uniqueness of inverse. ◯

2.4. Application to Matrix Multiplication

In this section, we shall discuss an application of invertible matrices. Detailed discussion of the technique involved will be covered in Chapter 7.

Definition. An $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix},$$

where $a_{ij} = 0$ whenever $i \neq j$, is called a diagonal matrix of order n.

Example 2.4.1. The 3×3 matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are both diagonal.

Given an $n \times n$ matrix A, it is usually rather complicated to calculate

$$A^k = \underbrace{A \ldots A}_{k}.$$

However, the calculation is rather simple when A is a diagonal matrix, as we shall see in the following example.

EXAMPLE 2.4.2. Consider the 3×3 matrix

$$A = \begin{pmatrix} 17 & -10 & -5 \\ 45 & -28 & -15 \\ -30 & 20 & 12 \end{pmatrix}.$$

Suppose that we wish to calculate A^{98}. It can be checked that if we take

$$P = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix},$$

then

$$P^{-1} = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 4/3 & 1 \\ 3 & -5/3 & -1 \end{pmatrix}.$$

Furthermore, if we write

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

then it can be checked that $A = PDP^{-1}$, so that

$$A^{98} = \underbrace{(PDP^{-1})\ldots(PDP^{-1})}_{98} = PD^{98}P^{-1} = P \begin{pmatrix} 3^{98} & 0 & 0 \\ 0 & 2^{98} & 0 \\ 0 & 0 & 2^{98} \end{pmatrix} P^{-1}.$$

This is much simpler than calculating A^{98} directly. Note that this example is only an illustration. We have not discussed here how the matrices P and D are found.

2.5. Finding Inverses by Elementary Row Operations

In this section, we shall discuss a technique by which we can find the inverse of a square matrix, if the inverse exists. Before we discuss this technique, let us recall the three elementary row operations we discussed in the previous chapter. These are: (1) interchanging two rows; (2) adding a multiple of one row to another row; and (3) multiplying one row by a non-zero constant.

Let us now consider the following example.

EXAMPLE 2.5.1. Consider the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \text{and} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Let us interchange rows 1 and 2 of A and do likewise for I_3. We obtain respectively

$$\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- Let us interchange rows 2 and 3 of A and do likewise for I_3. We obtain respectively

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- Let us add 3 times row 1 to row 2 of A and do likewise for I_3. We obtain respectively

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11} + a_{21} & 3a_{12} + a_{22} & 3a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11} + a_{21} & 3a_{12} + a_{22} & 3a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- Let us add -2 times row 3 to row 1 of A and do likewise for I_3. We obtain respectively

$$\begin{pmatrix} -2a_{31} + a_{11} & -2a_{32} + a_{12} & -2a_{33} + a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} -2a_{31} + a_{11} & -2a_{32} + a_{12} & -2a_{33} + a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- Let us multiply row 2 of A by 5 and do likewise for I_3. We obtain respectively

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- Let us multiply row 3 of A by -1 and do likewise for I_3. We obtain respectively

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let us now consider the problem in general.

DEFINITION. By an elementary $n \times n$ matrix, we mean an $n \times n$ matrix obtained from I_n by an elementary row operation.

We state without proof the following important result. The interested reader may wish to construct a proof, taking into account the different types of elementary row operations.

PROPOSITION 2L. *Suppose that A is an $n \times n$ matrix, and suppose that B is obtained from A by an elementary row operation. Suppose further that E is an elementary matrix obtained from I_n by the same elementary row operation. Then $B = EA$.*

We now adopt the following strategy. Consider an $n \times n$ matrix A. Suppose that it is possible to reduce the matrix A by a sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ of elementary row operations to the identity matrix I_n. If $E_1, E_2, \ldots, E_k$ are respectively the elementary $n \times n$ matrices obtained from I_n by the same elementary row operations $\alpha_1, \alpha_2 \ldots, \alpha_k$, then

$$I_n = E_k \ldots E_2 E_1 A.$$

We therefore must have

$$A^{-1} = E_k \ldots E_2 E_1 = E_k \ldots E_2 E_1 I_n.$$

It follows that the inverse A^{-1} can be obtained from I_n by performing the same elementary row operations $\alpha_1, \alpha_2, \ldots, \alpha_k$. Since we are performing the same elementary row operations on A and I_n, it makes sense to put them side by side. The process can then be described pictorially by

$$\begin{aligned} (A|I_n) &\xrightarrow{\alpha_1} (E_1 A|E_1 I_n) \\ &\xrightarrow{\alpha_2} (E_2 E_1 A|E_2 E_1 I_n) \\ &\xrightarrow{\alpha_3} \ldots \\ &\xrightarrow{\alpha_k} (E_k \ldots E_2 E_1 A|E_k \ldots E_2 E_1 I_n) = (I_n|A^{-1}). \end{aligned}$$

In other words, we consider an array with the matrix A on the left and the matrix I_n on the right. We now perform elementary row operations on the array and try to reduce the left hand half to the matrix I_n. If we succeed in doing so, then the right hand half of the array gives the inverse A^{-1}.

EXAMPLE 2.5.2. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}.$$

To find A^{-1}, we consider the array

$$(A|I_3) = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now perform elementary row operations on this array and try to reduce the left hand half to the matrix I_3. Note that if we succeed, then the final array is clearly in reduced row echelon form. We therefore follow the same procedure as reducing an array to reduced row echelon form. Adding -3 times row 1 to row 2, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Adding 2 times row 1 to row 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 5 & 4 & 2 & 0 & 1 \end{pmatrix}.$$

Multiplying row 3 by 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 15 & 12 & 6 & 0 & 3 \end{pmatrix}.$$

Adding 5 times row 2 to row 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Multiplying row 1 by 3, we obtain

$$\begin{pmatrix} 3 & 3 & 6 & 3 & 0 & 0 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding 2 times row 3 to row 1, we obtain

$$\begin{pmatrix} 3 & 3 & 0 & -15 & 10 & 6 \\ 0 & -3 & -3 & -3 & 1 & 0 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding -1 times row 3 to row 2, we obtain

$$\begin{pmatrix} 3 & 3 & 0 & -15 & 10 & 6 \\ 0 & -3 & 0 & 6 & -4 & -3 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Adding 1 times row 2 to row 1, we obtain

$$\begin{pmatrix} 3 & 0 & 0 & -9 & 6 & 3 \\ 0 & -3 & 0 & 6 & -4 & -3 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Multiplying row 1 by 1/3, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & -3 & 0 & 6 & -4 & -3 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Multiplying row 2 by $-1/3$, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & -2 & 4/3 & 1 \\ 0 & 0 & -3 & -9 & 5 & 3 \end{pmatrix}.$$

Multiplying row 3 by $-1/3$, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & -2 & 4/3 & 1 \\ 0 & 0 & 1 & 3 & -5/3 & -1 \end{pmatrix}.$$

Note now that the array is in reduced row echelon form, and that the left hand half is the identity matrix I_3. It follows that the right hand half of the array represents the inverse A^{-1}. Hence

$$A^{-1} = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 4/3 & 1 \\ 3 & -5/3 & -1 \end{pmatrix}.$$

EXAMPLE 2.5.3. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 5 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To find A^{-1}, we consider the array

$$(A|I_4) = \begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 2 & 4 & 5 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now perform elementary row operations on this array and try to reduce the left hand half to the matrix I_4. Adding -2 times row 1 to row 2, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Adding 1 times row 2 to row 4, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \end{pmatrix}.$$

Interchanging rows 2 and 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \end{pmatrix}.$$

At this point, we observe that it is impossible to reduce the left hand half of the array to I_4. For those who remain unconvinced, let us continue. Adding 3 times row 3 to row 1, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 0 & -5 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \end{pmatrix}.$$

Adding -1 times row 4 to row 3, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 0 & -5 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \end{pmatrix}.$$

Multiplying row 1 by 6 (here we want to avoid fractions in the next two steps), we obtain

$$\begin{pmatrix} 6 & 6 & 12 & 0 & -30 & 18 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \end{pmatrix}.$$

Adding -15 times row 4 to row 1, we obtain

$$\begin{pmatrix} 6 & 6 & 12 & 0 & 0 & 3 & 0 & -15 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \end{pmatrix}.$$

Adding -2 times row 2 to row 1, we obtain

$$\begin{pmatrix} 6 & 0 & 12 & 0 & 0 & 3 & -2 & -15 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \end{pmatrix}.$$

Multiplying row 1 by $1/6$, multiplying row 2 by $1/3$, multiplying row 3 by -1 and multiplying row 4 by $-1/2$, we obtain

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 1/2 & -1/3 & -5/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1/2 & 0 & -1/2 \end{pmatrix}.$$

Note now that the array is in reduced row echelon form, and that the left hand half is not the identity matrix I_4. Our technique has failed. In fact, the matrix A is not invertible.

2.6. Criteria for Invertibility

Examples 2.5.2–2.5.3 raise the question of when a given matrix is invertible. In this section, we shall obtain some partial answers to this question. Our first step here is the following simple observation.

PROPOSITION 2M. *Every elementary matrix is invertible.*

PROOF. Let us consider elementary row operations. Recall that these are: (1) interchanging two rows; (2) adding a multiple of one row to another row; and (3) multiplying one row by a non-zero constant.

These elementary row operations can clearly be reversed by elementary row operations. For (1), we interchange the two rows again. For (2), if we have originally added c times row i to row j, then we can reverse this by adding $-c$ times row i to row j. For (3), if we have multiplied any row by a non-zero constant c, we can reverse this by multiplying the same row by the constant $1/c$. Note now that each elementary matrix is obtained from I_n by an elementary row operation. The inverse of this elementary matrix is clearly the elementary matrix obtained from I_n by the elementary row operation that reverses the original elementary row operation. ○

Suppose that an $n \times n$ matrix B can be obtained from an $n \times n$ matrix A by a finite sequence of elementary row operations. Then since these elementary row operations can be reversed, the matrix A can be obtained from the matrix B by a finite sequence of elementary row operations.

DEFINITION. An $n \times n$ matrix A is said to be row equivalent to an $n \times n$ matrix B if there exist a finite number of elementary $n \times n$ matrices $E_1, \ldots, E_k$ such that $B = E_k \ldots E_1 A$.

REMARK. Note that $B = E_k \ldots E_1 A$ implies that $A = E_1^{-1} \ldots E_k^{-1} B$. It follows that if A is row equivalent to B, then B is row equivalent to A. We usually say that A and B are row equivalent.

The following result gives conditions equivalent to the invertibility of an $n \times n$ matrix A.

PROPOSITION 2N. *Suppose that*

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix},$$

and that

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \textit{and} \qquad \mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

are $n \times 1$ matrices, where $x_1, \ldots, x_n$ are variables.

(a) Suppose that the matrix A is invertible. Then the system $A\mathbf{x} = \mathbf{0}$ of linear equations has only the trivial solution.

(b) Suppose that the system $A\mathbf{x} = \mathbf{0}$ of linear equations has only the trivial solution. Then the matrices A and I_n are row equivalent.

(c) Suppose that the matrices A and I_n are row equivalent. Then A is invertible.

PROOF. (a) Suppose that $\mathbf{x}_0$ is a solution of the system $A\mathbf{x} = \mathbf{0}$. Then since A is invertible, we have

$$\mathbf{x}_0 = I_n\mathbf{x}_0 = (A^{-1}A)\mathbf{x}_0 = A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{0} = \mathbf{0}.$$

It follows that the trivial solution is the only solution.

(b) Note that if the system $A\mathbf{x} = \mathbf{0}$ of linear equations has only the trivial solution, then it can be reduced by elementary row operations to the system

$$x_1 = 0, \qquad \ldots, \qquad x_n = 0.$$

This is equivalent to saying that the array

$$\left(\begin{array}{ccc|c} a_{11} & \ldots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} & 0 \end{array}\right)$$

can be reduced by elementary row operations to the reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \end{array}\right).$$

Hence the matrices A and I_n are row equivalent.

(c) Suppose that the matrices A and I_n are row equivalent. Then there exist elementary $n\times n$ matrices $E_1,\ldots,E_k$ such that $I_n = E_k \ldots E_1 A$. By Proposition 2M, the matrices $E_1,\ldots,E_k$ are all invertible, so that

$$A = E_1^{-1} \ldots E_k^{-1} I_n = E_1^{-1} \ldots E_k^{-1}$$

is a product of invertible matrices, and is therefore itself invertible. ◯

2.7. Consequences of Invertibility

Suppose that the matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$

is invertible. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

are $n\times 1$ matrices, where $x_1,\ldots,x_n$ are variables and $b_1,\ldots,b_n \in \mathbb{R}$ are arbitrary. Since A is invertible, let us consider $\mathbf{x} = A^{-1}\mathbf{b}$. Clearly

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b},$$

so that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of the system. On the other hand, let $\mathbf{x}_0$ be any solution of the system. Then $A\mathbf{x}_0 = \mathbf{b}$, so that

$$\mathbf{x}_0 = I_n\mathbf{x}_0 = (A^{-1}A)\mathbf{x}_0 = A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{b}.$$

It follows that the system has unique solution. We have proved the following important result.

PROPOSITION 2P. *Suppose that*

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix},$$

and that

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

are $n\times 1$ matrices, where $x_1,\ldots,x_n$ are variables and $b_1,\ldots,b_n \in \mathbb{R}$ are arbitrary. Suppose further that the matrix A is invertible. Then the system $A\mathbf{x} = \mathbf{b}$ of linear equations has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

We next attempt to study the question in the opposite direction.

PROPOSITION 2Q. *Suppose that*

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix},$$

and that

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

are $n \times 1$ matrices, where $x_1, \ldots, x_n$ are variables. Suppose further that for every $b_1, \ldots, b_n \in \mathbb{R}$, the system $A\mathbf{x} = \mathbf{b}$ of linear equations is soluble. Then the matrix A is invertible.

PROOF. Suppose that

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \qquad \ldots, \qquad \mathbf{b}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

In other words, for every $j = 1, \ldots, n$, $\mathbf{b}_j$ is an $n \times 1$ matrix with entry 1 on row j and entry 0 elsewhere. Now let

$$\mathbf{x}_1 = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}, \qquad \ldots, \qquad \mathbf{x}_n = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

denote respectively solutions of the systems of linear equations

$$A\mathbf{x} = \mathbf{b}_1, \qquad \ldots, \qquad A\mathbf{x} = \mathbf{b}_n.$$

It is easy to check that

$$A\,(\,\mathbf{x}_1 \quad \ldots \quad \mathbf{x}_n\,) = (\,\mathbf{b}_1 \quad \ldots \quad \mathbf{b}_n\,);$$

in other words,

$$A \begin{pmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{pmatrix} = I_n,$$

so that A is invertible. ○

We can now summarize Propositions 2N, 2P and 2Q as follows.

PROPOSITION 2R. *In the notation of Proposition 2N, the following four statements are equivalent:*
(a) The matrix A is invertible.
(b) The system $A\mathbf{x} = \mathbf{0}$ of linear equations has only the trivial solution.
(c) The matrices A and I_n are row equivalent.
(d) The system $A\mathbf{x} = \mathbf{b}$ of linear equations is soluble for every $n \times 1$ matrix $\mathbf{b}$.

2.8. Application to Economics

In this section, we describe briefly the Leontief input-output model, where an economy is divided into n sectors.

For every $i = 1, \ldots, n$, let x_i denote the monetary value of the total output of sector i over a fixed period, and let d_i denote the output of sector i needed to satisfy outside demand over the same fixed period. Collecting together x_i and d_i for $i = 1, \ldots, n$, we obtain the vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{and} \qquad \mathbf{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \in \mathbb{R}^n,$$

known respectively as the production vector and demand vector of the economy.

On the other hand, each of the n sectors requires material from some or all of the sectors to produce its output. For $i, j = 1, \ldots, n$, let c_{ij} denote the monetary value of the output of sector i needed by sector j to produce one unit of monetary value of output. For every $j = 1, \ldots, n$, the vector

$$\mathbf{c}_j = \begin{pmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{pmatrix} \in \mathbb{R}^n$$

is known as the unit consumption vector of sector j. Note that the column sum

$$c_{1j} + \ldots + c_{nj} \le 1 \tag{5}$$

in order to ensure that sector j does not make a loss. Collecting together the unit consumption vectors, we obtain the matrix

$$C = (\, \mathbf{c}_1 \quad \ldots \quad \mathbf{c}_n \,) = \begin{pmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \ldots & c_{nn} \end{pmatrix},$$

known as the consumption matrix of the economy.

Consider the matrix product

$$C\mathbf{x} = \begin{pmatrix} c_{11}x_1 + \ldots + c_{1n}x_n \\ \vdots \\ c_{n1}x_1 + \ldots + c_{nn}x_n \end{pmatrix}.$$

For every $i = 1, \ldots, n$, the entry $c_{i1}x_1 + \ldots + c_{in}x_n$ represents the monetary value of the output of sector i needed by all the sectors to produce their output. This leads to the production equation

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}. \tag{6}$$

Here $C\mathbf{x}$ represents the part of the total output that is required by the various sectors of the economy to produce the output in the first place, and $\mathbf{d}$ represents the part of the total output that is available to satisfy outside demand.

Clearly $(I - C)\mathbf{x} = \mathbf{d}$. If the matrix $I - C$ is invertible, then

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$

represents the perfect production level. We state without proof the following fundamental result.

PROPOSITION 2S. *Suppose that the entries of the consumption matrix C and the demand vector $\mathbf{d}$ are non-negative. Suppose further that the inequality (5) holds for each column of C. Then the inverse matrix $(I-C)^{-1}$ exists, and the production vector $\mathbf{x}=(I-C)^{-1}\mathbf{d}$ has non-negative entries and is the unique solution of the production equation (6).*

Let us indulge in some heuristics. Initially, we have demand $\mathbf{d}$. To produce $\mathbf{d}$, we need $C\mathbf{d}$ as input. To produce this extra $C\mathbf{d}$, we need $C(C\mathbf{d}) = C^2\mathbf{d}$ as input. To produce this extra $C^2\mathbf{d}$, we need $C(C^2\mathbf{d}) = C^3\mathbf{d}$ as input. And so on. Hence we need to produce

$$\mathbf{d}+C\mathbf{d}+C^2\mathbf{d}+C^3\mathbf{d}+\ldots=(I+C+C^2+C^3+\ldots)\mathbf{d}$$

in total. Now it is not difficult to check that for every positive integer k, we have

$$(I-C)(I+C+C^2+C^3+\ldots+C^k)=I-C^{k+1}.$$

If the entries of C^{k+1} are all very small, then

$$(I-C)(I+C+C^2+C^3+\ldots+C^k)\approx I,$$

so that

$$(I-C)^{-1}\approx I+C+C^2+C^3+\ldots+C^k.$$

This gives a practical way of approximating $(I-C)^{-1}$, and also suggests that

$$(I-C)^{-1}=I+C+C^2+C^3+\ldots.$$

EXAMPLE 2.8.1. An economy consists of three sectors. Their dependence on each other is summarized in the table below:

	To produce one unit of monetary value of output in sector 1	2	3
monetary value of output required from sector 1	0.3	0.2	0.1
monetary value of output required from sector 2	0.4	0.5	0.2
monetary value of output required from sector 3	0.1	0.1	0.3

Suppose that the final demand from sectors 1, 2 and 3 are respectively 30, 50 and 20. Then the production vector and demand vector are respectively

$$\mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\qquad\text{and}\qquad\mathbf{d}=\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}=\begin{pmatrix}30\\50\\20\end{pmatrix},$$

while the consumption matrix is given by

$$C=\begin{pmatrix}0.3&0.2&0.1\\0.4&0.5&0.2\\0.1&0.1&0.3\end{pmatrix},\qquad\text{so that}\qquad I-C=\begin{pmatrix}0.7&-0.2&-0.1\\-0.4&0.5&-0.2\\-0.1&-0.1&0.7\end{pmatrix}.$$

The production equation $(I-C)\mathbf{x}=\mathbf{d}$ has augmented matrix

$$\left(\begin{array}{ccc|c}0.7&-0.2&-0.1&30\\-0.4&0.5&-0.2&50\\-0.1&-0.1&0.7&20\end{array}\right),\qquad\text{equivalent to}\qquad\left(\begin{array}{ccc|c}7&-2&-1&300\\-4&5&-2&500\\-1&-1&7&200\end{array}\right),$$

and which can be converted to reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3200/27 \\ 0 & 1 & 0 & 6100/27 \\ 0 & 0 & 1 & 700/9 \end{array}\right).$$

This gives $x_1 \approx 119$, $x_2 \approx 226$ and $x_3 \approx 78$, to the nearest integers.

2.9. Matrix Transformation on the Plane

Let A be a 2×2 matrix with real entries. A matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be defined as follows: For every $\mathbf{x} = (x_1, x_2) \in \mathbb{R}$, we write $T(\mathbf{x}) = \mathbf{y}$, where $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ satisfies

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Such a transformation is linear, in the sense that $T(\mathbf{x}' + \mathbf{x}'') = T(\mathbf{x}') + T(\mathbf{x}')$ for every $\mathbf{x}', \mathbf{x}'' \in \mathbb{R}^2$ and $T(c\mathbf{x}) = cT(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^2$ and every $c \in \mathbb{R}$. To see this, simply observe that

$$A \begin{pmatrix} x_1' + x_1'' \\ x_2' + x_2'' \end{pmatrix} = A \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + A \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} \qquad \text{and} \qquad A \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} = cA \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We shall study linear transformations in greater detail in Chapter 8. Here we confine ourselves to looking at a few simple matrix transformations on the plane.

Example 2.9.1. The matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{satisfies} \qquad A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

for every $(x_1, x_2) \in \mathbb{R}^2$, and so represents reflection across the x_1-axis, whereas the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{satisfies} \qquad A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$$

for every $(x_1, x_2) \in \mathbb{R}^2$, and so represents reflection across the x_2-axis. On the other hand, the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{satisfies} \qquad A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

for every $(x_1, x_2) \in \mathbb{R}^2$, and so represents reflection across the origin, whereas the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{satisfies} \qquad A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

for every $(x_1, x_2) \in \mathbb{R}^2$, and so represents reflection across the line $x_1 = x_2$. We give a summary in the table below:

Transformation	Equations	Matrix
Reflection across x_1-axis	$\begin{cases} y_1 = x_1 \\ y_2 = -x_2 \end{cases}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection across x_2-axis	$\begin{cases} y_1 = -x_1 \\ y_2 = x_2 \end{cases}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection across origin	$\begin{cases} y_1 = -x_1 \\ y_2 = -x_2 \end{cases}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection across $x_1 = x_2$	$\begin{cases} y_1 = x_2 \\ y_2 = x_1 \end{cases}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Problems for Chapter 2

1. Consider the four matrices

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 7 & 2 & 9 \\ 9 & 2 & 7 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 3 \\ 1 & 1 & 5 \\ 3 & 2 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 7 \\ 2 & 1 & 2 \\ 1 & 3 & 0 \end{pmatrix}.$$

Calculate all possible products.

2. In each of the following cases, determine whether the products AB and BA are both defined; if so, determine also whether AB and BA have the same number of rows and the same number of columns; if so, determine also whether $AB = BA$:

a) $A = \begin{pmatrix} 0 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 6 \\ 1 & 5 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -4 \\ 12 & 1 \end{pmatrix}$

d) $A = \begin{pmatrix} 3 & 1 & -4 \\ -2 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

3. Evaluate A^2, where $A = \begin{pmatrix} 2 & -5 \\ 3 & 1 \end{pmatrix}$, and find $\alpha, \beta, \gamma \in \mathbb{R}$, not all zero, such that the matrix $\alpha I + \beta A + \gamma A^2$ is the zero matrix.

4. a) Let $A = \begin{pmatrix} 6 & -4 \\ 9 & -6 \end{pmatrix}$. Show that A^2 is the zero matrix.

b) Find all 2×2 matrices $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that B^2 is the zero matrix.

5. Prove that if A and B are matrices such that $I - AB$ is invertible, then the inverse of $I - BA$ is given by the formula $(I - BA)^{-1} = I + B(I - AB)^{-1}A$.
[Hint: Write $C = (I - AB)^{-1}$. Then show that $(I - BA)(I + BCA) = I$.]

6. For each of the matrices below, use elementary row operations to find its inverse, if the inverse exists:

a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & -2 \\ 1 & 5 & 3 \\ 2 & 6 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \\ 2 & 3 & 3 \end{pmatrix}$

e) $\begin{pmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{pmatrix}$

7. a) Using elementary row operations, show that the inverse of

$$\begin{pmatrix} 2 & 5 & 8 & 5 \\ 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 2 \\ 1 & 3 & 5 & 3 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 3 & -2 & 1 & -5 \\ -2 & 5 & -2 & 3 \\ 0 & -2 & 1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

b) Without performing any further elementary row operations, use part (a) to solve the system of linear equations

$$\begin{aligned} 2x_1 + 5x_2 + 8x_3 + 5x_4 &= 0, \\ x_1 + 2x_2 + 3x_3 + x_4 &= 1, \\ 2x_1 + 4x_2 + 7x_3 + 2x_4 &= 0, \\ x_1 + 3x_2 + 5x_3 + 3x_4 &= 1. \end{aligned}$$

8. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 1 & 5 & 5 \\ 2 & 1 & 9 & 8 \\ 2 & 0 & 6 & 3 \end{pmatrix}.$$

a) Use elementary row operations to find the inverse of A.
b) Without performing any further elementary row operations, use your solution in part (a) to solve the system of linear equations

$$\begin{aligned} x_1 \qquad + 3x_3 + x_4 &= 1, \\ x_1 + x_2 + 5x_3 + 5x_4 &= 0, \\ 2x_1 + x_2 + 9x_3 + 8x_4 &= 0, \\ 2x_1 \qquad + 6x_3 + 3x_4 &= 0. \end{aligned}$$

9. In each of the following, solve the production equation $\mathbf{x} = C\mathbf{x} + \mathbf{d}$:

a) $C = \begin{pmatrix} 0.1 & 0.5 \\ 0.6 & 0.2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 50000 \\ 30000 \end{pmatrix}$

b) $C = \begin{pmatrix} 0 & 0.6 \\ 0.5 & 0.2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 36000 \\ 22000 \end{pmatrix}$

c) $C = \begin{pmatrix} 0.2 & 0.2 & 0 \\ 0.1 & 0 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 4000000 \\ 8000000 \\ 6000000 \end{pmatrix}$

10. Consider three industries A, B and C. For industry A to manufacture \$1 worth of its product, it needs to purchase 25c worth of product from each of industries B and C. For industry B to manufacture \$1 worth of its product, it needs to purchase 65c worth of product from industry A and 5c worth of product from industry C, as well as use 5c worth of its own product. For industry C to manufacture \$1 worth of its product, it needs to purchase 55c worth of product from industry A and 10c worth of product from industry B. In a particular week, industry A receives \$500000 worth of outside order, industry B receives \$250000 worth of outside order, but industry C receives no outside order. What is the production level required to satisfy all the demands precisely?

11. Suppose that C is an $n \times n$ consumption matrix with all column sums less than 1. Suppose further that $\mathbf{x}'$ is the production vector that satisfies an outside demand $\mathbf{d}'$, and that $\mathbf{x}''$ is the production vector that satisfies an outside demand $\mathbf{d}''$. Show that $\mathbf{x}' + \mathbf{x}''$ is the production vector that satisfies an outside demand $\mathbf{d}' + \mathbf{d}''$.

12. Suppose that C is an $n \times n$ consumption matrix with all column sums less than 1. Suppose further that the demand vector $\mathbf{d}$ has 1 for its top entry and 0 for all other entries. Describe the production vector $\mathbf{x}$ in terms of the columns of the matrix $(I - C)^{-1}$, and give an interpretation of your observation.

13. Consider a pentagon in $\mathbb{R}^2$ with vertices $(1,1)$, $(3,1)$, $(4,2)$, $(2,4)$ and $(1,3)$. For each of the following transformations on the plane, find the 3×3 matrix that describes the transformation with respect to homogeneous coordinates, and use it to find the image of the pentagon:
   a) reflection across the x_2-axis
   b) reflection across the line $x_1 = x_2$
   c) anticlockwise rotation by $90°$
   d) translation by the fixed vector $(3, -2)$
   e) shear in the x_2-direction with factor 2
   f) dilation by factor 2
   g) expansion in x_1-direction by factor 2
   h) reflection across the x_2-axis, followed by anticlockwise rotation by $90°$
   i) translation by the fixed vector $(3, -2)$, followed by reflection across the line $x_1 = x_2$
   j) shear in the x_2-direction with factor 2, followed by dilation by factor 2, followed by expansion in x_1-direction by factor 2