# LINEAR ALGEBRA

# Chapter 4

## VECTORS

### 4.1. Introduction

A vector is an object which has magnitude and direction.

EXAMPLE 4.1.1. We may be travelling north-east at 50 kph. In this case, the direction of the velocity is north-east and the magnitude of the velocity is 50 kph. We can describe our velocity in kph as

$$\left(\frac{50}{\sqrt{2}}, \frac{50}{\sqrt{2}}\right),$$

where the first coordinate describes the speed with which we are moving east and the second coordinate describes the speed with which we are moving north.

EXAMPLE 4.1.2. An object in the sky may be 100 metres away in the south-east direction 45 degrees upwards. In this case, the direction of its position is south-eastand 45 degrees upwards and the magnitude of its distance is 100 metres. We can describe the position of the object in metres as

$$\left(50, -50, \frac{100}{\sqrt{2}}\right),$$

where the first coordinate describes the distance east, the second coordinate describes the distance north and the third coordinate describes the distance up.

The purpose of this chapter is to study some relationship between algebra and geometry. We shall first study some algebra which is motivated by geometric considerations. We then use the algebra later to better understand some problems in geometry.

**4.2. Vectors in $\mathbb{R}^2$**

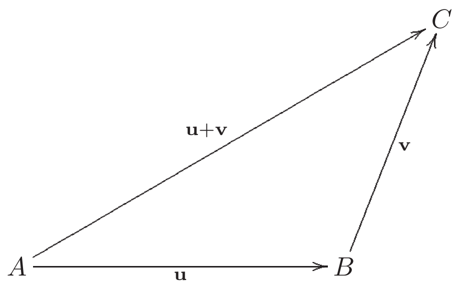
A vector on the plane $\mathbb{R}^2$ can be described as an ordered pair $\mathbf{u} = (u_1, u_2)$, where $u_1, u_2 \in \mathbb{R}$.

DEFINITION. Two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$ are said to be equal, denoted by $\mathbf{u} = \mathbf{v}$, if $u_1 = v_1$ and $u_2 = v_2$.

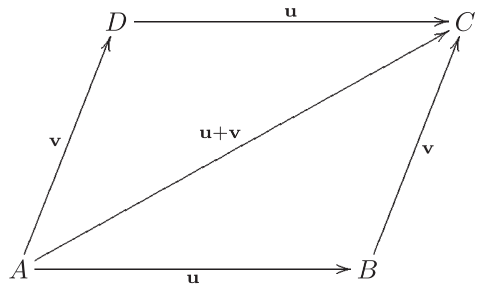
DEFINITION. For any two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$, we define their sum to be

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2).$$

Geometrically, if we represent the two vectors $\mathbf{u}$ and $\mathbf{v}$ by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively, then the sum $\mathbf{u} + \mathbf{v}$ is represented by $\overrightarrow{AC}$ as shown in the diagram below:




The next diagram demonstrates geometrically that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$:




**PROPOSITION 4A.** (VECTOR ADDITION)
(a) *For every* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, *we have* $\mathbf{u} + \mathbf{v} \in \mathbb{R}^2$.
(b) *For every* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, *we have* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
(c) *For every* $\mathbf{u} \in \mathbb{R}^2$, *we have* $\mathbf{u} + \mathbf{0} = \mathbf{u}$, *where* $\mathbf{0} = (0, 0) \in \mathbb{R}^2$.
(d) *For every* $\mathbf{u} \in \mathbb{R}^2$, *there exists* $\mathbf{v} \in \mathbb{R}^2$ *such that* $\mathbf{u} + \mathbf{v} = \mathbf{0}$.
(e) *For every* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, *we have* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

PROOF. Write $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, where $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$. To check part (a), simply note that $u_1 + v_1, u_2 + v_2 \in \mathbb{R}$. To check part (b), note that

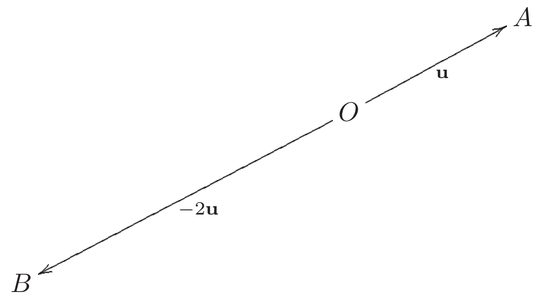
$$\begin{aligned}\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_1, u_2) + (v_1 + w_1, v_2 + w_2) = (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)) \\ &= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2) = (u_1 + v_1, u_2 + v_2) + (w_1, w_2) \\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}.\end{aligned}$$

Part (c) is trivial. Next, if $\mathbf{v} = (-u_1, -u_2)$, then $\mathbf{u} + \mathbf{v} = \mathbf{0}$, giving part (d). To check part (e), note that $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2) = \mathbf{v} + \mathbf{u}$. ◯

DEFINITION. For any vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$ and any scalar $c \in \mathbb{R}$, we define the scalar multiple to be

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2).$$

EXAMPLE 4.2.1. Suppose that $\mathbf{u} = (2, 1)$. Then $-2\mathbf{u} = (-4, 2)$. Geometrically, if we represent the two vectors $\mathbf{u}$ and $-2\mathbf{u}$ by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, then we have the diagram below:




**PROPOSITION 4B.** (SCALAR MULTIPLICATION)
*(a) For every $c \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^2$, we have $c\mathbf{u} \in \mathbb{R}^2$.*
*(b) For every $c \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, we have $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.*
*(c) For every $a, b \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^2$, we have $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.*
*(d) For every $a, b \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^2$, we have $(ab)\mathbf{u} = a(b\mathbf{u})$.*
*(e) For every $\mathbf{u} \in \mathbb{R}^2$, we have $1\mathbf{u} = \mathbf{u}$.*

PROOF. Write $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, where $u_1, u_2, v_1, v_2 \in \mathbb{R}$. To check part (a), simply note that $cu_1, cu_2 \in \mathbb{R}$. To check part (b), note that

$$\begin{aligned} c(\mathbf{u} + \mathbf{v}) &= c(u_1 + v_1, u_2 + v_2) = (c(u_1 + v_1), c(u_2 + v_2)) \\ &= (cu_1 + cv_1, cu_2 + cv_2) = (cu_1, cu_2) + (cv_1, cv_2) = c\mathbf{u} + c\mathbf{v}. \end{aligned}$$

To check part (c), note that

$$\begin{aligned} (a + b)\mathbf{u} &= ((a + b)u_1, (a + b)u_2) = (au_1 + bu_1, au_2 + bu_2) \\ &= (au_1, au_2) + (bu_1, bu_2) = a\mathbf{u} + b\mathbf{u}. \end{aligned}$$

To check part (d), note that

$$(ab)\mathbf{u} = ((ab)u_1, (ab)u_2) = (a(bu_1), a(bu_2)) = a(bu_1, bu_2) = a(b\mathbf{u}).$$

Finally, to check part (e), note that $1\mathbf{u} = (1u_1, 1u_2) = (u_1, u_2) = \mathbf{u}$. ◯

DEFINITION. For any vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$, we define the norm of $\mathbf{u}$ to be the non-negative real number

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}.$$

REMARKS. (1) The norm of a vector is simply its magnitude or length. The definition follows from the famous theorem of Pythagoras.

(2) Suppose that $P(u_1, u_2)$ and $Q(v_1, v_2)$ are two points on the plane $\mathbb{R}^2$. To calculate the distance $d(P, Q)$ between the two points, we can first find a vector from $P$ to $Q$. This is given by $(v_1 - u_1, v_2 - u_2)$. The distance $d(P, Q)$ is then the norm of this vector, so that

$$d(P, Q) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}.$$

(3) It is not difficult to see that for any vector $\mathbf{u} \in \mathbb{R}^2$ and any scalar $c \in \mathbb{R}$, we have $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$.

DEFINITION. Any vector $\mathbf{u} \in \mathbb{R}^2$ satisfying $\|\mathbf{u}\| = 1$ is called a unit vector.

EXAMPLE 4.2.2. The vector $(3, 4)$ has norm 5.

EXAMPLE 4.2.3. The distance between the points $(6, 3)$ and $(9, 7)$ is $\sqrt{(9-6)^2 + (7-3)^2} = 5$.

EXAMPLE 4.2.4. The vectors $(1, 0)$ and $(0, -1)$ are unit vectors in $\mathbb{R}^2$.

EXAMPLE 4.2.5. The unit vector in the direction of the vector $(1, 1)$ is $(1/\sqrt{2}, 1/\sqrt{2})$.

EXAMPLE 4.2.6. In fact, all unit vectors in $\mathbb{R}^2$ are of the form $(\cos\theta, \sin\theta)$, where $\theta \in \mathbb{R}$.

Quite often, we may want to find the angle between two vectors. The scalar product of the two vectors then comes in handy. We shall define the scalar product in two ways, one in terms of the angle between the two vectors and the other not in terms of this angle, and show that the two definitions are in fact equivalent.

DEFINITION. Suppose that $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors in $\mathbb{R}^2$, and that $\theta \in [0, \pi]$ represents the angle between them. We define the scalar product $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta & \text{if } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0}. \end{cases} \tag{1}$$

Alternatively, we write

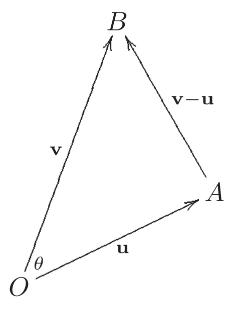
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2. \tag{2}$$

The definitions (1) and (2) are clearly equivalent if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. On the other hand, we have the following result.

**PROPOSITION 4C.** *Suppose that* $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$ *are non-zero vectors in* $\mathbb{R}^2$*, and that* $\theta \in [0, \pi]$ *represents the angle between them. Then*

$$\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = u_1 v_1 + u_2 v_2.$$

PROOF. Geometrically, if we represent the two vectors $\mathbf{u}$ and $\mathbf{v}$ by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, then the difference $\mathbf{v} - \mathbf{u}$ is represented by $\overrightarrow{AB}$ as shown in the diagram below:




By the Law of cosines, we have

$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 - 2\overline{OA}\,\overline{OB} \cos\theta;$$

in other words, we have

$$\|\mathbf{v}-\mathbf{u}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta,$$

so that

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta&=\tfrac{1}{2}(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-\|\mathbf{v}-\mathbf{u}\|^2)\\&=\tfrac{1}{2}(u_1^2+u_2^2+v_1^2+v_2^2-(v_1-u_1)^2-(v_2-u_2)^2)\\&=u_1v_1+u_2v_2\end{aligned}$$

as required. ○

REMARKS. (1) We say that two non-zero vectors in $\mathbb{R}^2$ are orthogonal if the angle between them is $\pi/2$. It follows immediately from the definition of the scalar product that two non-zero vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^2$ are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v}=0$.

(2) We can calculate the scalar product of any two non-zero vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^2$ by the formula (2) and then use the formula (1) to calculate the angle between $\mathbf{u}$ and $\mathbf{v}$.

EXAMPLE 4.2.7. Suppose that $\mathbf{u}=(\sqrt{3},1)$ and $\mathbf{v}=(\sqrt{3},3)$. Then by the formula (2), we have

$$\mathbf{u}\cdot\mathbf{v}=3+3=6.$$

Note now that

$$\|\mathbf{u}\|=2\qquad\text{and}\qquad\|\mathbf{v}\|=2\sqrt{3}.$$

It follows from the formula (1) that

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}=\frac{6}{4\sqrt{3}}=\frac{\sqrt{3}}{2},$$

so that $\theta=\pi/6$.

EXAMPLE 4.2.8. Suppose that $\mathbf{u}=(\sqrt{3},1)$ and $\mathbf{v}=(-\sqrt{3},3)$. Then by the formula (2), we have $\mathbf{u}\cdot\mathbf{v}=0$. It follows that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

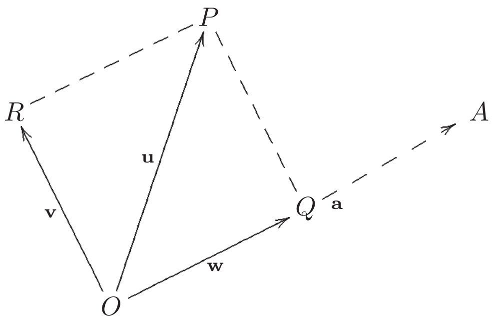
**PROPOSITION 4D.** (SCALAR PRODUCT) *Suppose that* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^2$ *and* $c\in\mathbb{R}$. *Then*
*(a)* $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$;
*(b)* $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=(\mathbf{u}\cdot\mathbf{v})+(\mathbf{u}\cdot\mathbf{w})$;
*(c)* $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})$;
*(d)* $\mathbf{u}\cdot\mathbf{u}\geq 0$; *and*
*(e)* $\mathbf{u}\cdot\mathbf{u}=0$ *if and only if* $\mathbf{u}=\mathbf{0}$.

PROOF. Write $\mathbf{u}=(u_1,u_2)$, $\mathbf{v}=(v_1,v_2)$ and $\mathbf{w}=(w_1,w_2)$, where $u_1,u_2,v_1,v_2,w_1,w_2\in\mathbb{R}$. Part (a) is trivial. To check part (b), note that

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=u_1(v_1+w_1)+u_2(v_2+w_2)=(u_1v_1+u_2v_2)+(u_1w_1+u_2w_2)=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}.$$

Part (c) is rather simple. To check parts (d) and (e), note that $\mathbf{u}\cdot\mathbf{u}=u_1^2+u_2^2\geq 0$, and that equality holds precisely when $u_1=u_2=0$. ○

Consider the diagram below:




(3)

Here we represent the two vectors **a** and **u** by $\overrightarrow{OA}$ and $\overrightarrow{OP}$ respectively. If we project the vector **u** on to the line $\overline{OA}$, then the image of the projection is the vector **w**, represented by $\overrightarrow{OQ}$. On the other hand, if we project the vector **u** on to a line perpendicular to the line $\overline{OA}$, then the image of the projection is the vector **v**, represented by $\overrightarrow{OR}$.

Definition. In the notation of the diagram (3), the vector **w** is called the orthogonal projection of the vector **u** on the vector **a**, and denoted by $\mathbf{w} = \text{proj}_{\mathbf{a}}\mathbf{u}$.

**PROPOSITION 4E.** (ORTHOGONAL PROJECTION) *Suppose that* $\mathbf{u}, \mathbf{a} \in \mathbb{R}^2$. *Then*

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

Remark. Note that the component of **u** orthogonal to **a**, represented by $\overrightarrow{OR}$ in the diagram (3), is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

Proof of Proposition 4E. Note that $\mathbf{w} = k\mathbf{a}$ for some $k \in \mathbb{R}$. It clearly suffices to prove that

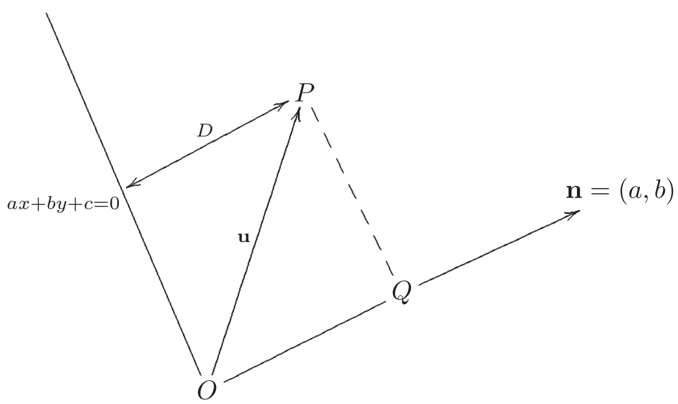
$$k = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}.$$

It is easy to see that the vectors $\mathbf{u} - \mathbf{w}$ and **a** are orthogonal. It follows that the scalar product $(\mathbf{u} - \mathbf{w}) \cdot \mathbf{a} = 0$. In other words, $(\mathbf{u} - k\mathbf{a}) \cdot \mathbf{a} = 0$. Hence

$$k = \frac{\mathbf{u} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}$$

as required. ○

To end this section, we shall apply our knowledge gained so far to find a formula that gives the perpendicular distance of a point $(x_0, y_0)$ from a line $ax + by + c = 0$. Consider the diagram below:



Suppose that $(x_1, y_1)$ is any arbitrary point $O$ on the line $ax+by+c=0$. For any other point $(x, y)$ on the line $ax+by+c=0$, the vector $(x-x_1, y-y_1)$ is parallel to the line. On the other hand,

$$(a, b) \cdot (x-x_1, y-y_1) = (ax+by) - (ax_1+by_1) = -c+c = 0,$$

so that the vector $\mathbf{n} = (a, b)$, in the direction $\overrightarrow{OQ}$, is perpendicular to the line $ax+by+c=0$. Suppose next that the point $(x_0, y_0)$ is represented by the point $P$ in the diagram. Then the vector $\mathbf{u} = (x_0-x_1, y_0-y_1)$ is represented by $\overrightarrow{OP}$, and $\overrightarrow{OQ}$ represents the orthogonal projection $\mathrm{proj}_{\mathbf{n}}\mathbf{u}$ of $\mathbf{u}$ on the vector $\mathbf{n}$. Clearly the perpendicular distance $D$ of the point $(x_0, y_0)$ from the line $ax+by+c=0$ satisfies

$$D = \|\mathrm{proj}_{\mathbf{n}}\mathbf{u}\| = \left\| \frac{\mathbf{u}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} \right\| = \frac{|(x_0-x_1, y_0-y_1)\cdot(a, b)|}{\sqrt{a^2+b^2}} = \frac{|ax_0+by_0-ax_1-by_1|}{\sqrt{a^2+b^2}} = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}.$$

We have proved the following result.

**PROPOSITION 4F.** *The perpendicular distance $D$ of a point $(x_0, y_0)$ from a line $ax+by+c=0$ is given by*

$$D = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}.$$

## Problems for Chapter 4

1. For each of the following pairs of vectors in $\mathbb{R}^2$, calculate $\mathbf{u} + 3\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u} - \mathbf{v}\|$ and find the angle between $\mathbf{u}$ and $\mathbf{v}$:
   a) $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (-5, 0)$
   b) $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (2, 1)$

2. For each of the following pairs of vectors in $\mathbb{R}^2$, calculate $2\mathbf{u} - 5\mathbf{v}$, $\|\mathbf{u} - 2\mathbf{v}\|$, $\mathbf{u} \cdot \mathbf{v}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ (to the nearest degree):
   a) $\mathbf{u} = (1, 3)$ and $\mathbf{v} = (-2, 1)$
   b) $\mathbf{u} = (2, 0)$ and $\mathbf{v} = (-1, 2)$

3. For the two vectors $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (5, 1)$ in the 2-dimensional euclidean space $\mathbb{R}^2$, determine each of the following:
   a) $\mathbf{u} - \mathbf{v}$
   b) $\|\mathbf{u}\|$
   c) $\mathbf{u} \cdot (\mathbf{u} - \mathbf{v})$
   d) the angle between $\mathbf{u}$ and $\mathbf{u} - \mathbf{v}$

4. For each of the following pairs of vectors in $\mathbb{R}^3$, calculate $\mathbf{u} + 3\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u} - \mathbf{v}\|$, find the angle between $\mathbf{u}$ and $\mathbf{v}$, and find a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$:
   a) $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (-5, 0, 5)$
   b) $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (3, 2, 1)$

5. Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v}$ is parallel to $(1, 2, 3)$, $\mathbf{v} + \mathbf{w} = (7, 3, 5)$ and $\mathbf{w}$ is orthogonal to $(1, 2, 3)$.

6. Let $ABCD$ be a quadrilateral. Show that the quadrilateral obtained by joining the midpoints of adjacent sides of $ABCD$ is a parallelogram.
   [Hint: Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be vectors representing the four sides of $ABCD$.]