# 1. An introduction to dynamic optimization -- Optimal Control

002 Math Econ - Summer 2012

## I. Overview of optimization

Optimization is the unifying paradigm in almost all economic analysis. So before we start, let's think about optimization. The tree below provides a very nice general representation of the range of optimization problems that you might encounter. There are two things to take from this. First, all optimization problems have a great deal in common: an objective function, constraints, and choice variables. Second, there are lots of different types of optimization problems and how you solve them will depend on the branch on which you find yourself.

In this part of course we will use both analytical & numerical methods to solve certain class of optimization problems. This class focuses on a set of optimization problems that have two common features: the objective function is a linear aggregation over time, and a set of variables, called the state variables, are constrained across time. And so we begin …

*Static Optimization*: single optimal magnitude for each choice variable and does not entail a schedule of optimal sequence of action.
*Dynamic Optimization*: it takes the form of an optimal time path for every choice variable (today, tomorrow etc.), and determines the optimal magnitude thereby.

## II. Introduction – A simple 2-period consumption model

Consider the simple consumer's optimization problem:

$$\max_{z}\ u(z_a, z_b)$$
$$s.t.\ p_a z_a + p_b z_b \le x$$

[pay attention to the notation: $z$ is the vector of choice variables and $x$ is the consumer's exogenously determined income.]

Solving the one-period problem should be familiar to you. What happens if the consumer lives for two periods, but has to survive off of the income endowment provided at the beginning of the first period? That is, what happens if her problem is

$$\max_{z} U(z_{1a}, z_{1b}, z_{2a}, z_{2b}) = U(z_1, z_2)$$
$$s.t.\ p' z_1 + p' z_2 \le x_1$$

where the constraint uses matrix notation with **p** = [$p_a$ , $p_b$] refers to a price vector and $z_1$ = [ $z_{1a}$ , $z_{1b}$]. We now have a problem of dynamic optimization. When we chose $z_1$, we must take into account how it will affect our choices in period 2.

We're going to make a **huge** (though common) assumption and maintain that assumption throughout the course: utility is additively separable across time1:

$$u(\mathbf{z}) = u(z_1) + u(z_2)$$

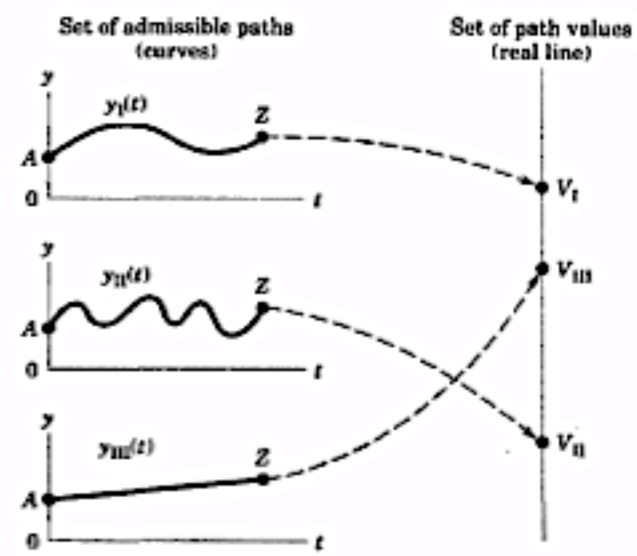
Clearly one way to solve this problem would be just as we would a standard static problem: set up a Lagrangian and solve for all optimal choices simultaneously. This may work here, where there are only 2 periods, but if we have 100 periods (or even an infinite number of periods) then this could get really messy. This course will develop methods to solve such problems.

**The Dynamic Optimization problem has 4 basic ingredients –**

1. A given ***initial point*** and a given ***terminal point***; X(0) & X(T)
2. A set of ***admissible paths*** from the initial point to the terminal point; 0 & T
3. A set of path values serving as performance indices (cost, profit, etc.) associated with the various paths; and
4. A specified ***objective*** - either to maximize or to minimize the path value or performance index by choosing the optimal path.

**The Concept of a Functional**

The relationship between paths and path values deserves our close attention, for it represents a special sort of mapping-not a mapping from real numbers to real numbers as in the usual function, but a mapping from paths (curves) to real numbers (performance indices). Let us think of the paths in question as time paths, and denote them by $Y_I(t)$, $Y_{II}(t)$, and so on and $V_I$, $V_{II}$ represent the associated path values. The general notation for the mapping should therefore be V[y(t)]. But it must be emphasized that this symbol fundamentally differs from the composite-function symbol g[f(x)]. In the latter, g is a function of f, and f is in turn a function of x; thus, g is in the final analysis a function of x. In the symbol V[y(t)], on the other hand, the y(t) component comes as an integral unit-to indicate time paths-and therefore we should not take V to be a function of t. Instead, V should be understood to be a function of "y(t)" as such.




**This is a good point to introduce some *very important* terminology:**

- All dynamic optimization problems have a **time horizon**. In the problem above $t$ is discrete, $t$={1,2}, but $t$ can also be continuous, taking on every value between $t_0$ and $T$, and we can solve problems where $T \to \infty$
- $x_t$ is what we call a **state variable** because it is the state that the decision-maker faces in period $t$. Note that $x_t$ is parametric (i.e., it is taken as given) to the decision-maker's problem in $t$, and $xt$+1 is parametric to the choices in period $t$+1. However, $x_{t+1}$ is determined by the choices made in $t$. *The state variables in a problem are the variables upon which a decision maker bases his or her choices in each period.* Another important characteristic of state variables is that typically the choices you make in one period will influence the value of the state

variable in the next period.

- A **state equation** defines the intertemporal changes in a state variable.
- $z_t$ is the vector of $t^{th}$ period **control (or choice) variables**. Choice variables determine the (expected) payoff in the current period and the (expected) state next period.
- $p_a$ and $p_b$ are **parameters** of the model. They are held constant or change exogenously and deterministically over time.
- Finally, we have what is called **intermediate variables**. These are variables that are really functions of the state and control variables and the parameters. For example, in the problem considered here, one-period utility might be carried as an intermediate variable. In firm problems, production or profit might be other intermediate variables while productivity or profitability (a firm's capacity to generate output or profits) could be state variables. *Do you see the difference? This is very important.* When you formulate a problem it is very important to distinguish state variables from intermediate variables.
- The **benefit function** [here $u(z_t)$] tells the instantaneous or single period ***net*** benefits that accrue to the planner during the planning horizon. Despite its name, the benefit function can take on positive or negative values. For example, a function that defines the cost in each period can be the benefit function.
- In many problems there are benefits (or costs) that accrue after the planning horizon. This is captured in models by including a **salvage value**, which is usually a function of the terminal stock. Since the salvage value occurs after the planning horizon, it can not be a function of the control variables, though it can be a separate optimization problem in which choices are made.
- The sum (or integral) over the planning horizon plus the salvage value determines the **objective function(al)**. We usually use discounting when we sum up over time.
- All of the problems that we will study in this course fall into the general category of **Markov decision processes** (MDP). In an MDP the probability distribution over the states in the next period is wholly determined by the current state and current actions. One important implication of limiting ourselves to MDPs is that, *typically, history does not matter*, i.e. $x_{t+1}$ depends on $z_t$ and $x_t$, irrespective of the value of $x_{t-1}$. When history is important in a problem then the relevant historical variables must be explicitly included as state variables.

In sum, the problems that we will study will have the following features. In each period or moment in time the decision maker looks at the state variables ($x_t$), then chooses the control variables ($z_t$). The combination of *xt* and *zt* generates immediate benefits and costs. They also determine the probability distribution over $x$ in the next period or moment.

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Instead of using brute force to find the solutions of all the $z$'s in one step, we reformulate the problem. Let $x_1$ be the endowment which is available in period 1, and $x_2$ be the endowment that remains in period 2. Following from the budget constraint, we can see that $x_2 = x_1 - \mathbf{p}'z_1$, with $x_2 \geq 0$. In this problem $x_2$ defines the **state** that the decision maker faces at the start of period 2. The equation which describes the change in the $x$ from period 1 to period 2, $x_2 - x_1 = -\mathbf{p}'z_1$, is called the **state equation**. This equation is also sometimes referred to as the *equation of motion* or the *transition equation*.

We now rewrite our consumer's problem, this time making use of the state equation:

$$\max_{z_t} \sum_{t=1}^{2} u_t(z_t) \;\; s.t.$$

$$\left.\begin{aligned} x_{t+1} - x_t &= -p' z_t \\ x_{t+1} &\geq 0 \end{aligned}\right\} \; t = 1,2 \tag{1.1}$$

$$x_t \text{ is fixed.}$$

We now have a nasty little optimization problem with four constraints, two of them inequality constraints – not fun. This course will help you solve and understand these kinds of problems. Note that this formulation is quite general in that you could easily write the *n*-period problem by simply replacing the 2's in (1) with *n*.

**III. The OC (optimal control) way of solving the problem**

We will solve dynamic optimization problems using two related methods. The first of these is called *optimal control*. Optimal control makes use of Pontryagin's maximum principle.

To see this approach, first note that for most specifications, economic intuition tells us that $x_2>0$ and $x_3=0$. Hence, for $t=1$ ($t+1=2$), we can suppress inequality constraint in (1). We'll use the fact that $x_3=0$ at the very end to solve the problem.

Write out the Lagrangian of (1):

$$L = \sum_{t=1}^{2} u_t(z_t, x_t) + \lambda_t(x_t - x_{t+1} - p' z_t)] \tag{1.2}$$

where we include $x_t$ in $u(.)$ for completeness, though $\partial u / \partial x = 0$ .

**More terminology**

In optimal control theory, the variable $\lambda_t$ is called the **co-state variable** and, following the standard interpretation of Lagrange multipliers, at its optimal value $\lambda_t$ is equal to the marginal value of relaxing the constraint. In this case, that means it is the marginal value of the state variable, $x_t$. The co-state variable plays a critical role in dynamic optimization.

The FOCs for (2) are standard:

$$\partial L / \partial z_{ti} = \partial u / \partial z - \lambda_t p_i = 0, \;\; i = a, b; \;\; t = 1,2$$

$$\partial L / \partial x_2 = \partial u / \partial x_2 - \lambda_1 + \lambda_2 = 0$$

$$\partial L / \partial \lambda_t = (x_t - x_{t+1} - p' z_t) = 0, \; t = 1,2$$

We now use a little notation change that simplifies this problem and adds some intuition (we'll see how the intuition arises in later lectures). That is, we define a function known as the **Hamiltonian** where

$$H = u(z_1, x_1) + \lambda_t(-p' z_t) \; .$$

Some things to note about the Hamiltonian:

- the $t^{th}$ Hamiltonian includes only $z_t$ and $\lambda_t$ ,

- Unlike in a Lagrangian, only the RHS of state equation appears in the parentheses.

In the left column of table below we present the first-order conditions of the Lagrangian specification. Then on the right we present the derivative of the Hamiltonian with respect to the same variables. By comparison, we then can see what we would have to place on the right-hand side of the first derivative to obtain the same optimum if using the Hamiltonian that we would reach if we used the Lagrangian approach.

| Lagrangian | | Hamiltonian |
|---|---|---|
| $L=\sum_{t=1}^{2}\left[u_t(z_{ta},z_{tb})+\lambda_t\left(x_t-x_{t+1}-(p_a z_{ta}+p_b z_{tb})\right)\right]$ | | $H=u\left(z_t,x_g\right)+\lambda_t\left(-\mathbf{p}'z_t\right)$ |
| Standard FOCs | | $\partial H/\partial$__ |
| $\frac{\partial L}{\partial z_{ti}}=\frac{\partial u_t}{\partial z_{ti}}-\lambda_t p_i=0$ , $t$=1,2, $i$=$a$,$b$ | $z$ | $\frac{\partial H}{\partial z_{ti}}=\frac{\partial u_t}{\partial z_{ti}}-\lambda_t p=$______ |
| $\frac{\partial L}{\partial x_2}=\frac{\partial u(\cdot)}{\partial x_2}-\lambda_1+\lambda_2=0$ | $x_2$ | $\frac{\partial H}{\partial x_2}=\frac{\partial u\left(z_2,x_2\right)}{\partial x_2}=$________ |
| $\frac{\partial L}{\partial \lambda_t}=x_t-x_{t+1}-\mathbf{p}'z_t=0$ , $t$=1,2, $i$=$a$,$b$ | $\lambda_t$ | $\frac{\partial H}{\partial \lambda_t}=-\mathbf{p}'z_t=$________ |

Hence, we see that for the solution using the Hamiltonian to yield the same maximum the following conditions must hold

1. $\frac{\partial H}{\partial z_t}=0$ => The Hamiltonian should be maximized w.r.t. the control variable at every point in time.
2. $\frac{\partial H}{\partial x_t}=\lambda_{t-1}-\lambda_t,\ for\ t>1$ => The co-state variable changes over time at a rate equal to minus the marginal value of the state variable to the Hamiltonian.
3. $\frac{\partial H}{\partial \lambda_t}=x_{t+1}-x_t$ => The state equation must always be satisfied.

When we combine these with a 4th condition, called the **transversality** condition (how we *transverse* over to the world beyond $t$=1,2) we're able to solve the problem. In this case the condition that $x3$ =0 (which for now we will assume to hold without proof) serves that purpose. We'll discuss the transversality condition in more detail in a few lectures.

These four conditions are the starting points for solving most optimal control problems and sometimes the FOCs alone are sufficient to understand the economics of a problem. However, if we want an explicit solution, then we would solve this system of equations.

Although in this class most of the OC problems we'll face are in continuous time, the parallels should be obvious when we get there.

**IV. The DP (Dynamic programming) way of solving the problem**

The second way that we will solve dynamic optimization problems is using Dynamic Programming. DP is about **backward induction**–thinking backwards about problems. Let's see how this is applied in the context of the 2-period consumer's problem.

Imagine that the decision-maker is now in period 2, having already used up part of her endowment in period 1, leaving $x_2$ to be spent. In period 2, her problem is simply

$$V_2(x_2) = \max_{z_2} u_2(z_2), \text{ s.t.}$$
$$p' z_2 \leq x_2$$

If we solve this problem, we can easily obtain the function $V(x_2)$, which tells us the maximum utility that can be obtained if she arrives in period 2 with $x_2$ dollars remaining. The function $V(.)$ is equivalent to the indirect utility function with $p_a$ and $p_b$ suppressed. The period 1 problem can then be written

$$\max_{z_1} u(z_1) + V_2(x_2) \text{ s.t.}$$
$$x_2 = x_1 - p' z_1 \qquad (1.3)$$

Note that we've implicitly assumed an interior solution so that the constraint requiring that $x_3 \geq 0$ is assumed to hold with an equality and can be suppressed. Once we know the functional form of $V(.)$, (3) becomes a simple static optimization problem and its solution is straightforward. Assume for a moment that the functional form of $V(x_2)$ has been found. We can then write out Lagrangian of the first period problem,

$$L = u(z_1) + V_2(x_2) + \lambda_1(x_1 - p' z_1 - x_2).$$

Again, we see that the economic meaning of the costate variable, l 1 is just as in the OC setup, i.e., it is equal to the marginal value of a unit of $x1$.

Of course the problem is that we do not have an explicit functional form for $V(.)$ and as the problem becomes more complicated, obtaining a functional form becomes more difficult, even impossible for many problems. Hence, the trick to solving DP problems is to find the function $V(.)$.

**V. Summary**

- OC problems are solved using the vehicle of the Hamiltonian, which must be maximized at each point in time.
- DP is about backward induction.
- Both techniques are equivalent to standard Lagrangian techniques and the interpretation of the shadow price, l, is the same.

**VI. References**

Deaton, Angus and John Muellbauer. 1980. *Economics of Consumer Behavior.* New York: Cambridge University Press.

## 2. Introduction to Optimal Control

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### I. Why we're not studying Calculus of Variations.

1) OC is better & much more widely used.
2) Parallels to DP are clearer in OC.
3) COV is tough but you can study it in Kamien and Schwartz (1991) Part I

### II. Optimal Control problems always contain

$z_t \Rightarrow$ the (set of ) choice variable(s),

$x_t \Rightarrow$ the (set of ) state variable(s),

$\dot{x} = f(t,x,z)$ $\Rightarrow$ the state equation(s),

$V = \int_0^T F(t,x,z)dt \Rightarrow$ an objective function in which $F(\cdot)$ is the benefit function introduced previously.

$x_0 \Rightarrow$ an initial condition for the state variable,

and sometimes explicit intratemporal constraints, e.g. $g(t,x,z) \leq 0$

As we saw in the two-period discrete-time model in lecture 1, OC problems can be solved more easily using the vehicle of the Hamiltonian. In the next lecture we'll see more formally why this holds and then explore the economic intuition behind the Hamiltonian. For now, take my word for it.

Generally, the Hamiltonian takes the form: $H = F(t,x,z) + \lambda_t f(t,x,z)$.

The *maximum principle*, due to Pontryagin, states that the following conditions, if satisfied, guarantee a solution to the problem (you should commit these conditions to memory)

1. $\max_z H(t,x,z,\lambda)$ for all $t \in [0,T]$
2. $\frac{\partial H}{\partial x} = -\dot{\lambda} = \frac{\partial \lambda}{\partial t}$
3. $\frac{\partial H}{\partial \lambda} = \dot{x}$
4. *Transversality condition* (such as, $\lambda(T) = 0$)

Points to note:

- the maximization condition, 1, is not equivalent to $\partial H/\partial z = 0$, since corner solutions are admissible and non-differential problems can be considered.
- the maximum criteria include 2 sets of differential equations (2&3), so there's one set of differential equations that was not present in the original problem.
- $\partial H/\partial \lambda$ = the state equation by the definition of $H$.
- There are no second-order partial differential equations

In general the *transversality condition* is a condition that specifies what happens as we *transverse* to time outside the planning horizon. Above we state $\lambda(T)=0$ as the condition for a problem in which there is no binding constraint on the terminal value of the state variable(s). This condition makes intuitive sense: since $\lambda_t$ is the marginal value of the state variable at time $t$, if you have complete flexibility in choosing $x_T$, you would want to choose that level so that its marginal value is zero, i.e., $\lambda_T$=0. We will spend more time discussing the meaning and derivation of transversality conditions in the next lecture.

**III. The Solution of an optimal problem** (An example from Chiang (1991) with slight notation changes).

$$\max_{z_t} \int_0^T -\left(1+z_t^2\right)^{1/2} dt$$
$$s.t. \quad \dot{x}_t = z_t$$
$$and \quad x_0 = A, \quad x_T \quad \text{free}$$

The Hamiltonian of this problem is

$$H = -\left(1+z^2\right)^{1/2} + \lambda z$$

Note that we can use the standard interior solution for the maximization of the Hamiltonian since the benefit function is concave and continuously differentiable. Hence, our maximization equations are

1. $\partial H/\partial z = -1/2\left(1+z^2\right)^{-1/2} 2z + \lambda = 0$

   (if you check the 2nd order conditions you can verify we've got a maximum)
2. $\partial H/\partial x = 0 = -\dot{\lambda}$
3. $\partial H/\partial \lambda = z = \dot{x}$
4. $\lambda_T$=0, the transversality of this problem (because of the free value for $x_T$).

Solving this problem is real easy.

1. 2 means that $\lambda$ is constant.
2. Together with 4, this means that $\lambda$ is constant at 0, i.e., $\lambda_t$=0 for all $t$.
3. To find $z_t^*$, solve 1 after dropping out $\lambda$ and we see that the only way $-1/2\left(1+z^2\right)^{-1/2} 2z = 0$ is if $z_t^* = 0$.
4. Plug this into the state equation, 3, and we find that $x$ remains constant at $A$.

Now that was easy, but not very interesting. Let's try something a little more challenging.

**IV. A simple consumption problem**

$$\max_{z_t} \int_0^1 \ln\left[z_t 4x_t\right] dt$$
$$s.t. \quad \dot{x}_t = 4x_t\left(1-z_t\right)$$
$$and \quad x_0 = 1, \quad x_1 = e^2$$

*What would a phase diagram for x in x-z space look like?*

*What is the transversality condition here?*

The Hamiltonian for this problem is
$H = \ln\left[z_t 4x_t\right] + \lambda\left[4x_t\left(1-z_t\right)\right]$

Maximum conditions:

1. $\frac{\partial H}{\partial z} = \frac{1}{z_t} - \lambda 4x_t = 0$ (check $2^{nd}$ order condition)
2. $\dot{\lambda} = -\frac{\partial H}{\partial x} = -\left[\frac{1}{x_t} + \lambda 4\left(1-z_t\right)\right]$
3. $\dot{x}_t = \frac{\partial H}{\partial \lambda} = 4x_t\left(1-z_t\right)$
4. $x_1 = e^2$

Simplifying the first equation yields
$\frac{1}{\lambda_t 4x_t} = z_t$ .

**At this point one can almost always get some economic intuition from the solution**
For example, in this problem we find that current consumption is
inversely related to the product of the state and costate variables. *Does this make intuitive sense?*

Substituting for $z_t$ in 2

$$\dot{\lambda}_t = -\frac{1}{x_t} - \lambda_t 4\left(1 - \frac{1}{\lambda_t 4x_t}\right)$$

$$\dot{\lambda}_t = -\frac{1}{x_t} - \left(\lambda_t 4 - \frac{1}{x_t}\right)$$

$$\dot{\lambda}_t = -\lambda_t 4$$

*Can you solve the differential equation to obtain $\lambda_t$ as a function of t?*

Now, substituting for $z_t$ in the state equation, we obtain

$$\dot{x}_t = 4x_t\left(1 - \frac{1}{\lambda_t 4x_t}\right)$$

So our three simplified equation are

5. $\dfrac{1}{\lambda_t 4x_t} = z_t$

6. $\dot{\lambda}_t = -\lambda_t 4$

7. $\dot{x}_t = 4x_t - \dfrac{1}{\lambda_t}$

*Is there an equilibrium where both* $\dot{\lambda}$ and $\dot{x}$ *equal zero?*

Notice that 6 involves one variable, 7 involves two variables and 5 involves three variables. This suggests an order in which we might want to solve the problem – start with 6.

The differential equation in 6 can be solved directly to obtain

8. $\lambda_t = \lambda_0 e^{-4t}$

(where $\lambda_0$ is the constant of integration, but clearly is also the value of $\lambda$ when $t$=0).

[$\Rightarrow$ check $\dot{\lambda}_t = -4\lambda_0 e^{-4t} = -4\lambda_t$ ]

This solution can then be substituted into 7 to get

$$\dot{x}_t = 4x_t - \frac{e^{4t}}{\lambda_0},$$

a linear FODE. Recall the way we solve linear FODE's is as follows.

$$e^{-4t}\left(\dot{x}_t - 4x_t\right) = -\frac{1}{\lambda_0}$$

$$e^{-4t}\dot{x}_t - 4e^{-4t}x_t = -\frac{1}{\lambda_0}$$

We can integrate both sides of this equation over $t$

LHS: $\int\left[e^{-4t}\dot{x}_t - 4e^{-4t}x_t\right]dt = x_t e^{-4t} + A_1$

RHS: $\int -\dfrac{1}{\lambda_0}dt = -\dfrac{t}{\lambda_0} + A_2$

so

$$x_t e^{-4t} + A_1 = -\frac{t}{\lambda_0} + A_2$$

or

$$e^{-4t}x_t = -\frac{t}{\lambda_0} + A$$

or

9. $x_t = -\dfrac{te^{4t}}{\lambda_0} + Ae^{4t}$

where $A$ is an unknown constant.

We are close to the solution, but we aren't finished until the values for all constants of integration have been identified. To do this we use the initial and terminal conditions (a.k.a. transversality condition).
Substituting in, $x_0$=1, and $t$=0, yields

$$1 = -\frac{0 \cdot e^{4\bullet 0}}{\lambda_0} + A \cdot e^{4\bullet 0} = A$$

so $A$=1.

Now use the condition $x_1$=$e^2$

$$e^2 = -\frac{e^{4\cdot 1}}{\lambda_0} + e^{4\cdot 1}$$

$$\frac{e^4}{\lambda_0} = e^4 - e^2$$

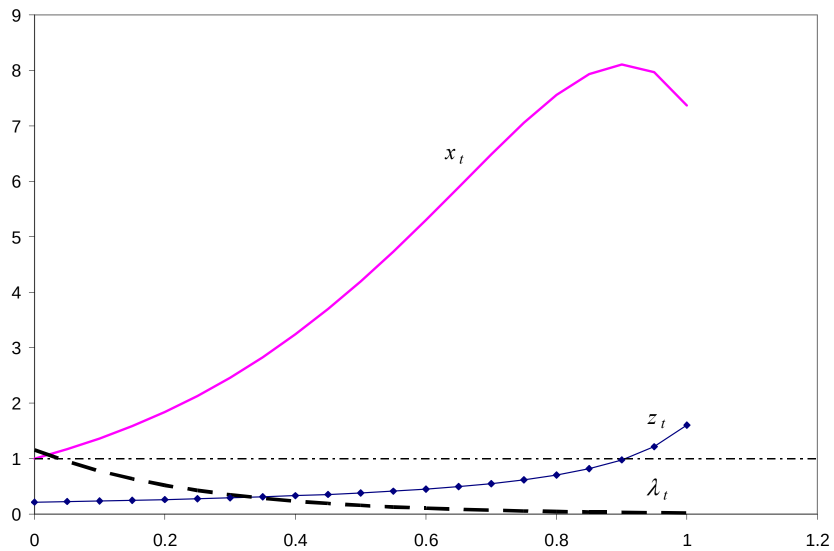
$$\frac{e^4}{e^4 - e^2} = \lambda_0$$

$$\lambda_0 \approx 1.156$$

Now plug the values for $A$ and $\lambda_0$ into 8 and 9 to get the complete time line for $\lambda$ and $x$: $\lambda_t = (1.156)e^{-4t}$ and $x_t = e^{4t}$-.865$te^{4t}$. These can then be substituted into 5 to get

$$z_t = \frac{1}{4.624 - 4t}$$

So this is the solution to the problem can be graphed as follows.




*Are these curves consistent with our intuition?*

**V. An infinite horizon resource management problem**

Consider the case of a fishery in which the stock of fish in a lake, $x_t$, changes continuously over time according to the following equation of motion:

$$\dot{x}_t = ax_t - b(x_t)^2 - z_t$$

where $a$>0 and $b$>0 are parameters of the species' biological growth function and $z_t$ is the rate of harvest. Society's utility comes from fish consumption at the rate ln($z_t$), and the goal is to maximize the discounted present value of its utility over an infinite horizon, discounting at the rate $r$.

A formal statement of the planner's problem, therefore is:

$$\max_{z_t} \int_0^{\infty} e^{-rt} \ln(z_t) \qquad s.t.$$

$$\dot{x}_t = ax_t - bx_t^2 - z_t$$

$$x_t \geq 0$$

We solve this problem using a Hamiltonian:

$$H = e^{-rt} \ln(z_t) + \lambda_t \left(ax_t - b(x_t)^2 - z_t\right)$$

yielding the first-order conditions:

1. $\dfrac{e^{-rt}}{z_t} = \lambda_t$
2. $\lambda_t(a - 2bx_t) = -\dot{\lambda}_t$
3. $\dot{x}_t = \left(ax_t - b(x_t)^2 - z_t\right)$
4. $\lim_{t\to\infty} \lambda_t = 0$

In this case, let's jump directly to the phase diagram exploring the dynamics of the system. The state equation gives tells us the dynamic relationship between $x_t$ and $z_t$. We can use FOCs 1 and 2, to uncover the dynamic relationships of $z_t$. Using 2 we see that

$$-\frac{\dot{\lambda}_t}{\lambda_t} = (a - 2bx_t)$$

We can then use 1 to identify the 1:1 relationship between $\dot{\lambda}_t$ and $\dot{z}_t$:

$$\lambda_t = \frac{e^{-rt}}{z_t}$$

$$\ln(\lambda_t) = -rt - \ln(z_t)$$

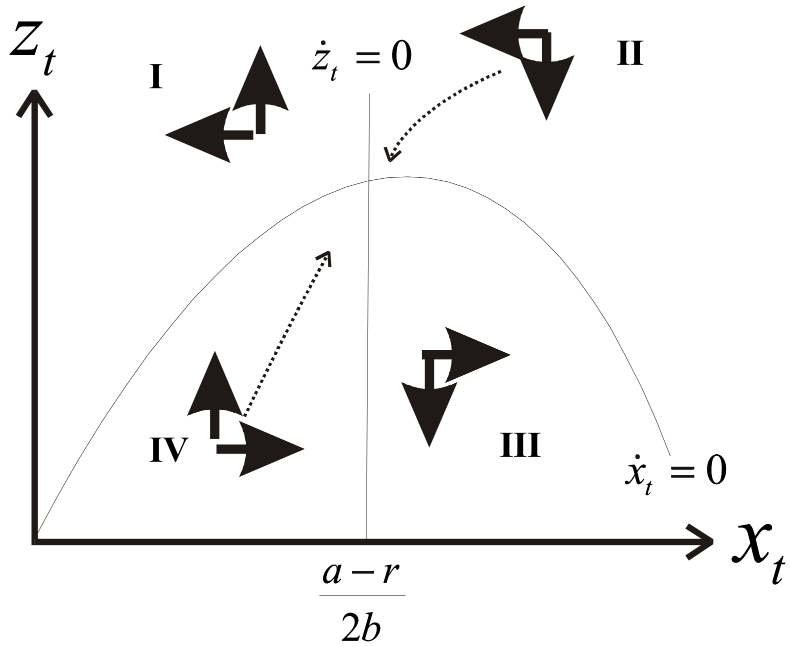
$$\frac{\dot{\lambda}_t}{\lambda_t} = -r - \frac{\dot{z}_t}{z_t}$$

Hence we can write

$$r + \frac{\dot{z}_t}{z_t} = (a - 2bx_t) \Rightarrow \dot{z}_t = (a - r - 2bx_t)z_t .$$

The two equations for our phase diagram, therefore, are

| | and | |
|---|---|---|
| $\dot{z}_t = (a - r - 2bx_t) z_t$ | | $\dot{x}_t = \left(ax_t - b(x_t)^2 - z_t\right)$ |
| $\dot{z}_t \geq 0 \Rightarrow (a - r - 2bx_t) z_t \geq 0$ <br> since, $z_t > 0$ by the $\ln(\cdot)$ function <br> $\Rightarrow a - r - 2bx_t \geq 0$ <br> $\Rightarrow \frac{a-r}{2b} \geq x_t$ | | $\dot{x}_t \geq 0 \Rightarrow ax_t - bx_t^2 - z_t \geq 0$ <br> $\Rightarrow ax_t - bx_t^2 \geq z_t$ |




It is clear from the diagram that we have a saddlepath equilibrium with paths in quadrants II and IV, but all of the dynamics presented in the phase diagram are consistent with the first order conditions 1 – 3. However, we can now use the constraint $x_t \geq 0$ and the transversality condition to show that only points that are actually on the saddlepaths are optimal by ruling out all other points.

First, in quadrant I all paths lead to decreasing values of *x* and increasing values of *z*. Along such paths $\dot{x}_t = ax_t - b(x_t)^2 - z_t$ is negative and growing in absolute value; eventually *x* would have to become negative. But this violates the constraint on *x*; so such paths are not admissible in the optimum.

In quadrant III, harvests are declining and the stock is increasing. Eventually this will lead to a point where *x* reaches the biological steady state where natural growth is zero so harvests, $z_t$ must also be zero. This will occur in finite time. But that means at such a point $\lambda_t = \infty$, which violates the transversality condition. Hence as with quadrant I, no point in quadrant II is consistent with the optimum.

Finally, we can also rule out any point in quadrants II or IV that are not on the saddle path because if the path does not lead to the equilibrium it will cross over to quadrant I or III. Hence, only points on the separatrices are optimal.

## VI. References

Chiang, Alpha C. 1991. *Elements of Dynamic Optimization*. McGraw Hill

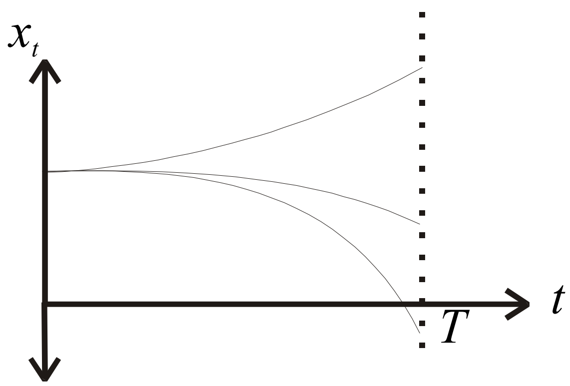
**3. End points and transversality conditions**
002 - Math. Econ. Summer 2012

In this lecture we consider a variety of alternative ending conditions for continuous-time dynamic optimization problems. For example, it might be that the state variable, *x*, must equal zero at the terminal time *T*, i.e., $x_T = 0$, or it might be that it must be less than some function of *t*, $x_T \le \phi(T)$. We also consider problems where the ending time is flexible or $T \rightarrow \infty$. In the process, we will provide a more formal development of Pontryagin's maximum principle.

## I. Transversality conditions for a variety of ending points

(Based on Chiang pp. 181-184)

*A. Vertical or Free-endpoint problems*




By vertical end point, we mean that *T* is fixed and $x_T$ can take on any value. This would be appropriate if you are managing an asset or set of assets over a fixed horizon and it doesn't matter what condition the assets are in when you reach *T*. This case we have considered previously. When looked at from the perspective of the beginning of the planning horizon, the value that *t* takes on at *T* is free and, moreover, it has no effect on what happens in the future. So it is a fully free variable and we would maximize *V* over $x_T$. Hence, it follows that the shadow price of $x_T$ must equal zero, giving us our transversality condition, $\lambda_T = 0$.

We will now confirm this intuition by deriving the transversality condition for this particular problem and at the same time giving a more formal presentation of Pontryagin's maximum principle.

The objective function is

$$V \equiv \int_0^T F(t,x,z) \quad dt$$

now, setting up an equation as a Lagrangian with the state-equation constraint, we have

$$L = \int_0^T \left[ F(t,x,z) + \lambda_t \left( f(t,x,z) - \dot{x}_t \right) \right] dt .$$

We put the constraint inside the integral because it must hold at every point in time. Note that the shadow price variable, $\lambda_t$, is actually not a single variable, but is instead defined at every point in time in the interval 0 to *T*. Since the state equation must be satisfied at each point in time, at the optimum, it follows that $\lambda_t\left(f\left(t,x,z\right)-\dot{x}_t\right)=0$ at each instant *t*, so that the value of *L* must equal the value of *V*. Hence, we might write instead

$$V=\int_0^T\left[F\left(t,x,z\right)+\lambda_t\left(f\left(t,x,z\right)-\dot{x}_t\right)\right]dt$$

or

$$V=\int_0^T\left[\underbrace{\left\{F\left(t,x,z\right)+\lambda_t f\left(t,x,z\right)\right\}}-\lambda_t\dot{x}_t\right]dt$$

$$V=\int_0^T\left[\overbrace{H\left(t,x,z,\lambda\right)}-\lambda_t\dot{x}_t\right]dt$$

It will be useful to reformulate the last term, $\lambda_t\dot{x}_t$, by integrating by parts:

$$\int udv=vu-\int vdu$$

with $\lambda=u$ and $x=v$, so that $dv=\dot{x}$, we get

$$-\int_0^T\lambda_t\dot{x}_t dt=-\left[\lambda_t x_t\right]_0^T+\int_0^T\dot{\lambda}_t x_t dt$$

$$=\int_0^T\dot{\lambda}_t x_t dt+\lambda_0 x_0-\lambda_T x_T$$

so, we can rewrite *V* as

1. $V=\int_0^T\left[H(t,x,z,\lambda)+\dot{\lambda}_t x_t\right]dt+\lambda_0 x_0-\lambda_T x_T$

> **Derivation of the maximum conditions** (Based on Chiang chapter 7)
>
> From 1, we can easily derive the first two conditions of the maximum principal. Assuming an interior solution and twice-differentiability, a necessary condition for an optimum is that the first derivatives of choice variables are equal to zero.
>
> First consider our choice variable, $z_t$. At each point in time it must be that $\partial V/\partial z_t=0$. This reduces to $\partial H/\partial z=0$, which is the first of the conditions stated without proof in lecture 3.
>
> Next, for all $t\in[0,T]$, $x_t$ is also a choice variable in 1, so it must also hold that $\partial V/\partial x_t=0$. This reduces to if $-H_x=\dot{\lambda}$, which is the second of the conditions stated in lecture 3.
>
> Finally, the FOC with respect to $\lambda_t$ is more directly derived from the Lagrangian above. $\partial L/\partial\lambda_t=f\left(t,x,z\right)-\dot{x}_t$, so this implies that $\partial L/\partial\lambda_t=0\Rightarrow\dot{x}_t=f\left(t,x,z\right)$.

If the terminal condition is that $x_T$ can take on any value, then it must be that the marginal value of a change in $x_T$ must equal to zero, i.e., $\partial V/\partial x_T=0$. Hence, the first-order condition with respect to $x_T$ is

$$\frac{\partial V}{\partial x_T}=\int_0^T\left[H_t\frac{\partial t}{\partial x_T}+H_x\frac{\partial x_t}{\partial x_T}+H_z\frac{\partial z_t}{\partial x_T}+H_\lambda\frac{\partial \lambda_t}{\partial x_T}+\dot{\lambda}_t\frac{\partial x_t}{\partial x_T}+x_t\frac{\partial \dot{\lambda}_t}{\partial x_T}\right]dt\quad -\lambda_T=0$$

Several terms in this derivative must equal zero. First, clearly it holds that $\partial t/\partial x_T=0$ so

$$H_t\frac{\partial t}{\partial x_T}=0.$$

Second, as stated above when we converted from $L$ to $V$, $\lambda_t$ will have no effect on $V$ as long as the constraint is satisfied, i.e., as long as the state equation is satisfied. Hence, the terms that involve $\frac{\partial V}{\partial \lambda_t}$ or $\frac{\partial V}{\partial \dot{\lambda}_t}$ can be ignored. Hence,

$$\frac{\partial V}{\partial x_T}=\int_0^T\left[\cancel{H_t\frac{\partial t}{\partial x_T}}+H_x\frac{\partial x_t}{\partial x_T}+H_z\frac{\partial z_t}{\partial x_T}+\cancel{H_\lambda\frac{\partial \lambda_t}{\partial x_T}}+\dot{\lambda}_t\frac{\partial x_t}{\partial x_T}+\cancel{x_t\frac{\partial \dot{\lambda}_t}{\partial x_T}}\right]dt\quad -\lambda_T=0$$

or

$$\frac{\partial V}{\partial x_T}=\int_0^T\left[H_x\frac{\partial x_t}{\partial x_T}+H_z\frac{\partial z_t}{\partial x_T}+\dot{\lambda}_t\frac{\partial x_t}{\partial x_T}\right]dt-\lambda_T=0$$

$$\frac{\partial V}{\partial x_T}=\int_0^T\left[\left(H_x+\dot{\lambda}_t\right)\frac{\partial x_t}{\partial x_T}+H_z\frac{\partial z_t}{\partial x_T}\right]dt\quad -\lambda_T=0$$

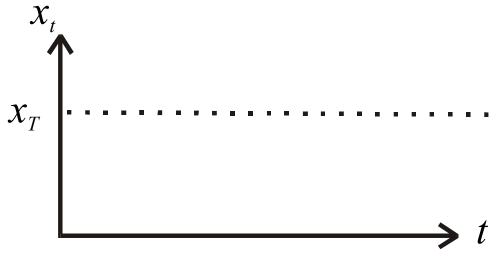
As we derived above, the maximum principle requires that $H_x=-\dot{\lambda}_t$ and $H_z$=0, so both of the terms inside the integral equal zero at the optimum. Hence, we are left with

$$\frac{\partial V}{\partial x_T}=-\lambda_T=0.$$

The minus sign on the LHS is there because it reflects the marginal cost of leaving a marginal unit of the stock at time $T$. In general, we can show that $\lambda_t$ is the value of an additional unit of the stock at time $t$. Setting this FOC equal to zero, we obtain the transversality condition, $\lambda_T$=0.

This confirms our intuition that since we're attempting to maximize $V$ over our planning horizon, from the perspective of the beginning of that horizon $x_T$ is a variable to be chosen, it must hold that $\lambda_T$, the marginal value of an additional unit of $x_T$, must equal zero. Note that this is the marginal value to $V$, i.e., to the sum of all benefits over time for 0 to $T$, not the value to the benefit function, $F(\cdot)$. Although an additional unit may add value if it arrived at time $T$, i.e., $\partial F(\cdot)/\partial x_T>0$, the costs that are necessary for that marginal unit of $x$ to arrive at $T$ must exactly balance the marginal benefit.

*B. Horizontal terminal line or fixed-endpoint problem*




In this case there is no fixed endpoint, but the ending state variables must have a given level. For example, you can keep an asset as long as you wish, but at the end of your use it must be in a certain state. Again, we will use equation 1:

$$V = \int_0^T \left[H(t,x,z,\lambda) + \dot{\lambda}_t x_t\right]dt + \lambda_0 x_0 - \lambda_T x_T \; .$$

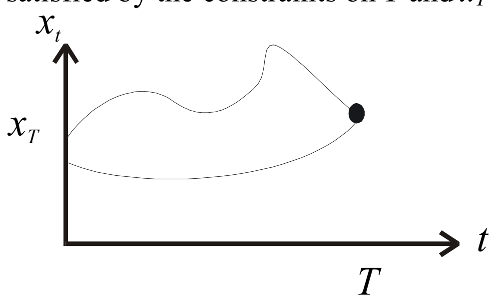
Now, if we have the right terminal time, it must be the case that $\partial V/\partial T=0$, for otherwise it would certainly be the case that a change in $T$ would increase $V$; if $V/\partial T>0$ we would want to increase the time horizon, and if $V/\partial T<0$ it should be shortened. (Note that this is a necessary, but not sufficient condition -- for the sufficient condition we'll have to wait until we introduce an infinite horizon framework). Evaluating this derivative (remember Leibniz's rule), we get -

$$\frac{\partial V}{\partial T} = \left[H\left(T, x_T, z_T, \lambda_T\right) + \dot{\lambda}_T x_T\right] - \left(\dot{\lambda}_T x_T + \lambda_T \dot{x}_T\right) = 0$$

The second and third terms cancel and, since we are restricted to have $x_T$ equal to a specific value, it follows that $\dot{x}_T = 0$. Hence, the condition reduces to $H(T,x_T,z_T,\lambda_T)=0$, i.e., $H=F(T,x_T,z_T)+\lambda_T(f(T, x_T,z_T))=0$

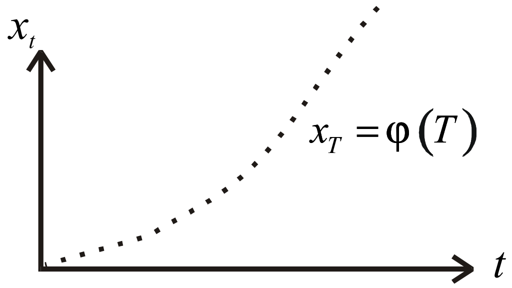
*C. Fixed Terminal Point*

In this case both $x_T$ and $T$ are fixed. Such would be the case if you're managing the asset and, at the end of a fixed amount of time you have to have the asset in a specified condition. A simple case: you rent a car for 3 days and at the end of that time the gas tank has to have 5 gallons in it. There's nothing complicated about the transversality condition here, it is satisfied by the constraints on $T$ and $x_T$ , i.e. $x_3=5$.




When added to the other optimum criteria, this transversality equation gives you enough equations to solve the system and identify the optimal path.

*D. Terminal Curve*




In this case the terminal condition is a function, $x_T = \varphi(T)$. Again, we use

1 $\quad V = \int_0^T \left[ H(t,x,z,\lambda) + \dot{\lambda}_t x_t \right] dt \;\; + \lambda_0 x_0 - \lambda_T x_T$ .

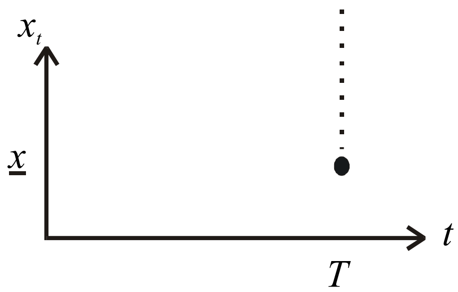
Taking the derivative with respect to $T$ and substituting in $\dot{x}_T = \phi'(T)$

$$\frac{\partial V}{\partial T} = H(T, x_T, z_T, \lambda_T) \quad + \dot{\lambda}_T x_T - \dot{\lambda}_T x_T - \lambda_T \phi'(T) = 0$$

which can be simplified to the transversality condition,

$$\frac{\partial V}{\partial T} = H(T, x_T, z_T, \lambda_T) - \lambda_T \phi'(T) = 0$$

*E. Truncated Vertical Terminal Line*




In this case the terminal time is fixed, but $x_T$ can only take on a set of values, e.g. $x_T \geq \underline{x}$. This would hold, for example, in a situation where you are using a stock of inputs that must be used before you reach time $T$ and $x_T \geq 0$. You can use the input from 0 to $T$, but $x_t$ can never be negative.

For such problems there are two possible transversality conditions. If $x_T > \underline{x}$, then the transversality condition $\lambda_T = 0$ applies. On the other hand, if the optimal path is to reach the constraint on $x$, then the terminal condition would be $x_T = \underline{x}$. In general, the Kuhn-Tucker specification is what we want. That is, our maximization objective is the same, but we now have an inequality constraint, i.e., we're seeking to maximize

$V = \int_0^T \left[ H(t,x,z,\lambda) + \dot{\lambda}_t x_t \right] dt \; + \lambda_0 x_0 - \lambda_T x_T$ s.t. $x_T \geq \underline{x}$.

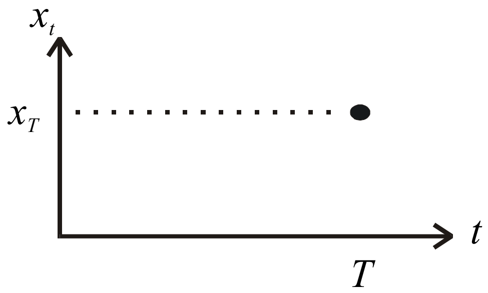
The Kuhn-Tucker conditions for the optimum then are:

$\lambda_T \geq 0$, $x_T \geq \underline{x}$, and $(x_T - \underline{x})\lambda_T = 0$

where the last of these is the complementary slackness condition of the Kuhn-Tucker conditions.

As a practical matter, rather than burying the problem in calculus and algebra, I suggest that you would typically take a guess, Is $x_T$ going to be greater than $\underline{x}$? If you think it is, then solve, the problem first using $\lambda_T$=0. If your solution leads to $x_T \geq \underline{x}$, you're done. If not, substitute in $x_T = \underline{x}$ and solve again. This will usually work. *When would this approach not work?*

*F. Truncated Horizontal Terminal Line*




In this case the time is flexible up to a point, e.g., $T \leq T_{max}$, but the state is fixed at a given level, say $x_T$ is fixed. Again there are two possibilities, $T = T_{max}$ or $T < T_{max}$. Using the horizontal terminal line results from above, the transversality condition takes on a form similar to the Kuhn-Tucker conditions above,
$T \leq T_{max}$, $H(T, x_T, z_T, \lambda_T) \geq 0$, and $(T - T_{max})H_T = 0$.

## II. First, a word on salvage value

The problems above have assumed that all benefits and costs accrue during the planning horizon. However, for finite horizon problems it is often the case that there are benefits or costs that are functions of $x_T$ at $T$. For example, operating a car is certainly a dynamic problem and there is typically some value (perhaps negative) to your vehicle when you're finally finished with it. Similarly, farm production problems might be thought of as a dynamic optimization problem in which there are costs during the growing season, followed by a salvage value at harvest time.

Values that accrue to the planner outside of the planning horizon are referred to as *salvage values*. The general optimization problem with salvage value becomes

$$\max_z \int_0^T F(t,x,z)\,dt \quad + S(x_T, T) \quad \text{s.t.}$$
$$\dot{x}_t = f(t,x,z)$$
$$x_0 = x_0$$

Rewriting equation 1 with the salvage value, we obtain:

1′ $$V = \int_0^T \left[ H(t,x,z,\lambda) + \dot{\lambda}_t x_t \right] dt + \lambda_0 x_0 - \lambda_T x_T + S(T, x_T).$$

Following the same derivation as for the vertical end-point problem above, we can obtain

$$\lambda_T = \frac{\partial S(T, x_T)}{\partial x_T}.$$

Intuitively, this makes sense: $\lambda_T$ is the marginal value of the stock and $\frac{\partial S(T, x_T)}{\partial x_T}$ is the marginal value of the stock outside the planning horizon. When these are equal, it means that the marginal value of the stock over the planning horizon is equal to zero and all of the value is captured by the salvage value.

Note that the addition of the salvage value does not affect the Hamiltonian, nor will it affect the first 3 of the criteria that must be satisfied. *What would be the transversality condition for a horizontal end-point problem with a salvage value?*

## III. An important caveat

Most of the results above will not hold exactly if there are additional constraints on the problem or if there is a salvage value. However, you should be able to derive similar transversality conditions equation 1 and similar logic.

## IV. Infinite horizon problems

It is frequently the case (I would argue, usually the case) that the true problem of interest has an infinite horizon. The optimality conditions for an infinite horizon problem are identical to those of a finite horizon problem with the exception of the transversality condition. Hence, in solving the problem the most important change is how we deal with the need for the transversality conditions. [Obviously, in infinite horizon problems the mnemonic of *transversing* to the other side doesn't really work because there is no "other side" to which we might transverse.]

### *A. Fixed finite x*

If we have a value of $x$ to which we must arrive, i.e., $x_\infty \equiv \lim_{t\to\infty} x_t = k$, then the problem is identical to the horizontal terminal line case considered above.

### *B. Flexible $x_T$*

Recall from above that for the finite horizon problem we used equation 1:

$$V = \int_0^T \left[ H(t, x, z, \lambda) + \dot{\lambda}_t x_t \right] dt + \lambda_0 x_0 - \lambda_T x_T .$$

In the infinite horizon case this equation is rewritten:

$$V = \int_0^\infty \left[ H(t, x, z, \lambda) + \dot{\lambda}_t x_t \right] dt + \lambda_0 x_0 - \lim_{t\to\infty} \lambda_t x_t$$

and, for problem in which $x_\infty$ is free, the condition analogous to the transversality condition in the finite horizon case is $\lim_{t\to\infty} \lambda_t = 0$. Note that if our objective is to maximize the present-value of benefits, this means that the **present value** of the marginal value of an additional unit of $x$ must go to zero as $t$ goes to infinity. Hence, the current value (at time $t$) of an additional unit of $x$ must either be finite or grow at a rate slower than $r$ so that the discount factor, $e^{-rt}$, pushes the present value to zero.

One way that we frequently present the results of infinite horizon problems is to evaluate the equilibrium where $\dot{\lambda} = \dot{x} = 0$. Using these equations (and evaluating convergence and stability via a phase diagram) we can then solve the problem. See the fishery problem in Lecture 3.

## V. Summary

The central idea behind all transversality conditions is that if there is any flexibility at the end of the time horizon, then the marginal benefit from taking advantage of that flexibility must be zero at the optimum. You can apply this general principal to problems with more than one variable, to problems with constraints and, as we have seen, to problems with a salvage value.

## 4. An economic understanding of optimal control as explained by Dorfman (1969)

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The purpose of this lecture and the next is to help us understand the intuition behind the optimal control framework. We draw first on Dorfman's seminal article in which he explained OC to economists. .

***(For this lecture, I will use Dorfman's notation so k is the state variable and x is the choice variable)***

### *A. The problem*

Dorfman's problem is to maximize

(1) $W\left(k_t, \vec{x}\right) = \int_t^T u\left(k, x, \tau\right) d\tau$

where $\vec{x}$ is the stream of all choices made between $t$ and $T$.
the state equation is

$$\dot{k} = \frac{\partial k}{\partial t} = f\left(k, x, t\right)$$

### *B. Step 1. Divide time into two pieces*

In order to help us understand this problem, Dorfman divides the time from $t$ to $T$ into two pieces, from $t$ to $t+\Delta$ and from $t+\Delta$ to $T$. If $\Delta$ is small, then there is little loss of accuracy if we linearize utility over the interval from $t$ to $t+\Delta$, i.e., assume that $u(k,x,t)$ is constant over this interval. Technically, all the "=" signs below should be replaced by "≈" signs, but we will assume the approximation error is trivial. Hence, we rewrite

$$W\left(k_t, \vec{x}\right) = u\left(k, x_t, t\right) \cdot \Delta + \int_{t+\Delta}^T u\left(k, x, \tau\right) d\tau$$

Let's look just at this second term. If we assume that we maximize over the second interval from $t+\Delta$ to $T$, then we can eliminate the control variable, $\vec{x}$, from the second term to obtain

$$V^*\left(k_{t+\Delta}, t+\Delta\right) = \max_{\vec{x}} W\left(k_{t+\Delta}, \vec{x}, t+\Delta\right) = \int_{t+\Delta}^T u\left(k^*, x^*, \tau\right) d\tau,$$

where $k^*$ and $x^*$ are the optimal paths of the state and control variables.

Following a policy of $x_t$ constant for the initial period from $t$ to $t+\Delta$, and then optimizing beyond that point can then be written

(2) $V\left(k_t, x_t, t\right) = u\left(k_t, x_t, t\right)\Delta + V^*\left(k_{t+\Delta}, t+\Delta\right)$.

(note that the $V$ on the LHS does not have a *, i.e., it is not necessarily at the optimum).

### *C. Step 2. Evaluate the FOC w.r.t. the control variable, $x_t$*

Problem (2) can be solved by applying standard tools of calculus. Dorfman takes the FOC, directly with respect to the choice variable $x_t$

(3) $$\Delta\frac{\partial}{\partial x_t}u(k,x_t,t)+\frac{\partial}{\partial x_t}V^*(k_{t+\Delta},t+\Delta)=0.$$

We can then rewrite the second term

(4) $$\frac{\partial V^*}{\partial x_t}=\frac{\partial V^*}{\partial k_{t+\Delta}}\frac{\partial k_{t+\Delta}}{\partial x_t}$$

Since we assume that the interval Δ is quite short, we can approximate the state equation

$k_{t+\Delta}=k_t+\dot{k}\Delta=k_t+f(k,x_t,t)\Delta$

so that

(5) $$\frac{\partial k_{t+\Delta}}{\partial x_t}=0+\frac{\partial f}{\partial x_t}\Delta$$

Dorfman then substitutes (5) into (4), and also writes $V'=\lambda$, so that (3) can be rewritten

$$\Delta\frac{\partial u}{\partial x_t}+\lambda_{t+\Delta}\Delta\frac{\partial f}{\partial x_t}=0.$$

> Note: we can get the same results if we start with a Lagrangian, i.e.,
>
> $L=u(k_t x_t,t)\Delta+V^*(k_{t+\Delta},t+\Delta)-\lambda_{t+\Delta}\left(k_{t+\Delta}-\left(k_t+f(k,x_t,t)\Delta\right)\right)$
>
> and then the FOCs would be,
>
> $\Delta\frac{\partial u}{\partial x_t}+\lambda_{t+\Delta}\Delta\frac{\partial f}{\partial x_t}=0$, and
>
> $\frac{\partial V(\cdot)}{\partial k_{t+\Delta}}=\lambda_{t+\Delta}$.
>
> In the context of the Lagrangian we know that $\lambda$ is the value of marginally relaxing the constraint, i.e., the change in $V$ that would be achieved by an extra unit $k$. Hence, $V'$ and $\lambda$ are equivalent.

If we take the limit as Δ→0, $\lambda_{t+\Delta}=\lambda_t$. Then Δ can then be canceled to obtain

(6) $$\frac{\partial u}{\partial x_t}=-\lambda_t\frac{\partial f}{\partial x_t}$$

This is the first of the optimality conditions of the maximum principle, (i.e., $\partial H/\partial z=0$).
Dorfman (822-23) provides a clear and succinct economic interpretation of this term:

> [Equation (6)] says that the choice variable at every instant should be selected so that the marginal immediate gains are in balance with the value of the marginal contribution to the accumulation of capital.

Put another way, $z$ should be increased as long as the marginal immediate benefit is greater than the marginal future costs. In problems where $z$ is discrete or constrained, it may not be possible to actually achieve the equi-marginal condition, but the intuition remains the same.

So now we've got a nice intuitive explanation for the first of the maximum conditions. *The central principle of dynamic optimization is that optimal choices are made when a balance is struck between the immediate and future marginal consequences of our choices.*

*D. Step 3. Look at the value of $\lambda_t$ by taking $\partial V^*/\partial k_t$*

We now assume that the optimal choice of $x$ has been made over our short interval, $t$ to $\Delta$.

$$V^*(k_t, t) = u(k_t, x_t^*, t)\Delta + V^*(k_{t+\Delta}, t+\Delta)$$

Differentiating this expression w.r.t. $k$ and substituting $\lambda_t$ for $V_t'$, we get

$$\lambda_t = \Delta\frac{\partial u}{\partial k} + \frac{\partial}{\partial k}V^*(k_{t+\Delta}, t+\Delta)$$

$$\lambda_t = \Delta\frac{\partial u}{\partial k} + \frac{\partial V^*(k_{t+\Delta}, t+\Delta)}{\partial k_{t+\Delta}}\frac{\partial k_{t+\Delta}}{\partial k}$$

$$\lambda_t = \Delta\frac{\partial u}{\partial k} + \lambda_{t+\Delta}\frac{\partial k_{t+\Delta}}{\partial k}$$

Since this is over a short period, we can approximate

$\lambda_{t+\Delta} = \lambda_t + \dot{\lambda}\Delta$ and $k_{t+\Delta} = k_t + \Delta\dot{k}$, so that $\dfrac{\partial k_{t+\Delta}}{\partial k_t} = 1 + \Delta\dfrac{\partial f}{\partial k}$

Hence,

$$\lambda_t = \Delta\frac{\partial u}{\partial k} + (\lambda_t + \dot{\lambda}\Delta)\left(1 + \Delta\frac{\partial f}{\partial k}\right)$$

$$\cancel{\lambda_t} = \Delta\frac{\partial u}{\partial k} \cancel{+\lambda_t} + \dot{\lambda}\Delta + \lambda_t\Delta\frac{\partial f}{\partial k} + \dot{\lambda}\Delta^2\frac{\partial f}{\partial k}$$

$$0 = \cancel{\Delta}\frac{\partial u}{\partial k} + \dot{\lambda}\cancel{\Delta} + \lambda_t\cancel{\Delta}\frac{\partial f}{\partial k} + \dot{\lambda}\Delta^{\cancel{2}}\frac{\partial f}{\partial k}$$

or,

$$-\dot{\lambda} = \frac{\partial u}{\partial k} + \lambda_t\frac{\partial f}{\partial k} + \dot{\lambda}\Delta\frac{\partial f}{\partial k}.$$

Taking the limit at $\Delta \to 0$, the last term falls out and we're left with

(7) $$-\dot{\lambda} = \frac{\partial u}{\partial k} + \lambda\frac{\partial f}{\partial k}$$

which is the second maximum condition, $-\dot{\lambda} = \partial H/\partial k$.

What does Dorfman (p. 821) tell us about the economic intuition behind this equation?

> To an economist, it $[\dot{\lambda}]$ is the rate at which the capital is appreciating.
>
> $-\dot{\lambda}$ is therefore the rate at which a unit of capital depreciates at time $t$. … In other words, [1] a unit of capital loses value or depreciates as time passes at the rate at which its potential contribution to profits becomes its past contribution. … [or] [2] Each unit of the capital good is gradually decreasing in value at precisely the same rate at which it is giving rise to

valuable outputs. [3] We can also interpret $-\dot{\lambda}$ as the loss that would be incurred if the acquisition of a unit of capital were postponed for a short time [which at the optimum must be equal to the instantaneous marginal value of that unit of capital].

So we see that since the value of the capital stock at the beginning of the problem is equal to the sum of the contributions of the capital stock across time. As we move across time, therefore, the capital stock's ability to contribute to $V$ is "used up".

*E. Step 4. Summing up*

Hence, each of the optimality conditions associated with the Hamiltonian has a clear economic interpretation.
Let $H = u(k,x,t) + \lambda_t f(k,x,t)$

| **FOC#** | **Equation** | **Interpretation** |
|---|---|---|
| Choice | $\frac{\partial H}{\partial x} = 0$ | Finds the optimal balance between current and future welfare. |
| State | $\frac{\partial H}{\partial k} = -\dot{\lambda}$ | The marginal value of the state variable is decreasing at the same rate at which it is generating benefits. |
| Costate | $\frac{\partial H}{\partial \lambda} = \dot{k}$ | The state equation must hold. |

## II. A word about discounting

**Discounting:** Recall that if $r$ is the annual rate of discount, then $(1+r)^{-T}$ is the discount factor applied to benefits or costs $T$ years in the future. If we break each year into $n$ periods, then the periodic discount factor becomes $r/n$ so over $n$ periods (i.e., a year) the one-year discount factor becomes $(1+r/n)^{-n}$. As $n\to\infty$, this converges to $e^{-r}$, the continuous-time discount factor.

Consider a modification of Dorfman's problem with the assumption that we will maximize the present value of $u(k,x,t)=e^{-rt}w(k,x)$ over the interval 0 to $T$, i.e.,

$$W = \int_0^T e^{-rt} w(k,x)\,dt$$

This is a restrictive specification of (1), so the optimality conditions must still hold. The Hamiltonian now is

(8) $$H = e^{-rt} w(k,x) + \lambda_t f(k,x,t)$$

The interpretation of $\lambda_t$ is the same: it is a measure of the contribution to $W$ of an additional unit of $k$ in period $t$. However, because of discounting there is a tendency for $\lambda_t$ to fall over time. If $W_t$ is the present value (back to year zero) of all the benefits from $t$ to $T$, then $W_t$ will tend to be much smaller far in the future than it is for $t$ close to zero. Correspondingly, $\partial W_t/\partial k_t = \lambda_t$ will also tend to fall over time.

Hence, the value of $\lambda_t$ is influenced by two effects: the current (in period $t$) marginal value of $k$, which could either be increasing or decreasing, and the discounting effect, which is always falling. Hence, even if the marginal value of capital is increasing over time (in current dollars), $\lambda$ might be falling. Because of these two factors, it often happens that the economic meaning of $\lambda_t$ is not easily seen. An alternative way to specify discounted optimal control problems that leads to more helpful solution is called the *current value Hamiltonian*.

*A. The Current Value Hamiltonian*

We begin by defining an alternative shadow price variable, $\mu_t$, which is equal to the value of an additional unit of $k$ to the benefit stream, valued in period $t$ units, i.e.,

$\mu_t=e^{rt}\lambda_t$

that is to get $\mu_t$ we have to **<u>in</u>**flate $\lambda_t$ to get it into period $t$ (current) values.

How could we solve for $\mu_t$ directly in Dorfman's model?

The current value Hamiltonian is obtained by inflating (8) to obtain

(9) $H_c = w(k,x)+\mu_t f(k,x,t)= H\cdot e^{rt}$ .

As a simple matter of algebra, we can derive the maximum conditions corresponding to $H_c$ and $\mu$ instead of $H$ and $\lambda$.

The first condition, can be rewritten,

$$\frac{\partial H}{\partial x}=e^{-rt}\frac{\partial H_c}{\partial x}$$

so, $\frac{\partial H}{\partial x}=0$ if and only if $\frac{\partial H_c}{\partial x}=0$.

Hence the analogous principle holds w.r.t. the control variable, i.e.,

1') $\frac{\partial H_c}{\partial x}=0$

or, more generally, maximize $H_c$ with respect to $x$.

Now look at the FOC w.r.t. the state variable:

The standard formulation is

$$\frac{\partial H}{\partial k}=-\dot{\lambda}\,.$$

Looking at the LHS of this equation, we see that for the current value Hamiltonian, $H_c$,

$$\frac{\partial H}{\partial k}=e^{-rt}\frac{\partial H_c}{\partial k}$$

and, on the RHS, since $\lambda_t=e^{-rt}\mu_t$

$$-\dot{\lambda}=-\left(-re^{-rt}\mu_t+e^{-rt}\dot{\mu}_t\right)=re^{-rt}\mu_t-e^{-rt}\dot{\mu}_t$$

Putting the LHS and RHS together, we get

$$\begin{aligned}\frac{\partial H}{\partial k} &= -\dot{\lambda}_t\\ e^{-rt}\frac{\partial H_c}{\partial k} &= re^{-rt}\mu_t-e^{-rt}\dot{\mu}_t\end{aligned}$$

cancelling $e^{-rt}$ gives the second optimum condition,

2') $\frac{\partial H_c}{\partial k} = r\mu_t - \dot{\mu}_t$.

Obviously the third condition, that the state equation must hold, remains unchanged. The Transversality condition might change by a discount factor, but in many cases analogous conditions hold. For example, if the TC is $\lambda_T = 0$, and $\lambda_T = \mu_T e^{-rT}$ then it must also hold that $\mu_T = 0$. (Note that if $T$=∞, then for $r$>0, this would be satisfied if $\mu_t$ does not go to infinity as $t \rightarrow \infty$).

Hence, we can use the current value Hamiltonian, but it is important to use the correct optimality conditions.

In summary: We seek to maximize

$W = \int_0^T e^{-rt} w(k,x)dt$ subject to the state equation $\dot{k} = f(k,x,t)$.

We can do this using the vehicle of the current value Hamiltonian,

$H_c = w(k,x) + \mu_t f(k,x,t)$.

where the maximum criteria are:

1') $\frac{\partial H_c}{\partial x} = 0$

2') $\frac{\partial H_c}{\partial k_t} = r\mu_t - \dot{\mu}_t$

3') $\frac{\partial H_c}{\partial \mu} = \dot{k}$

*B. An economic interpretation of the current-value Hamiltonian*

As in the standard case, the condition that $H_c$ be maximized over time requires that we strike a balance at every point in time; the only difference is that now we're considering this tradeoff at future points in time, rather than in present value terms.

The second condition is a bit trickier. Recall that 2' requires

$$\frac{\partial H_c}{\partial k} = \frac{\partial u}{\partial k} + \mu \frac{\partial f}{\partial k} = r\mu - \mu$$

which we will rewrite

$$\frac{\partial u}{\partial k} + \mu \frac{\partial f}{\partial k} + \dot{\mu} = r\mu$$

The three terms of LHS of this equation reflect the benefits of holding a marginal unit of the capital stock for an instant longer. The first term indicates the marginal immediate benefit of the capital stock. The second term is the capital stock's marginal value, in terms of its contribution to future benefits. Finally, the third term indicates that the marginal value of the capital increases over time. The sum of these three tell us the benefit of holding a marginal unit of capital for one more instant. The RHS of $r\mu$, can be thought of as the opportunity cost of holding capital. For example, suppose that our capital good can be easily transformed into dollars and we discount at the rate $r$ because it is the market interest rate. Then $r\mu$ is the immediate opportunity cost of holding capital, since we could sell it and earn interest at the rate $r$. Hence, at the optimum, we will hold our state variable up to the point where its marginal value is equal to the marginal cost.

### *C. Summary*

The current value formulation is very attractive for economic analysis because current values are usually more interesting than discounted values. For example, in a simple economy, the market price of a capital stock will equal the current-value co-state variable. As economists we are usually more interested in such actual prices than we are in their discounted present value. Hence, very often the current-value Hamiltonian is more helpful than the present-value variety.

Also, as a practical matter, for analysis it is often the case that the differential equation for $\mu$ will be autonomous (independent of $t$) while that for $\lambda$ will not be. Hence, the dynamics of a system involving $\mu$ can be interpreted using phase-diagram and steady-state analysis, while this does not hold for $\lambda$.

One note of caution: we have stated and derived many of the basic results for the present-value formulation (e.g., transversality conditions). When you are using the current-value formulation, you need to be careful to ensure that everything is modified consistently.

## III. Reference

Dorfman, Robert. 1969. An Economic Interpretation of Optimal Control Theory. American Economic Review 59(5):817-31.

**5. Lessons in the optimal use of natural resource from optimal control theory**
002 - Math. Econ. Summer 2012

## I. The model of Hotelling 1931

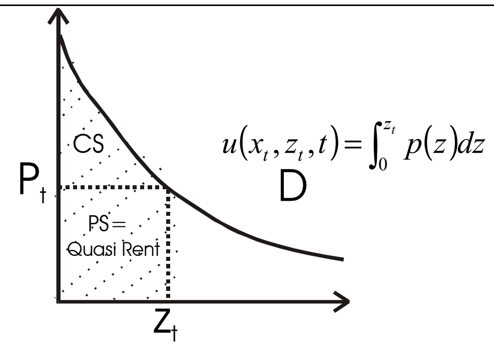
Hotelling's 1931 article, "The Economics of Exhaustible Resources" is a classic that provides very important intuition that applies not only to natural resources, but any form of depletable asset. Hotelling does not use the methodology of optimal control (since it wasn't discovered yet), but this methodology is easily applicable to the problem.

### *A. The basic Hotelling model*

Hotelling considers the problem of a depletable resource and how might it be optimally used over time. *What are the state and control variables of such a problem?*

Let $x_t$ be the stock of the resource remaining at time $t$ and let $z_t$ be the rate at which the stock is being depleted. For simplicity, first assume that extraction costs are zero, and that the market is perfectly competitive. In this case, the representative owner of the resource will receive $p_t z_t$ from the extraction of $z_t$ in period $t$ and this will be pure profit or, more accurately, *quasi-rents*.

> **Definitions** (from http://www.bized.ac.uk/)
>
> *Economic rent*: A surplus paid to any factor of production over its supply price. Economic rent is the difference between what a factor of production is earning (its return) and what it would need to be earning to keep it in its present use. It is, in other words, the amount a factor is earning over and above what it could be earning in its next best alternative use (its transfer earnings).
>
> *Quasi-rent*: Short-term economic rent arising from a temporary inelasticity of supply.




We consider the problem of a social planner who wants to maximize the present value of consumer surplus plus rents (= producer surplus in this case). CS + PS at any instant in time is equal to the area under the inverse demand curve, i.e., $u(x_t, z_t, t) = \int_0^{z_t} p(z)\,dz$, where $p(z)$ is the inverse demand curve for extractions of the resource.

The problem is constrained by the fact that the original supply of the resource is finite, $x(t=0)=x_0$ and any extraction of the resource will reduce the available stock, $\dot{x}=-z$. We know that in any period $x_t\geq0$ and simple intuition assures us that $x_T=0$. *Do you see why* $x_T=0$*?*

A formal statement of the planner's problem, then, is as follows:

$$\max_{z_t}\int_0^T e^{-rt}u(x_t,z_t,t)\,dt=\max_{z_t}\int_0^T e^{-rt}\left[\int_0^{z_t}p(z)\,dz\right]dt \text{ s.t.}$$

$\dot{x}_t=-z_t$

$x(t=0)=x_0$

$x_t\geq0$

The Hamiltonian of this problem is, therefore,

$H=e^{-rt}u(\cdot)+\lambda(-z_t)$

and the maximization criteria are:

1. $H_z=0$: $\quad e^{-rt}u'(\cdot)-\lambda_t=0\Rightarrow e^{-rt}p(z_t)-\lambda_t=0$

2. $H_x=-\dot{\lambda}$: $\quad -\dot{\lambda}=0$

3. $H_\lambda=\dot{x}$: $\quad \dot{x}_t=-z_t$

The transversality condition in this case is found by the terminal point condition,

4. $x_T=0$

Looking at 1 and using the intuition developed by Dorfman, we see that the marginal benefit of extraction in $t$, $e^{-rt}p(z_t)$, must be equal to the marginal cost in terms of foregone future net benefits, $\lambda_t$.

From 2 we see that $\lambda$ is constant at, say, $\lambda_0$ so we can drop the subscript. This is true in any dynamic optimization problem in which neither the benefit function nor the state equation depend on the state variable. This too is consistent with the intuition of Dorfman – since the state variable does not give rise to benefits at $t$, its marginal value does not change over time.

Substituting $\lambda_t=\lambda$ into 1, we obtain

$p(z_t)=\lambda e^{rt}$.

This is important. It shows that the optimal price will grow at the discount rate, and this is true regardless of the demand function (as long as we have an interior solution). [Note that in this example the marginal extraction cost is set at zero so that the price is equal to the marginal quasi-rents earned by the producer. More generally, the marginal quasi-rents would be equal to price minus marginal cost, and this would grow at the rate of interest.]

Another thing that is interesting in this model is that the value of $\lambda$ does not change over time. That means that the marginal increment to the objective function (the whole integral) of a unit of the resource stock never changes. In other words, looking at the entire time horizon, the planner would be completely indifferent between receiving a marginal unit of the resource at time 0 and the instant before *T*, as long as it is known in advance that at some point the unit will be arriving. However, note that this is the present value co-state variable, $\lambda$. *What would the path of the current-value costate variable look like? How does the economic meaning of* $\mu$ *differ from that of* $\lambda$*?*

If we want to proceed further, it is necessary to define a particular functional form for our demand equation. Suppose that $p(z)=e^{-\gamma z}$ so that the inverse demand curve looks like the figure above.

Hence, from 1, $H_z=0 \Rightarrow e^{-rt}e^{-\gamma z_t} = \lambda$, or $e^{-\gamma z_t} = \lambda e^{rt}$ so that,

$$-\gamma z_t = \ln\lambda + rt$$

or

5 $$z_t = -\frac{\ln\lambda + rt}{\gamma}$$

At any point in time it will always hold that $x_t = x_0 + \int_{\tau=0}^{t} \dot{x}_\tau d\tau$. Hence, from our transversality condition, 4,

$$x_T = 0 \Rightarrow x_0 = -\int_{\tau=0}^{T} \dot{x}_\tau d\tau .$$

From 3 and 5 this can be rewritten $\int_0^T z_t dt = x_0$ or $\int_0^T \left( -\frac{\ln(\lambda)+rt}{\gamma} \right) dt = x_0$ .

Evaluating this integral leads to

$$\frac{1}{\gamma}\left( -\ln(\lambda)t - \frac{r}{2}t^2 \right)_0^T = x_0$$

$$\left( -\ln\lambda - \frac{r}{2}T \right)\frac{T}{\gamma} = x_0$$

$$-\ln\lambda = \frac{\gamma}{T}x_0 + \frac{r}{2}T$$

Hence, we can solve for the unknown value of $\lambda$,

$$\lambda = e^{\left( -\frac{\gamma}{T}x_0 - \frac{r}{2}T \right)} .$$

In this case we can then solve explicitly for *z* by substituting into 5, yielding

$$z_t = -\frac{\ln\lambda + rt}{\gamma}$$

$$z_t = -\frac{\ln\left(e^{\left(-\frac{\gamma}{T}x_0 - \frac{r}{2}T\right)}\right) + rt}{\gamma}$$

$$z_t = \frac{\gamma}{\gamma T}x_0 + \frac{r}{\gamma 2}T - \frac{r}{\gamma}t$$

6. $z_t = \frac{x_0}{T} + \frac{r}{2\gamma}T - \frac{r}{\gamma}t$

---

To verify that this is correct, check the integral of this, from 0 to $T$

$$\int_0^T z_t dt = \frac{x_0}{T}T + \frac{r}{2\gamma}T^2 - \frac{r}{2\gamma}T^2 = x_0 .$$

---

Looking at 6, we see that the rate of consumption at any point in time is determined by two parts: a constant portion of the total stock, $\frac{x_0}{T}$, plus a portion that declines linearly over time $\frac{r}{\gamma}\left(\frac{T}{2} - t\right)$. This second portion is greater than zero until $t = \frac{T}{2}$, and is then less than zero for the remainder of the period.

Note that, that $z_T < 0$ if

7. $T > \sqrt{2\gamma x_0 / r}$ .

So that if this inequality is satisfied, along the optimal path defined by 6 $x_t$ will become negative and then needs to be rebuilt so that it reaches zero at $T$. This violates the constraint $x_t \geq 0$. Hence, if 7 holds, we need to re-solve the problem with an explicit constraint on the optimization problem. We will evaluate how to solve such constrained problems later on.

*B. Some variations on the theme and other results*

Hotelling's analysis certainly doesn't end here.

**Q**: Consider again the question, “What would happen if we used the current-value instead of the present-value Hamiltonian?”

**A**: Well, you can be sure that the current value co-state variable, $\mu_t$, would not be constant over time – how would the change in the shadow price of capital evolve? What’s the economic interpretation of $\mu$?

**Q**: What if there are costs to extraction $c(z_t)$ so that the planner's problem is to maximize the area under the demand curve minus the area under the marginal cost curve?

**A**: First recognize that if we define $\tilde{u}(\cdot)=\int_0^{z_t} p(z)-c'(z)dz$ , where *c'* is the marginal cost function, then the general results will be exactly the same as in the original case after substituting "marginal quasi rents" for "price". That is, in this case the marginal surplus will rise at the rate of interest. Obviously getting a nice clean closed-for solution $z^*$ will not be as easy as it was in the first case, but the economic intuition does not change. This economic principle is a central to a wide body of economic analysis.

**Q**: Would the social optimum be achieved in a competitive market?
**A**: First, assuming that both consumers and producers are interested in maximizing the present value of their respective welfare, then we've maximized total surplus, i.e., it is a Pareto Efficient outcome. So we can then ask, *Do the assumptions of the 2nd Welfare Theorem hold? If they do, then what does that tell us about the social optimum?* If these hold, then for a Pareto efficient there exists a price vector for which any Pareto efficient allocation will be a competitive equilibrium. Finding the Pareto optimal allocation also gives a competitive equilibrium. Hence, our findings are not only normative, but more importantly, they're positive; i.e. a prediction of what choices would actually occur in a perfectly competitive economy.

Now, let's look at this question a little more intuitively. We know that one of the basic results is that the price (or marginal quasi rents) grow at the rate of interest? Is this likely to occur in a competitive economy as well? In the words of Hotelling, "it is a matter of indifference to the owner of a mine whether he receives for a unit of his product a price $p_0$ now or a price $p_0e^{\gamma t}$ after time $t$" (p. 140). That is, price takers will look at the future and decide to extract today, or a unit tomorrow at a higher price. The price must increase by at least the rate of interest in this simple model because, if not, the market would face a glut today. If the price rose faster than the rate of interest, then the owners would choose to extract none today. Assuming that the inverse-demand curve is downward sloping, supply and demand can be equal only if each individual is completely indifferent as to when he or she extracts which also explains the constancy of $\lambda$.

This also gets at an important difference between *profit* and *rents*. We all know that in a perfectly competitive economy with free entry, profits are pushed to zero -- so why do the holders of the resource still make money in this case? Because there is not free entry. The total resource endowment is fixed at $x_0$. An owner of a portion of that stock is able to make *resource rents* because he or she has access to a restricted profitable input. Further, the owner is able to exploit the tradeoffs between current and future use to make economic gains. This is what is meant by *Hotelling rents*.

**II. Hartwick's model of national accounting and the general interpretation of the Hamiltonian**

Hartwick (1990) has a very nice presentation of the Hamiltonian's intuitive appeal as a measure of welfare in a growth economy. The analogies to microeconomic problems will be considered at the end of this section. Hartwick's paper builds on Weitzman (1976) and is a generalization of his more often cited 1977 paper.

*A. The general case*

We'll first present the general case and then look at some of the Hartwick's particulars. Consider the problem of optimal growth in an economy maximizing

$$\int_0^\infty U(C)e^{-\rho t}dt$$

subject to a state equation for a malleable capital stock, $x_0$, that can either be consumed or saved for next period

$$\dot{x}_0 = g_0(\mathbf{x},\mathbf{z}) - C$$

and $n$ additional state equations for the $n$ other assets in the economy (e.g., infrastructure, human capital, environmental quality, etc.).

$\dot{x}_i = g_i(\mathbf{x},\mathbf{z})$, $i$=1,…,$n$.

> Please excuse the possibly confusing notation. Here the subscript is an index of the good and the time subscript is suppressed.

**z** is a vector of control variables and $C$ is the numeraire choice variable (think consumption). The vector of state variables is denoted **x**.

The general current value Hamiltonian of this optimization problem is

$$H_c = U(C) + \mu_0\left(g_0(\mathbf{x},\mathbf{z}) - C\right) + \sum_{j=1}^{n}\mu_j g_j(\mathbf{x},\mathbf{z}).$$[1]

This is our first exposure to the problem of optimal control with multiple state and control variables, but the maximization conditions are the simple analogues of the single variable case:

$$\frac{\partial H}{\partial C} = \frac{\partial H}{\partial z_i} = 0 \text{ for all } i$$ [or in general, maximize $H$ with respect $C$ and all the $z_i$'s]

$$\frac{\partial H}{\partial x_j} = \rho\mu_j - \dot{\mu}_j \text{ for all } j$$

$$\frac{\partial H}{\partial \mu_j} = \dot{x}_j \text{ for all } j$$

Given the specification of utility, $\frac{\partial H}{\partial C} = U' - \mu_0 = 0 \Rightarrow \mu_0 = U'$.

(remember, $\mu_0$ is the costate variable on the numeraire good, not the costate variable at $t$=0.)

Similar to the approach used by Dorfman, Hartwick uses a linear approximation of current utility, $U(C) \approx U' \cdot C$, and, if we measure consumption in terms of dollars, $U'$ is the marginal utility of income. He then presents an approximation of the Hamiltonian in terms of the marginal utility of consumption.

[1] Again to write more concisely, $H$ is the current value Hamiltonian, which we typically write $H_c$.

$$\frac{H}{U'} = C + \dot{x}_0 + \sum_{j=1}^{n} \frac{\mu_j}{\mu_0} \dot{x}_j$$

If you look at the RHS of this equation, you will see that this is equivalent to net national product in a closed economy without government. NNP is equal to the value of goods and services ($C$) plus the net change in the value of the assets of the economy, $\left( \dot{x}_0 + \sum_{j=1}^{n} \frac{\mu_j}{\mu_0} \dot{x}_j \right)$.

The first lesson from this model, therefore, is a general one and, as we will discuss below, it carries over quite nicely to microeconomic problems: maximizing the Hamiltonian is equivalent to maximizing NNP, which seems like a pretty reasonable goal.

Using some simplistic economies, Hartwick helps us understand what the appropriate shadow prices on changes in an economy's assets should be, i.e., what are $\frac{\mu_j}{\mu_0}$?

*B. The case of a non-renewable resource*

The first case to consider is an economy in which there are two state variables.

- First there's the fungible capital stock, $x_0$ which we will now call $K$.
- Second, there's a nonrenewable resource or mine, $S$ which falls as the resource is extracted, $R$, and grows when there are discoveries, $D$. Extractions, $R$ are used in the production function $F(\cdot)$ but cost $f(R,S)$.
- Discovery costs rise over time as a function of cumulative discoveries so that the marginal cost of finding more of the resource increases over time. The total cost of discovery in a period is $v(D)$, linearly approximated as $v_D \cdot D$ with $v_D$ changing over time.[2]
- Hartwick also includes labor, $L$, although since the economy is always assumed to be at full employment and the growth rate of labor is exogenous, labor can be treated as an intermediate variable and can, therefore, be largely ignored.

The three state equations are, therefore,

Capital stock: $\dot{K} = F(K,L,R) - C - f(R,S) - v_D D$

Resource Stock: $\dot{S} = -R + D$

Discovery Cost: $\dot{v}_D = g(D)$

and the resulting current value Hamiltonian is

$$H = U(C) + \mu_K \left[ F(K,L,R) - C - f(R,S) - v_D D \right] + \mu_S \left[ -R + D \right] + \mu_D g(D)$$

The FOCs w.r.t. the choice variables are:

$H_C$=0: $U' = \mu_K$

$H_R$=0: $\mu_K \left[ F_R - f_R \right] - \mu_S = 0$.

[2] This is a refinement of the specification in Hartwick (1990) as proposed Hamilton (1994).

$H_D$=0: $-\mu_K v_D + \mu_S + \mu_D g' = 0$.

A linear approximation of the current-value Hamiltonian can be written
$H = U'C + \mu_K \dot{K} + \mu_S[-R + D] + \mu_D g'D$
Dividing by $U'=\mu_k$, we get
$$\frac{H}{U'} = C + \dot{K} - \frac{\mu_S}{\mu_K}R + \frac{\mu_S}{\mu_K}D + \frac{\mu_D}{\mu_K}g'D$$
Using the $H_R$ and $H_D$ conditions, it follows that $\frac{\mu_S}{\mu_K} = [F_R - f_R]$ and

$$\mu_D = \frac{\mu_K v_D}{g'} - \frac{\mu_S}{g'} \text{ or } \mu_D = \frac{\mu_K v_D}{g'} - \frac{\mu_K[F_R - f_R]}{g'}$$
Hence the linear approximation of the Hamiltonian can be rewritten
$$\frac{H}{U'} = C + \dot{K} - \frac{\mu_K[F_R - f_R]}{\mu_K}R + \frac{\mu_K[F_R - f_R]}{\mu_K}D + \left(\frac{\mu_K v_D}{\mu_K g'} - \frac{\mu_K[F_R - f_R]}{\mu_K g'}\right)g'D$$
$$\frac{H}{U'} = C + \dot{K} - [F_R - f_R]R + v_D D$$
We know that in a competitive economy, the price paid for the resource would equal $F_R$ (resources are paid their marginal value product). Hence, to arrive at NNP current 'Hotelling Rents' from extractions, namely $[F_R - f_R]R$, should be netted out of GNP, and discoveries, priced at the marginal cost of discovery, should be added back in.[3]

Is this common practice in national accounting? No. The depreciation of natural resource assets is ignored in the system of national accounts leading to a misrepresentation of national welfare. One reason for this is the ability to actually implement the necessary accounting practice. Hartwick elaborates, "The principal problem of implementing the accounting rule above is in obtaining *marginal* extraction costs for minerals extracted."

*C. An economy with annoying pollution*

The final example that Hartwick presents is that of an economy in which there is a disutility associated with pollution. The case Hartwick considers is where national welfare is affected by changes in the pollution stock. That is, if the stock of pollution is increasing, welfare goes down. If the stock of pollution is falling, welfare goes up. In this case we would have $U = U(C, \dot{X})$, where $\dot{X}$ is the change in the pollution stock, with $\frac{\partial U}{\partial \dot{X}} < 0$.

Production is assumed to be affected by pollution, i.e., $F(K,L,X)$ so, for example, more pollution makes production more difficult. The pollution stock is assumed to increase

[3] This result differs from that presented in Hamilton (1994). I have not attempted to determine where the difference comes from.

with the production at the rate $\gamma$, and decrease with choices made regarding the level of cleanup, $b$, which costs $f(b)$, i.e., $\dot{X} = -bX + \gamma F(K,L,X)$ and the evolution of the numeraire capital stock follows $\dot{K} = F(K,L,R) - C - f(b)$.

The current value Hamiltonian with this stock change incorporated in the utility function, therefore is

$$H = U(C, -bX + \gamma F(K,L,X)) + \mu_K[F(K,L,X) - C - f(b)] + \mu_X[-bX + \gamma F(K,L,X)]$$

Again the FOC w.r.t. the control variables, $C$ and $b$, yield

$$\frac{\partial H}{\partial C} = 0 \Rightarrow U_C = \mu_K$$

$$\frac{\partial H}{\partial b} = 0 \Rightarrow -U_x X - \mu_K f_b - \mu_X X = 0 \Rightarrow \frac{-U_x}{\mu_K} - \frac{f_b}{X} = \frac{\mu_X}{\mu_K}$$

Using the linear approximation of $H$, therefore, yields

$$\frac{H}{U'} = C + \dot{K} + \frac{\mu_X}{\mu_K}\dot{X}$$

$$= C + \dot{K} - \left[\frac{U_X}{\mu_K} + \frac{f_b}{X}\right]\dot{X}$$

Hence, if we want to correctly incorporate changes in the stock of pollution in the calculation of welfare, the price that should be placed on these changes is a function not only of the marginal damage of changes in the stock of pollution, but the marginal cost of clean-up as well.

*D. Implications beyond the realm of national income accounting*

If you're not particularly interested in the national income accounts or environmental and natural resource economics, the above discussion may seem academic. However, clearly, the correct measurement of income is not an academic pursuit limited to the national income accounts.

Hicks' (1939, *Value and Capital*) defined income as, to paraphrase, the maximum amount that an individual can consume in a week without diminishing his or her ability to consume next week. Clearly, just as for a national account, farmers and managers also need to be aware of the distinction between investment, capital consumption, and true income. Hartwick's Hamiltonian formulation of NNP, therefore, with its useful presentation of the correct prices for use in the calculation of income, might readily be applied to a host of microeconomic problems of concern to applied economists.

## III. References

Hartwick, John M. 1977. Intergenerational Equity and the Investing of Rents from Exhaustible Resources. *American Economic Review* 67(5):972-74.

## 6. Optimal control with constraints and MRAP/Bang-Bang problems

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We now return to an optimal control approach to dynamic optimization. This means that our problem will be characterized by continuous time and will be deterministic.

It is usually the case that we are not *Free to Choose*.[1] The choice set faced by decision makers is almost always constrained in some way and the nature of the constraint frequently changes over time. For example, a binding budget constraint or production function might determine the options that are available to the decision maker at any point in time. In general, this implies that we will need to reformulate the simple Hamiltonian problem to take account of the constraints. Fortunately, in many cases, economic intuition will tell us that the constraint will not bind (except for example at $t=T$), in which case our life is much simplified. We consider here cases where we're not so lucky, where the constraints cannot be ruled out ex ante.

We will assume throughout that a feasible solution exists to the problem. Obviously, this is something that needs to be confirmed before proceeding to waste a lot of time trying to solve an infeasible problem.

In this lecture we cover constrained optimal control problems rather quickly looking at the important conceptual issues. For technical details I refer you to Kamien & Schwartz, which covers the technical details of solving constrained optimal control problems in various chapters. We then go on to consider a class of problems where the constraints play a particularly central role in the solution.

### I. Optimal control with equality constraints

#### *A. Theory*

Consider a simple dynamic optimization problem

$$\max_z \int_0^T e^{-rt} u(z,x,t)\,dt \quad \text{s.t.}$$

$$\dot{x} = g(z,x,t)$$

$$h(z,x,t) = c$$

$$x(0) = x_0$$

In this case we cannot use the Hamiltonian alone, because this would not take account of the constraint, $h(z,x,t)=c$. Rather, we need to maximize the Hamiltonian subject to a constraint ⇒ so we use a Lagrangian[2] in which $H_c$ is the objective function, i.e.,

$$L = H_c + \phi\big(h(z,x,t) - c\big)$$

$$= u(z,x,t) + \mu g(z,x,t) + \phi\big(c - h(z,x,t)\big).$$

Equivalently, you can think about embedding a Lagrangian, within a Hamiltonian, i.e.

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[1] This is an obtuse reference to the first popular book on economics I ever read, *Free to Choose* by Milton and Rose Friedman.

[2] This Lagrangian is given a variety of names in the literature. Some call it an augmented Hamiltonian, some a Lagrangian, some just a Hamiltonian. As long as you know what you're talking about, you can pretty much call it whatever you like.

$H_c = u(z,x,t) + \phi(c - h(z,x,t)) + \mu g(z,x,t)$. We'll use the first notation here.

Assuming that everything is continuously differentiable and that concavity assumptions hold, the FOC's of this problem, then, are:

1. $\frac{\partial L}{\partial z} = 0$
2. $\frac{\partial L}{\partial x} = r\mu - \dot{\mu}$

and, of course, the constraints must be satisfied:

$$\frac{\partial L}{\partial \mu} = \dot{x}$$

$$\frac{\partial L}{\partial \phi} = c - h(z,x,t) = 0$$

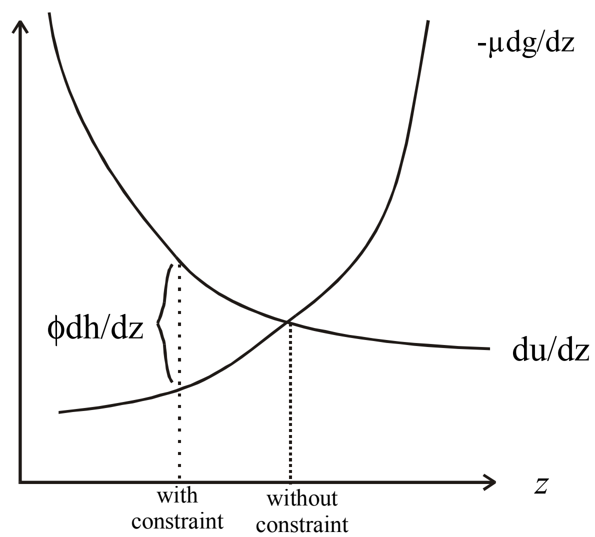
Let's look at these in more detail. The FOC w.r.t. *z* is

1'. $\frac{\partial L}{\partial z} = \frac{\partial u}{\partial z} + \mu \frac{\partial g}{\partial z} - \phi \frac{\partial h}{\partial z} = 0$

which can be rewritten

1". $\frac{\partial u}{\partial z} - \phi \frac{\partial h}{\partial z} = -\mu \frac{\partial g}{\partial z}$.

As Dorfman showed us, the FOC w.r.t. the control variable tells us that at the optimum we balance off the marginal current benefit and marginal future costs. In this case the RHS is the cost to future benefits of a marginal increase in *z*. The LHS, therefore, must indicate the benefit to current utility from marginal increments to *z*. If $\partial u/\partial z$>RHS, then this implies that there is a cost to the constraint and $\phi \partial h/\partial z$ is the cost to current utility of the intratemporal constraint, *h*. If $h(\cdot)$ were marginally relaxed, then *z* could be changed to push it closer to balancing off the value of *z* in the future.



In principle, the problem can then be solved based on these equations. It is important to note that $\phi$ will be a function of time and will typically change over time. *What is the economic significance of $\phi$?*

*B. Optimal control with multiple equality constraints*

The extension to the case of multiple equality constraints, is easy; with n constraints the Lagrangian will take the form

$$L = u(z,x,t) + \lambda g(z,x,t) + \sum_{i=1}^{n} \phi_i \left(c_i - h_i(z,x,t)\right).$$

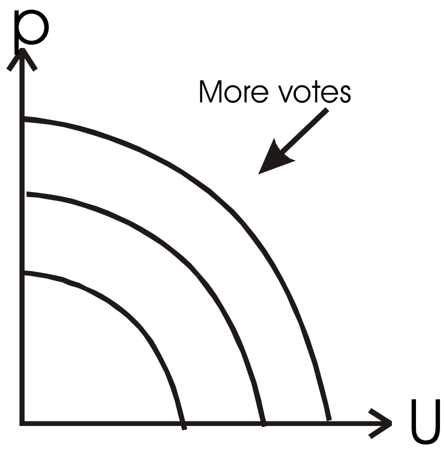
Obviously, if n is greater than the cardinality of $z$, there may not be a feasible solution unless some of the constraints do not bind or are redundant.

*C. Example: The political business cycle model (Chiang's (Elements of Dynamic Optimization) presentation of Nordhaus 1975)*

This model looks at macroeconomic policy. Two policy variables are available, U , the rate of unemployment, and p, the rate of inflation. It is assumed that there is a trade-off between these two so that support for the current administration can be defined by the equation

$$v = v(U,p)$$

so that the relationship between the two policies can be described by the iso-vote curves in the figure below.




Following standard Phillips-curve logic, there is an assumed trade-off between these two objectives,

$$p = \gamma(U) + \alpha\pi$$

where $\pi$ is the expected rate of inflation. Expectations evolve according to the differential equation

$$\dot{\pi} = b(p - \pi)$$

We assume that the votes obtained at time T are a weighted sum of the support that is obtained from 0 to *T*, with support nearer to the voting date being more important. Votes obtained at *T* are equal to $\int_0^T v(U,p)e^{rt}dt$.

The optimization problem then is

$$\max_{U,p} \int_0^T v(U,p)e^{rt}dt \text{ s.t.}$$

$$p = \gamma(U) + \alpha\pi$$

$$\dot{\pi} = b(p - \pi)$$

$$\pi(0) = \pi_0, \text{ and } \pi(T) \text{ free.}$$

Now clearly the first constraint could be used to substitute out for p and convert the problem to a single control problem, but let's consider the alternative, explicitly including the constraint.

The Lagrangian for this optimal control problem would be

$$L = v(U,p)e^{rt} + \lambda(b(p-\pi)) + \phi[\gamma(U) + \alpha\pi - p]$$

The optimum conditions would then be

$$\frac{\partial L}{\partial p} = \frac{\partial v}{\partial p}e^{rt} + \lambda b - \phi = 0$$

$$\frac{\partial L}{\partial U} = \frac{\partial v}{\partial U}e^{rt} + \phi\gamma' = 0$$

$$\frac{\partial L}{\partial \phi} = \gamma(U) + \alpha\pi - p = 0$$

$$\dot{\lambda} = \lambda b - \phi\alpha$$

$$\dot{\pi} = b(p - \pi)$$

If we specify a functional form (see Chiang chapter 7) we can find that the optimal path for policy, which shows that the political process creates a business cycle. In most problems, however, it is easier to find the solution by using equality constraints to eliminate variables before getting started.

## II. Optimal control with inequality constraints

### *A. Theory*

Suppose now that the problem we face is one in which we have inequality constraints, $h_i(t, x, z) \leq c_i$, with $i=1,\ldots,$ n

for n constraints and *x* and *z* are assumed to be vectors of the state and control variables respectively. For each $x_j \in x$, the state equation takes the form $\dot{x}_j = g_j(t,x,z)$.

As with standard constrained optimization problems, the Kuhn-Tucker conditions will yield a global maximum if any one of the Arrow-Hurwicz-Uzawa constraint qualifications is met (see Chiang p. 278). The way this is typically satisfied in most economic problems is for the $g_j$ to be concave or linear in the control variables.

Assuming that the constraint qualification is met, we can then proceed to use the Lagrangian specification using a Hamiltonian which takes the form

$$H = u(t,x,z) + \sum_{j=1}^{m} \lambda_{tj} g_j(t,x,z)$$

which we then plug into the Lagrangian with the constraints,

$$L = u(t,x,z) + \sum_{j=1}^{m} \lambda_{tj} g_j(t,x,z) + \sum_{i=1}^{n} \phi_{ti}(c_i - h_i(t,x,z)).$$

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Note: For maximization problems I always write the constraint term of the Lagrangian so that the argument inside the parentheses is greater than zero, or for minimization problems you write it so that the argument is less than zero. If you follow this rule, your Lagrange multiplier will always be positive.

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The FOC's for this problem are:

$$\frac{\partial L}{\partial z_k} = 0 \Rightarrow \frac{\partial u}{\partial z_k} + \sum_{j=1}^{n} \lambda_{tj} \frac{\partial g_j}{\partial z_k} - \sum_{i=1}^{m} \phi_{ti} \frac{\partial h_i}{\partial z_k} = 0 \quad \text{for all } z_k \in z$$

$$\frac{\partial L}{\partial x_j} = -\dot{\lambda}_{tj} \quad \text{for all j}$$

$$\frac{\partial L}{\partial \lambda_j} = \dot{x}_j$$

and, for the constraints

$$\frac{\partial L}{\partial \phi_i} \geq 0 \Rightarrow h_i(x_t, z_t) \leq c_{ti}$$

with the complementary slackness conditions:

$$\phi_i \geq 0 \text{ and } \phi_i \frac{\partial L}{\partial \phi_i} = 0 \text{ for all } i.$$

As with all such problems, the appropriate transversality conditions must be used and, if you choose to use a current-value Hamiltonian, the necessary adjustments must be made. Note that in the current value specification, the interpretation of both the costate variable and the shadow price on the intratemporal constraint would be altered.

B. Example: Hotelling's optimal extraction problem

We return to Hotelling's problem. The planner's problem is to maximize

$$\max_z \int_0^T e^{-rt} \left[ \int_0^{z_t} p(z)\,dz \right] dt \quad \text{s.t.}$$

$$\dot{x} = -z$$

$$x(0) = x_0, \quad x_t \geq 0$$

Economic intuition tells us that $x_T$=0. Hence, $x_t \geq 0$ for all t if $z_t \geq 0$. Hence, we can convert the problem to one of having a constraint on the control variable. The associated Lagrangian would then be
$L=e^{-rt}u(\cdot)+\lambda(-z_t)+\phi_t\cdot z_t$.

(note that we've converted a state constraint to a control constraint. We cover constraints on the state variable below)

The associated maximization criteria are:
3. $L_z=0$: $\quad e^{-rt}u'(\cdot)-\lambda_t+\phi_t=0 \Rightarrow e^{-rt}p(z_t)-\lambda_t+\phi_t=0$
4. $L_x=-\dot{\lambda}$: $\quad -\dot{\lambda}=0$
5. $L_\lambda=\dot{x}$: $\quad \dot{x}_t=-z_t$
6. $L_\phi \geq 0$: $\quad z_t \geq 0$
7. $\quad \phi_t \geq 0$
8. $\quad \phi_t z_t=0$
The transversality condition is $x_T=0$.

From 4 it still holds that $\lambda$ is constant. However, 3 can be rewritten
$p(z_t)=(\lambda-\phi_t)e^{rt}$.
Using the assumed functional form for inverse demand curve, $p(z)=e^{-\gamma z}$, we obtain $e^{-\gamma z}=(\lambda-\phi_t)e^{rt}$. Taking logs we get $-\gamma z=\ln(\lambda-\phi_t)+rt$
or

9. $$z=-\frac{\ln(\lambda-\phi_t)+rt}{\gamma}.$$

Now, using the complementary slackness conditions, we know that if $z>0$ then $\phi=0$ and if $z=0$, $\phi>0$. The state path can, therefore, be broken into two parts, the first part from 0 to $T_1$ during which $z>0$ and the second part, from $T_1$ to $T$, where $z=0$ and $\phi>0$.

From 0 to $T_1$
$$z=-\frac{\ln(\lambda-0)+rt}{\gamma}=-\frac{\ln(\lambda)+rt}{\gamma}$$
and from $T_1$ to $T$,
$$0=-\frac{\ln(\lambda-\phi_t)+rt}{\gamma}\Rightarrow \ln(\lambda-\phi_t)=rt$$

10. $$\phi_t=\lambda-e^{-rt}.$$

Now, we can speculate about the solution. It seems likely that at $T_1$, $\phi$ will equal zero and will then increase over time from that time onward. If not, then the paths of $z$ and $\phi$ will be discontinuous at $T_1$. So let's use this possibility and then later confirm that it holds.
If $\phi_{T_1}=0$, then

11. $$\lambda_{T_1}=e^{-rT_1}.$$

Furthermore, we must exhaust the resource by $T_1$ so that

$\int_0^{T_1} z_t dt = x_0$ or $\int_0^{T_1} \left( -\frac{\ln(\lambda) + rt}{\gamma} \right) dt = x_0$

Which we solved in lecture 6 to obtain

$\lambda = e^{\left( -\frac{\gamma}{T_1} x_0 - \frac{r}{2} T_1 \right)}$.

Now, substituting in from 11, we obtain

$$e^{-rT_1} = e^{\left( -\frac{\gamma}{T_1} x_0 - \frac{r}{2} T_1 \right)}$$

$$\frac{r}{2} T_1 = \frac{\gamma}{T_1} x_0$$

$$T_1^2 = \frac{2\gamma}{r} x_0$$

$$T_1 = \sqrt{\frac{2\gamma}{r} x_0}$$

Hence, if our assumption that $\phi = 0$ at $T_1$ is valid, the optimal solution is for consumption to decline from 0 to $T_1$ and then stay constant at zero from that point onward.

Is the assumption correct? Without a formal proof, we can see using economic intuition that it is. Suppose $z_{T_1} > 0$. A feasible option would be to reduce $z_{T_1}$ and consume for a little longer. Since $u(\cdot)$ is concave ($u''<0$) it will hold that the marginal cost of a slight reduction in $z$ at $T_1$ will be less than the marginal benefit of a slight increase in $z$ a moment later. Hence, it will never be optimal to consume a strictly positive amount of $z$ at $T_1$ so the assumption that $\phi=0$ at $T_1$ is valid and our solution is the optimum

## III. Constraints on the state space

### *A. Theory*

Suppose now that we have constraints on the state variables which define a feasible range. This is likely to be common in economic problems. You may, for example, have limited storage space so that you cannot accumulate your inventory forever. Or if you were dealing with a biological problem, you might be constrained to keep your stock of a species above a lower bound where reproduction begins to fail, and an upper bound where epidemics are common.

The approach to such problems is similar to that of the control problems.
Suppose we have an objective function

$\max \int_0^T u(t,x,z)dt$ s.t.

$\dot{x} = g(t,x,z)$, $x(0) = x_0$ and

$h(t,x) \geq 0$.

The augmented Hamiltonian for this problem is

$$L = u(t,x,z) + \lambda g(t,x,z) + \phi h(t,x)$$

and the necessary conditions for optimality include, the constraints plus

$$\frac{\partial L}{\partial z} = 0$$

$$\dot{\lambda} = -\frac{\partial L}{\partial x}$$

$\phi \geq 0$ and $\phi h = 0$

and the transversality condition.

Solving problems like this by hand can be quite difficult, even for very simple problems. (See K&S p.232 if you want to convince yourself). (An alternative approach presented in Chiang (p. 300) is often easier and we follow this approach below). For much applied analysis, however, there may be no alternative to setting a computer to the problem to find a numerical solution.

*B. Example: Hotelling's optimal extraction problem*

Clearly, Hotelling's problem can also be modeled as a restriction that $x_t \geq 0$. In this case our Lagrangian would take the form

$L=e^{-rt}u(\cdot)+\lambda(-z_t)+\phi_t \cdot x_t$.

And the associated maximization criteria are:

12. $L_z=0$: $e^{-rt}u'(\cdot)-\lambda_t=0 \Rightarrow e^{-rt}p(z_t)-\lambda_t=0$
13. $L_x=-\dot{\lambda}$: $-\dot{\lambda}=\phi_t$
14. $L_\lambda=\dot{x}$: $\dot{x}_t=-z_t$
15. $L_\phi \geq 0$: $x_t \geq 0$
16. $\phi_t \geq 0$
17. $\phi_t x_t = 0$

We won't solve this problem in all its detail, but the solution method would follow a similar path. We divide time into two portions, from 0 to $T_1$ where $\phi$=0 and $\lambda$ is constant, and from $T_1$ to $T$, where $x_t$=0 and $\lambda$ falls with the increase in $\phi$. To solve the problem we note that $\phi_{T_1}=0$ and then solve for $T_1$.

One thing that is interesting in this specification is that the costate variable is no longer constant over time. This makes sense: between 0 and $T_1$ we're indifferent about when we get the extra unit of the resource. But after $T_1$ it clearly makes a difference – the sooner we get the additional unit the more valuable (in PV terms) it will be. When $t>T_1$, we know that $z_t=0 \Rightarrow p=1$ and $\lambda_t=e^{-rt}$. A marginal increase in the stock over this range would allow the immediate sale of that stock at a price of 1 and the present value of this marginal change in stock would, therefore, be $e^{-rt}$.

## IV. An example of constrained optimal control

A clear and useful example of applied constrained optimal control is the paper by Chavas, Kliebenstein and Crenshaw (1985).

## V. Bang-bang OC problems

There are some problems for which the optimal path does not involve a smooth approach to the steady state or gradual changes over time. Two important classes of such problems are known as "bang-bang" problems and most rapid approach problems. In such problems the constraints play a central role in the solution.

### *A. Bang-bang example #1: A state variable constraint*

Consider the following problem in which we seek to maximize discounted linear utility obtained from a nonrenewable stock (sometimes referred to as a cake-eating problem):

$$\max_z \int_0^T e^{-rt} z_t dt \quad \text{s.t.}$$

$$\dot{x} = -z$$

$$x(t) \geq 0$$

$$x(0) = x_0$$

*What does intuition suggest about the solution to the problem?* Will we want to consume the resource stock $x$ gradually? Why or why not? Let's check our intuition.

Following the framework from above, we set up the Lagrangian by adding the constraint on the state variable to the Hamiltonian, i.e., $L=H+\phi$(constraint). Using the current-value specification, this give us

$$L = z_t - \mu_t z_t + \phi_t x_t$$

The FOCs for the problem are:

$$\frac{\partial L}{\partial z} = 0: \qquad 1 - \mu_t = 0 \qquad (i)$$

$$\frac{\partial L}{\partial x} = r\mu_t - \dot{\mu}_t: \qquad \phi_t = r\mu_t - \dot{\mu}_t \qquad (ii)$$

Because of the constraint, the complementary slackness condition must also hold:

$$\phi_t x_t = 0 \qquad (iii).$$

The first of these implies that $\mu_t=1$. Since this holds no matter the value of $t$, we know that $\dot{\mu}_t = 0$ for all $t$. Conditions i and ii together indicate that

$\mu_t=1$ and $\phi_t=r$.

The second of these is most interesting. It shows us that $\phi_t$, the Lagrange multiplier, is always positive. From the complementary slackness condition, it follows that $x_t$ must equal 0 <u>always</u>. But wait! We know this isn't actually true at $t=0$. However, at $t=0$, $x_t$ is not variable – it is parametric to our problem, so that point in time doesn't count. But at every instant except the immediate starting value, $x_t=0$.

So how big is $z$ at zero? The first thought is that it must equal $x_0$ but this isn't quite right. To see this, suppose that we found that the constraint started to bind, not immediately, but after 10 seconds. To get the $x$ to zero in 10 seconds, $z$ per second would have to equal $x_0/10$. Now take the limit of this at the denominator goes to zero $\Rightarrow$ $z$ goes to infinity.

Hence, what happens is that for one instant there is a spike of $z_t$ of infinite height and zero length that pushes $x$ exactly to zero. This type of solution is known as a bang-bang problem because the state variable jumps discontinuously at a single point – BANG-BANG! Since, in the real world it's pretty difficult to push anything to infinity, we would typically interpret this solution as "consume it as fast as you can." This is formalized in the framework of most-rapid-approach path problems below.

*B. Bang-Bang Example #2 (based on Kamien and Schwartz p. 205) A control variable constraint*

*Let* $x_t$ be a productive asset that generates output at the rate $rx_t$. This output can either be sold or reinvested. The portion that is reinvested will be called $z_t$ so $[1\text{-}z_t]$ is the portion that is sold. We assume that the interest can be consumed, but the principal cannot be touched.[3] Our question is, What portion of the interest should be invested and what portion should be consumed over the interval $[0,T]$?

Formally, the problem is:

$$\max_z \int_0^T [1-z_t]rx_t dt \text{ s.t.}$$
$$\dot{x}_t = z_t r x_t$$
$$0 \leq z_t \leq 1$$
$$x(0) = x_0$$

This time we have two constraints: $z_t \leq 1$ and $z_t \geq 0$. Hence, our Lagrangian is

$$L = [1-z_t]rx_t + \lambda z_t r x_t + \phi_{1t}(1-z_t) + \phi_{2t} z_t$$

So that the necessary conditions are

$$\frac{\partial L}{\partial z} = 0 \Leftrightarrow -rx_t + \lambda r x_t - \phi_1 + \phi_2 = 0$$
$$\frac{\partial L}{\partial x} = -\dot{\lambda}_t \Leftrightarrow -\dot{\lambda} = [1-z_t]r + \lambda z_t r$$

The transversality condition in this problem is $\lambda_T=0$ since $x_T$ is unconstrained with the Kuhn-Tucker conditions,

$KT_1$: $\phi_1 \geq 0$ & $\phi_1(1-z_t)=0$, and

$KT_2$: $\phi_2 \geq 0$ & $\phi_2 z=0$.

From the $KT_1$, we know that if $\phi_1>0$, then the first constraint binds and $z_t=1$. Similarly, from $KT_2$, if $\phi_2>0$, then the second constraint binds and $z=0$. i.e.

| | |
|---|---|
| $\phi_1>0 \Rightarrow z = 1$ | $\phi_2>0 \Rightarrow z = 0.$ |
| $\phi_1=0 \Leftarrow z < 1$ | $\phi_2=0 \Leftarrow z > 0.$ |

Clearly, it is not possible for both $\phi_1$ and $\phi_2$ to be positive at the same time.

[3] This problem is very similar to one looked at in Lecture 3. Comparing the two you'll see one key difference is that here utility is linear, while in lecture 3 utility was logarithmic.

The first FOC can be rewritten
$(\lambda_t - 1)rx_t - \phi_1 + \phi_2 = 0$ .

We know that $rx_t$ will always be positive since consumption of the capital stock is not allowed. Hence, we can see that three cases are possible:
1) if $\lambda$=1 $\Rightarrow \phi_1$=0 $\phi_2$=0 $\Rightarrow$ no constraint binds
2) if $\lambda$>1 $\Rightarrow \phi_1$>0 $\phi_2$=0 $\Rightarrow z_t$=1
3) if $\lambda$<1 $\Rightarrow \phi_1$=0 $\phi_2$>0 $\Rightarrow z_t$=0.

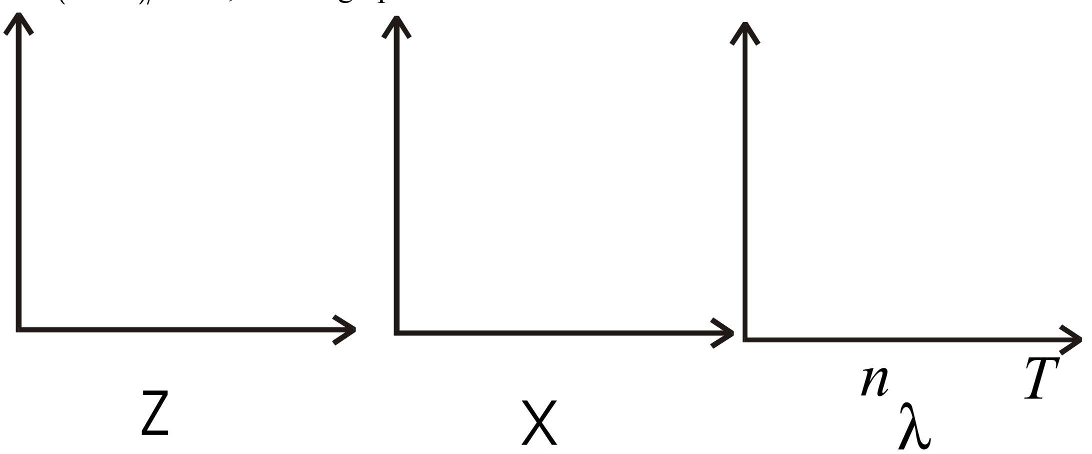
From the second FOC,
$\dot{\lambda} = -\{[1 - z_t]r + \lambda_t z_t r\}$.
Since everything in the brackets is positive, the RHS of the equation is negative $\Rightarrow \lambda$ is always falling.

By the transversality condition we know that eventually $\lambda$ must hit $\lambda_T$=0. Hence, eventually we'll reach case 3 where, $\lambda_t$<1 and $z_t$=0 and we sell all of our output. But when do we start selling – right away or after $x$ has grown for a while? We know from equation 2 that at $\lambda_t$=1 neither constraint binds.

- Suppose that at $t$=$n$ $\lambda_t$=1.
- For $t$<$n$ $\lambda_t$>1 and $z$=1.
- For $t$>$n$ $\lambda_t$<1 and $z$=0.

An important question then is when is $n$? We can figure this out by working backwards from $\lambda_T$=0. From the second FOC, we know that in the final period, (when $\lambda_t$<1) $z$=0, in which case
$\dot{\lambda} = -r$ .
Solving this differential equation yields
$\lambda_t = -rt + A.$
Using the transversality condition,
$\lambda_T = -rT + A = 0$
$A = rT$
$\lambda_t = -rt + rT = r(T - t)$
Hence, $\lambda_n$=1 if
$r(T - n) = 1$
$n = (rT - 1)/r$
Hence, we find that the optimal strategy is to invest everything from $t$=0 until $t = n = (rT - 1)/r$ . After $t$=$n$ consume all of the interest. If $(rT - 1)/r < 0$ then it would be optimal to sell everything from the very outset.

For $(rT-1)/r>0$, we can graph the solution:




What would be the solution as $T \to \infty$? Does this make intuitive sense? What is it about the specification of the problem that makes it inconsistent with our economic intuition?

## VI. Most Rapid Approach Path problems

Bang-bang problems fit into a general class of problems that are commonly found in economics: most-rapid-approach path problems (MRAP).[4] Here, the optimal policy is to get as quickly as possible to steady state where benefits are maximized. Consider the first example bang-bang example above. Wouldn't a solution in which we move toward the equilibrium as fast as possible rather than impossibly fast be more intuitively appealing?

### *A. MRAP example (Kamien & Schwartz p. 211)*

A very simple firm generates output from its capital stock with the function $f(x_t)$ with the property that $\lim_{x\to 0} f'(x)=\infty$. The profit rate, therefore, is

$$\pi_t = p \cdot f(x_t) - c \cdot z_t$$

where $x$ is the firm's capital stock and $z$ is investment, $p$ and $c_t$ are exogenously evolving unit price and unit cost respectively. The capital stock that starts with $x(0)=x_0$, depreciates at the rate $b$ so that

$\dot{x}_t = z_t - bx_t$.

The firm's problem, therefore, is to maximize the present value of its profits,

$\int_0^\infty e^{-rt}\left[p \cdot f(x_t) - c \cdot z_t\right]dt$ subject to

$\dot{x}_t = z_t - bx_t$,

with three additional constraints:

i) $x(t) \geq 0$

ii) $z_t \geq 0$

iii) $p \cdot f(x_t) - c \cdot z_t \geq 0$

Let's use economic intuition to help us decide if we need to explicitly include all the constraints in solving the problem?

[4] Sometimes the term "bang-bang" is also used to describe MRAP problems.

- The constraint on $x$ almost certainly does not need to be imposed because as long as $f'$ gets big as $x\to 0$, the optimal solution will always avoid zero.
- The constraints on $z$, on the other hand might be relevant. But, we'll start by assuming that neither constraint binds, and then see if we can figure out actual the solution based on the assumed interior solution or, if not, we'll need to use the Kuhn-Tucker specification. Note that if there does exist a steady state in $x$, then, as long as $b>0$, $z$ must be greater than zero. Hence, we anticipate that much might be learned from the interior solution.
- Similarly, the profit constraint might also bind, but we would expect that in the long run, profits would be positive. So again, we start by solving for an interior solution, assuming $\pi>0$ where $\pi = p\cdot f\left(x_t\right)-c\cdot z_t$.

*B. The interior solution*

The current value Hamiltonian of the problem (assuming an interior solution w.r.t. $z$ and $x$ with $\pi>0$) is

$$H_c = p\cdot f\left(x_t\right)-c\cdot z_t+\mu_t\left(z_t-bx_t\right)$$

The necessary conditions for an interior are:

$$\frac{\partial H_c}{\partial z_t}=0\Rightarrow -c+\mu_t=0$$

$$\frac{\partial H_c}{\partial x_t}=r\mu_t-\dot{\mu}_t\Rightarrow p\frac{\partial f\left(x_t\right)}{\partial x_t}-\mu_t b=r\mu_t-\dot{\mu}_t$$

Over any range where the constraints on $z$ do not bind, therefore, we have

$c=\mu_t$

and, therefore, it must also hold that

$\dot{\mu}_t=\dot{c}=0$.

Substituting $c$ for $\mu$ and rearranging, the second FOC becomes

18. $$p_t\frac{\partial f\left(x_t\right)}{\partial x_t}=\left(r+b\right)c-\dot{c}$$

over any interval where $z>0$.

We see, therefore, that the optimum conditions tell us about the optimal level of $x$, say $x^*$. We can then use the state equation to find the value of $z$ that maintains this relation.

Since $c$ and $p$ are constant, this means that the capital stock will be held at a constant level and 18 reduces to $\frac{pf'(x)}{r+b}=c$. This is known as the *modified golden rule*.

Let's think about this condition for a moment.

- In a static economy, the optimal choice would be to choose $x$ where the marginal product of increasing $x$ is equal to the marginal cost, i.e., where $pf'=c$.

- In an infinite-horizon economy, if we could increase $x$ at all points in time this would have a discounted present value of $\frac{pf'}{r}$. However, since the capital stock depreciates over time, this depreciation rate diminishes the present value of the gains that can be obtained from an increase in $x$ today, hence the present value of the benefit of a marginal increase in $x_t$ is $\frac{pf'}{r+b}$.

If $p$ and $c$ are not constant but grow in a deterministic way (e.g., constant and equal inflation) then we could de-trend the values and find a real steady state. If $p$ and $c$ both grow at a constant rate, say $w$, then there will be a unique and steady optimal value of $x$ for all $z$>0.

*C. Corner solutions*

All of the discussion above assumed that we are at an interior solution, where $0 < z_t < p \cdot f(x_t)/c$. But, we ended up finding that the interior solution only holds when the state variable $x$ is at the point defined by equation 18. Hence, if we're not at $x^*$ at $t$=0, then it must be that we're at a corner solution, either $z_t$=0 or $p \cdot f(x_t) - c \cdot z_t = 0$.

If $x_0 > x^*$ then it will follow that $z$ will equal zero until $x_t$ depreciates to $x^*$. If $x_0 < x^*$ then $z$ will be as large as possible $\frac{p}{c} f(x_t) = z_t$ until $x^*$ is reached.

Hence, economic intuition and a good understanding of the steady state can tell us where we want to get and how we're going to get there – in the most rapid approach possible.

*D. Some theory and generalities regarding MRAP problems*

The general principles of MRAP problems are discussed by Wilen (1985, p. 64)

> Spence and Starrett show that for *any* problem whose augmented integrand (derived by substituting the dynamic constraint into the original integrand) can be written as
>
> $$\Pi_A(K, \dot{K}) = M(K) + N(K)\dot{K}$$
>
> the optimal solution reduces to one of simply reaching a steady state level $K = K^*$ as quickly as possible.

Where $K$ is the state variable and by "integrand" they mean the objective function, profits in the case considered here.

How does this rule apply here? The integrand is $p_t f(x_t) - c_t z_t$. Using the state equation $bx_t + \dot{x}_t = z_t$, the integrand can be written

$p_t f(x_t) - c_t(bx_t + \dot{x}_t) = p_t f(x_t) - c_t b x_t - c_t \dot{x}_t$.

Converting this to the notation used by Wilen,

$M\left(K\right) = p_t f\left(x_t\right) - c_t b x_t$

and

$N\left(K\right)\dot{K} = c_t \dot{x}_t$.

Hence this problem fits into the general class of MRAP problems.

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