Issues in Economic Systems and Institutions: Part IV: Information Aggregation

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Madness of Crowds

"No one in this world, so far as I know, has ever lost money by underestimating the intelligence of the great masses of the common people."
H. I. Mencken.

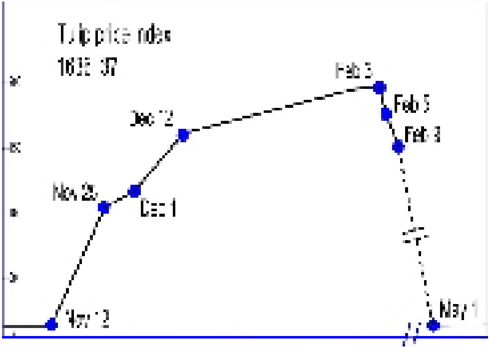
"[Physicians], like lemmings, episodically and with a blind, infectious enthusiasm, push certain diseases and treatments primarily because everyone else is doing the same."
John Burnum, New England Journal of Medicine.

"If the blind lead the blind, both shall fall into the ditch."
Matthew 15:14.

The Dutch Tulip Mania

- The Viceroy: a prized tulip.
- Cost of a bulb in 1637: 3,000 – 4,200 guilders.
- Skilled artisan's annual salary: 300 guilders.
- You could buy a house with the price of a bulb.
- Futures contracts traded. Often, the bulbs didn't even exist physically.

The Dutch Tulip Mania



A Simple Investment Model (Bikhchandani et al 1992)

- Investors 1, 2, 3... sequentially decide whether to invest ($a = I$) or not ($a = N$).
- State-of-the-world θ is the net return from investment. $\theta \in \{-1, 1\}$. $\Pr[\theta = 1] = \frac{1}{2}$.
- Each investor receives a conditionally independent private signal $s \in \{H, L\}$, with probabilities:

	$s = H$	$s = L$
$\theta = 1$	p	$1 - p$
$\theta = -1$	$1 - p$	p

- Investors observe previous players' actions, but not signals.
- Investor t's strategy is a mapping from histories $h_t = (a_1 a_2 ... a_{t-1})$ and private signals $\{H, L\}$ to $\{I, N\}$.

Benchmark: Observable Signals

- Posterior on high return after one H signal:

$$\lambda = \frac{\frac{1}{2}p}{\frac{1}{2}p + \frac{1}{2}(1-p)} = p$$

- Posterior on high return after one L signal:

$$\lambda = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}p + \frac{1}{2}(1-p)} = 1-p$$

Benchmark: Observable Signals

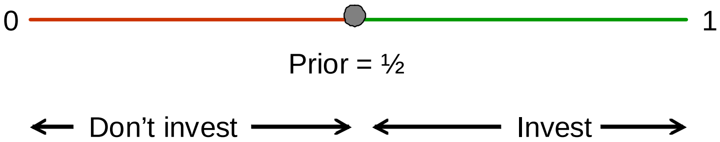
- Posterior on high return after *HL* signal:

$$\lambda = \frac{\frac{1}{2}p(1-p)}{\frac{1}{2}p(1-p) + \frac{1}{2}(1-p)p} = \frac{1}{2}$$

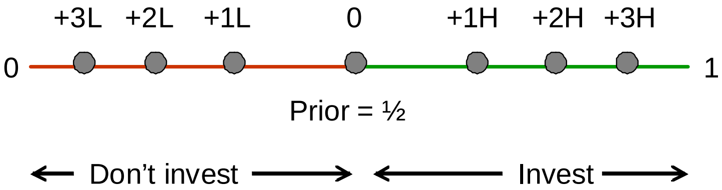
- Posterior on high return after k more H signals than L:

$$\lambda = \frac{p^k}{p^k + (1-p)^k}$$

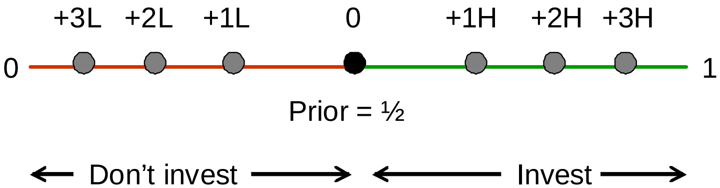
Observable Signals in Pictures



Observable Signals in Pictures

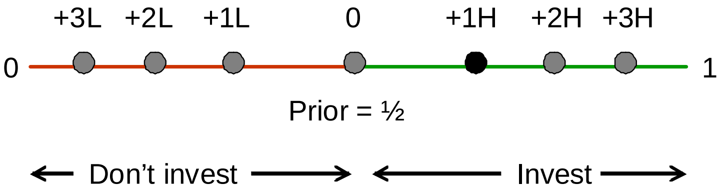


Observable Signals in Pictures



Signals:

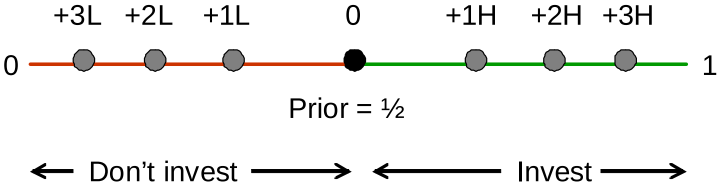
Observable Signals in Pictures



Signals: H

Decisions: I

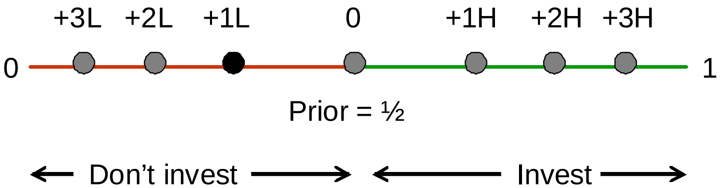
Observable Signals in Pictures



Signals: H, L

Decisions: I, N

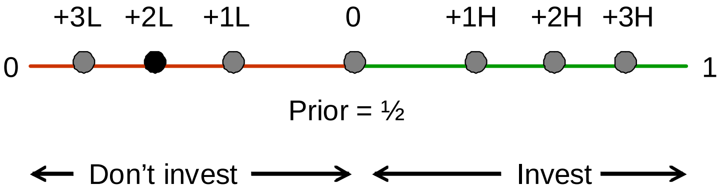
Observable Signals in Pictures



Signals: H, L, L

Decisions: I, N, N

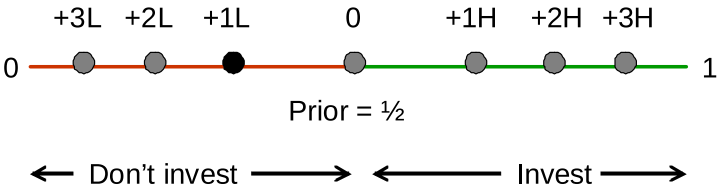
Observable Signals in Pictures



Signals: H, L, L, L

Decisions: I, N, N, N

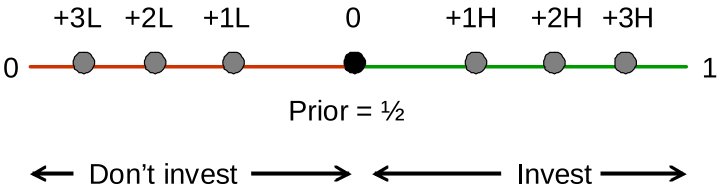
Observable Signals in Pictures



Signals: H, L, L, L, H

Decisions: I, N, N, N, N

Observable Signals in Pictures



Signals: H, L, L, L, H, H

Decisions: I, N, N, N, N, I

Learning When Signals Are Observable

- If $\theta = 1$, after enough periods, a majority of the signals will almost surely be H (law of large numbers).
- Beliefs will almost surely put nearly all the weight on the true state and almost all agents will take the right decision.
- There is "herding" but no "informational cascade", i.e. decisions are still sensitive to arrival of fresh information.
- All decisions are efficient, given the information.

Observable Actions

- Exp. return from investment: $\lambda.1 + (1-\lambda)(-1) = 2\lambda - 1$.
- Invest if $\lambda > \frac{1}{2}$, don't invest if $\lambda < \frac{1}{2}$ and toss a coin if $\lambda = \frac{1}{2}$.
- Investor 1: $a(H) = I$ and $a(L) = N$.
- Investor 2: can infer 1's information from his action

$$\begin{aligned} a(I,H) &= a(HH) = I \\ a(N,L) &= a(LL) = N \\ a(I,L) &= a(N,H) = a(HL) = \left(\frac{1}{2} \circ I, \frac{1}{2} \circ N\right) \end{aligned}$$

- Investor 2 is influenced by investor 1, but does not blindly mimic her.

The Third and Fourth Investors

- Posterior beliefs (after observing both predecessors invest):

$$
\begin{aligned}
\lambda(II, H) &= \lambda(HIH) \\
&= \frac{p^2\left[p + \frac{1}{2}(1-p)\right]}{p^2\left[p + \frac{1}{2}(1-p)\right] + (1-p)^2\left[1 - p + \frac{1}{2}p\right]} > \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\lambda(II, L) &= \lambda(HIL) > \frac{1}{2} \\
&= \frac{p(1-p)\left[p + \frac{1}{2}(1-p)\right]}{p(1-p)\left[p + \frac{1}{2}(1-p)\right] + (1-p)p\left[1 - p + \frac{1}{2}p\right]}
\end{aligned}
$$

- If the first two players invest, the third will mimic them regardless of her private information (H or L)!
- Same if the first two players both do not invest.

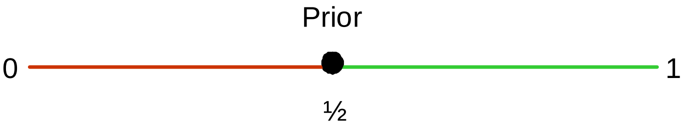
The Third Investor

- Investor 4 learns nothing about investor 3's private signal from her action.
- If the two predecessors take opposite actions, player 3 is in the same position as player 1:

$$\begin{aligned} \lambda(IN, H) &= \lambda(NI, H) = \lambda(HLH) = \lambda(H) = p \\ \lambda(IN, L) &= \lambda(NI, L) = \lambda(HLL) = \lambda(L) = 1 - p \end{aligned}$$

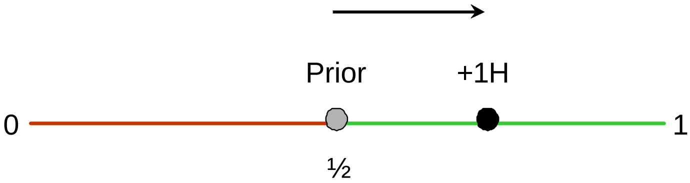
- Opposite actions reveal contradictory signals and therefore "cancel out" each other.
- If player 3 is in the same position as player 1, player 4 is in the same position as player 2.

Cascade in Pictures



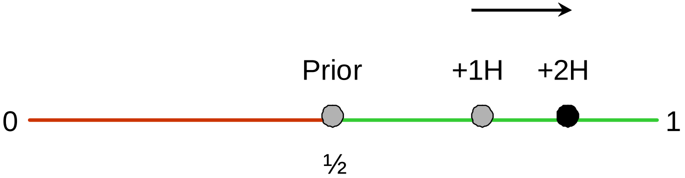
Investor	Signal	Net Info	Action	Inference

Cascade in Pictures



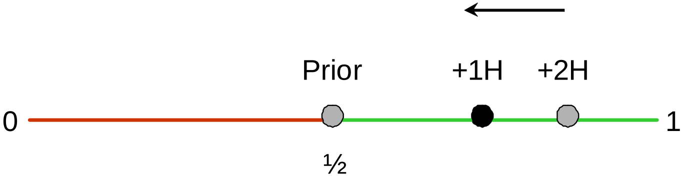
Investor	Signal	Net Info	Action	Inference
1	H	$+1H$	I	$+1H$

Cascade in Pictures



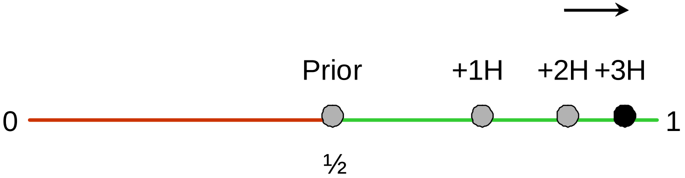
Investor	Signal	Net Info	Action	Inference
1	H	$+1H$	I	$+1H$
2	H	$+2H$	I	$> +1H$

Cascade in Pictures



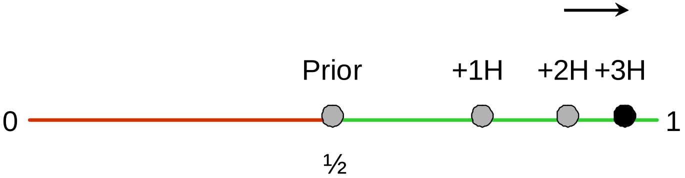
Investor	Signal	Net Info	Action	Inference
1	H	$+1H$	I	$+1H$
2	H	$+2H$	I	$> +1H$
3	L	$+1H$	I	?

Cascade in Pictures



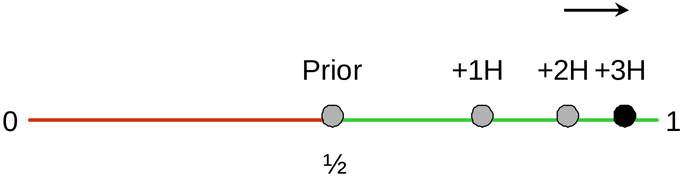
Investor	Signal	Net Info	Action	Inference
1	H	$+1H$	I	$+1H$
2	H	$+2H$	I	$> +1H$
3	L	$+1H$	I	?
3	H	$+3H$	I	?

Cascade in Pictures



Investor	Signal	Net Info	Action	Inference
1	H	$+1H$	I	$+1H$
2	H	$+2H$	I	$> +1H$
3	L	$+1H$	I	?
3	H	$+3H$	I	?

Cascade in Pictures



Investor	Signal	Net Info	Action	Inference
1	H	$+1H$	I	$+1H$
2	H	$+2H$	I	$> +1H$
3	L	$+1H$	**I**	$> +$**1H**
3	H	$+3H$	**I**	$> +$**1H**

Cascade After Two Rounds

- As long as decisions alternate, the next player's decision depends on her private information.
- As soon as two successive decisions are the same, all subsequent actions mimic them (cascade).
- After two rounds:

$$\Pr[\text{correct cascade}] = p\left[p + \frac{1}{2}(1-p)\right] = \frac{1}{2}p(1+p)$$

$$\Pr[\text{incorrect cascade}] = (1-p)\left[1 - p + \frac{1}{2}p\right] = \frac{1}{2}(1-p)(2-p)$$

$$\Pr[\text{no cascade}] = p(1-p)$$

Long Run Probabilities

- After n rounds (let $x = p(1-p)$):

$$\begin{aligned} \Pr[\text{cascade}] &= (1-x) + x(1-x) + x^2(1-x) + \ldots \\ &\to 1 \quad \text{as } n \to \infty \end{aligned}$$

- Other asymptotic probabilities ($n \to \infty$):

$$\Pr[\text{correct cascade}] = \frac{p(1+p)}{2(1-p+p^2)}$$

$$\Pr[\text{incorrect cascade}] = \frac{(1-p)(2-p)}{2(1-p+p^2)}$$

Numerical Examples

p	Pr cascade (2 rounds)		Pr Cascade (long run)	
	Right	Wrong	Right	Wrong
≈ 0.5	0.375	0.375	0.5	0.5
0.7	0.595	0.195	0.753	0.247
0.9	0.855	0.055	0.940	0.060

Examples

- Michael Reacy and Fred Wiersema secretly purchased 50,000 copies of their book *The Discipline of Market Leaders*.
- Hanson and Putler (1996) inflated download statistics for game software on AOL's site.
- In U.S primaries, early states like Iowa and New Hampshire are supposed to have disproportionate influence.
- Medical fads: tonsillectomy (no tangible benefits, idiosyncratic regional variations).
- Scientific consensus: are you sure the earth is round, not flat?
- Popular restaurants don't raise prices; IPOs are underpriced.
- Crime rates show large regional variations, even after controlling for income, race, etc.

Further Observations

- Fashion leaders: if investor 1 has precision $p_1 > p$, then everyone follows what she did (cascade with probability 1).
- Later players better off if the leader (investor 1) is less well informed ($p_1 < p$).
- Public release of information (e.g. product information or disease advisory) may make everyone worse off by precipitating a herd.
- Cascades are fragile (fads) because they are based on very little information. A small amount of new information, or drift in the state, can completely overturn a cascade.
- If timing is endogenous, there are often long periods of waiting followed by an avalanche of investments (booms and crashes).

Condorcet Jury Theorem

1. The decision of a jury will be correct more often than the decision of any single individual.
2. The decision of a jury is correct with probability approaching 1 a the size of the jury grows to infinity.

- Conditions apply for the conclusions to hold.
- Can apply to various voting rules: majority, super-majority and unanimity.
- Statistical versus strategic jury theorems: different assumptions about voting behaviour.
- No communication: voters' *interim* preferences differ due to differential private information, not conflicting interests.

Statistical Jury Theorem

- State of the world (s) = guilty (G) or innocent (I).
- Decision (d) = convict (C) or acquit (A).
- Correct decision: C when G, A when I.
- Voters $1, 2, 3..., n$. Probability of j voting correctly = $p_j \in [\frac{1}{2}, 1]$. Probabilities are independent.
- Voting rule = $\alpha \in [\frac{1}{2}, 1]$ (minimum fraction of votes needed for a decision).
- Let $x_j = 1$ when j's vote is correct; $x_j = 0$ when wrong.
- Probability that the jury's decision is correct:

$$P(n, \alpha) = \Pr\left[X = \sum_{j=1}^{n} x_j \geq \alpha n\right]$$

Statistical Jury Theorem

Theorem

Assume $p_j = p$ *for all* j*. Then*
(1) If $p > \alpha$*, then there exists* N *such that for all* $n > N$*,*
$P(n, \alpha) > p$ *and* $\lim_{n\to\infty} P(n, \alpha) = 1$*.*
(2) If $p \leq \alpha$*, then there exists* N *such that for all* $n > N$*,*
$P(n, \alpha) < p$ *and* $\lim_{n\to\infty} P(n, \alpha) = 0$*.*

- Under majority rule, the jury theorems hold whenever individual voters can do even slightly better than chance.
- Under super-majority rule, individual voters must be sufficiently accurate for the theorems to be valid.
- The *ex ante* probability of a decision (e.g., conviction) or an error (e.g., convicting the innocent) is lower the higher is the number of votes needed.

Proof

- By the Weak Law of Large Numbers, $\forall \epsilon, \delta > 0, \exists N(\epsilon)$ such that for all $n > N(\epsilon)$

$$\Pr\left[\left|\frac{X}{n} - p\right| > \delta\right] < \epsilon$$

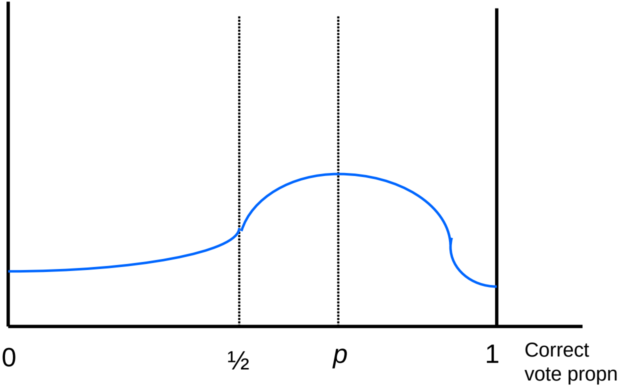
- Put $\delta = p - \alpha$ and $\epsilon = 1 - p$:

$$\begin{aligned} \Pr\left[p - \frac{X}{n} > p - \alpha\right] &< 1 - p \\ \text{or} \quad 1 - \Pr\left[\frac{X}{n} < \alpha\right] &= P(n, \alpha) > p \end{aligned}$$

- Put $\delta = p - \alpha$. For any $\epsilon > 0$:

$$\Pr\left[\frac{X}{n} < \alpha\right] < \epsilon \Rightarrow P(n, \alpha) \to 1 \text{ as } n \to \infty$$

Increasing Jury Size



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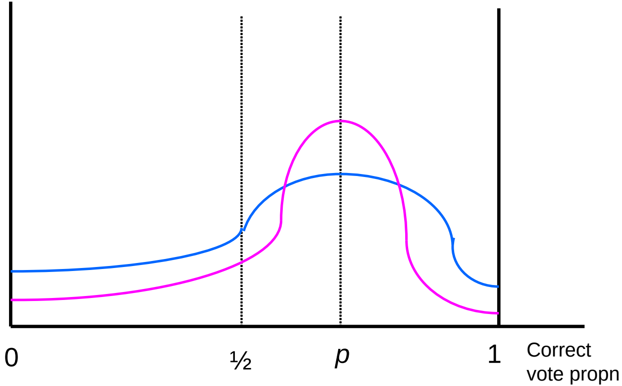

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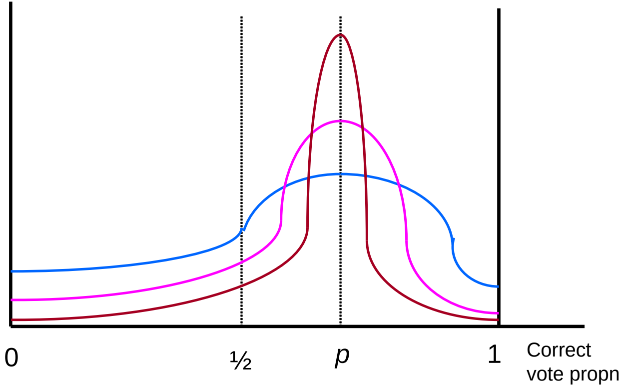

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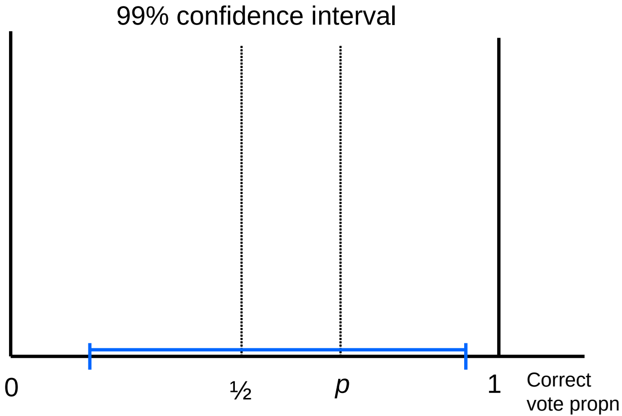

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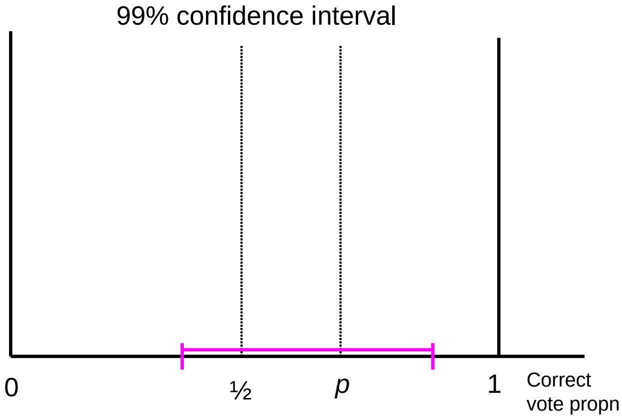

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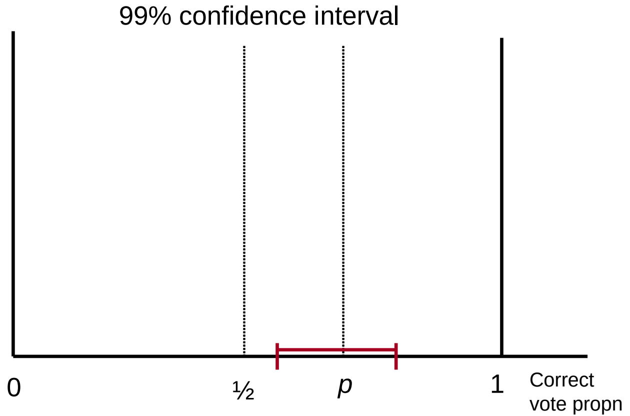

A Model of Rational Voters

- Voter preferences:

$$u(s,d) = \begin{cases} -q & \text{if } s = I, d = C, \\ -(1-q) & \text{if } s = G, d = A, \\ 0 & \text{otherwise,} \end{cases}$$

where $q \in (0,1)$.

- q is the "threshold of doubt": C is optimal iff the voter believes there is a greater than q chance the state is G.
- Let prior on "guilt" $= \pi$.

A Model of Rational Voters

- Voter private information: conditionally independent private signal $t_j \in \{g, i\}$, with distribution

	g	i
G	p_G	$1 - p_G$
I	$1 - p_I$	p_I

- Signals are informative: $p_G \neq 1 - p_I$.
- Voters **cannot communicate**; they must vote independently.
- Since there is common interest, the game with communication is trivial: voters have the incentive to share their signals truthfully.

Strategies

- Mappings from signal to vote: $v : \{g, i\} \to \Delta\{C, A\}$.
- A strategy is **informative** if $v(g) = C$ and $v(i) = A$.
- A strategy is **responsive** if $v(g) \neq v(i)$.
- A strategy is **sincere** if it is the same way the juror would have voted if she were making the decision alone ($n = 1$).

An Example

- Two voters or one
- Threshold of doubt: $q = \frac{1}{2}$
- Prior on $G = \pi = \frac{2}{3}$
- Signal accuracy: $p = \frac{3}{4}$

A Judge $(n = 1)$

- $\Pr\left(G|g\right) = \frac{\pi p}{\pi p + (1-\pi)(1-p)} = \frac{6}{7} > \frac{1}{2}$
- $\Pr(G|i) = \frac{\pi(1-p)}{\pi(1-p)+(1-\pi)p} = \frac{2}{5} < \frac{1}{2}$
- Optimal decision is informative: $v(g) = C$ and $v(i) = A$
- Expected payoff $= -\frac{1}{2}.\frac{1}{4} = -0.125$

Jury of Two

- Assume unanimity required for C, otherwise A
- Expected payoff:

$$-\frac{1}{2}[\pi(1-p^2)+(1-\pi)(1-p)^2] = -0.15625$$

- But sincere voting is not a Nash equilibrium
- Assume sincere voting. Then

$$\Pr(G|piv, i) = \Pr(G|g, i) = \frac{2}{3} < \frac{1}{2}$$

- Voter receiving i signal will want to deviate and vote for C

Symmetric Mixed Equilibrium

- Let $v(g) = 1$ and $v(i) = \sigma$

$$\Pr(G|piv) = \frac{\pi\left[p + (1-p)\sigma\right]}{\pi\left[p + (1-p)\sigma\right] + (1-\pi)\left[p\sigma + 1 - p\right]}$$

$$= \frac{6 + 2\sigma}{7 + 5\sigma} = \lambda$$

$$\Pr(G|piv, i) = \frac{\lambda(1-p)}{\lambda(1-p) + (1-\lambda)p}$$

$$= \frac{\lambda}{3 - 2\lambda}$$

- Indifference $\Rightarrow \frac{\lambda}{3-2\lambda} = \frac{1}{2} \Rightarrow \lambda = \frac{3}{4} \Rightarrow \sigma = \frac{3}{7}$

Symmetric Mixed Equilibrium

- Error probabilities:
  - In state G: $1 - \left[p^2 + 2p(1-p)\sigma + (1-p)^2\sigma^2\right] = \frac{13}{49}$
  - In state I: $(1-p)^2 + 2p(1-p)\sigma + p^2\sigma^2 = \frac{16}{49}$
- Expected payoff $= -\frac{1}{2}.\left[\frac{2}{3}.\frac{13}{49} + \frac{1}{3}.\frac{16}{49}\right] = -0.143 < 0.125$
- Jury still does worse than judge, even with sophisticated voters
- Is this adverse consequence a result of equilibrium selection?
- Another equilibrium: both voters vote for A regardless of their signal. Since neither is pivotal, best response property is not violated!
- Is there a better equilibrium than all of these?

Asymmetric Pure Equilibrium

- Voter 1 votes for C regardless of signal
- Voter 2 votes sincerely
- Since voter 2 is always pivotal, he is effectively a judge. Hence sincere and informative voting is a best response for voter 2.
- Checking best response property for voter 1:

$$\Pr(G|piv, i) = \Pr(G|g, i) = \pi = \frac{2}{3} < \frac{1}{2}$$

- The equilibrium mimics trial by judge ($n = 1$)
- Expected payoff $= -0.125$
- Jury does no worse than judge under this equilibrium selection.

Full Information and Sincere Voting

- If all the signals were known, posterior belief:

$$\Pr[s = G \mid \#g \text{ signals is } k] = \frac{\pi p_G^k (1-p_G)^{n-k}}{\pi p_G^k (1-p_G)^{n-k} + (1-\pi)(1-p_I)^k p}$$

$$= \frac{1}{1 + \frac{1-\pi}{\pi}\left[\frac{1-p_I}{p_G}\right]^k \left[\frac{p_I}{1-p_G}\right]^{n-k}}$$

- There is a critical number of g signals, k^*, such that the posterior is λ or higher iff $k \geq k^*$.

Theorem

If $p_I = p_G$, sincere voting is informative and rational iff the minimum number of votes needed for conviction is exactly k^*.

Inferiority of Unanimous Verdicts (Feddersen and Pesendorfer 1997)

- Under sincere voting, raising the minimum votes needed for conviction lowers the probability of wrongful conviction.
- Under strategic voting, *both* error probabilities may go up.
- Relies on the information content of the event: "the voter is pivotal". A vote matters only in this case.
- Under unanimity, being pivotal is a strong signal in favour of guilt (everyone else has voted for conviction!). This makes voters more inclined to vote for conviction.
- The greater willingness to convict may dominate the effect of more votes needed for conviction.
- Under unanimity, error probabilities remain bounded away from zero even as jury size goes to infinity. Under any "interior rule", error they approach zero.

Symmetric Mixed Equilibria

- Let $\pi = \frac{1}{2}$ and $p_G = p_I = p$.
- Let $\sigma(g), \sigma(i)$ be probability of voting for C when signal is and, i respectively.
- An equilibrium is responsive if $\sigma^*(g) \neq \sigma^*(i)$.
- Probabilities of voting for C:

$$\begin{aligned}\gamma_G &= p\sigma(g) + (1-p)\sigma(i)\\ \gamma_I &= (1-p)\sigma(g) + p\sigma(i)\end{aligned}$$

- Since posterior after a g signal $>$ posterior after an i signal,

$$\begin{aligned}\sigma(g) &\in (0,1) \Rightarrow \sigma(i) = 0\\ \sigma(i) &\in (0,1) \Rightarrow \sigma(g) = 1\end{aligned}$$

Mixed Equilibrium: Type 1

- $\sigma^*(g) = 1$ and $\sigma^*(i) = 0$.
- Arises if

$$\underbrace{\frac{p^{\alpha n-1}(1-p)^{(1-\alpha)n+1}}{p^{\alpha n-1}(1-p)^{(1-\alpha)n+1} + (1-p)^{\alpha n-1}p^{(1-\alpha)n+1}}}_{\Pr(G|piv,i)} \begin{array}{l} \leq \quad q \\ \leq \quad \text{doubt threshold} \end{array}$$

- and

$$\underbrace{\frac{p^{\alpha n}(1-p)^{(1-\alpha)n}}{p^{\alpha n}(1-p)^{(1-\alpha)n} + (1-p)^{\alpha n}p^{(1-\alpha)n}}}_{\Pr(G|piv,g)} \begin{array}{l} \geq \quad q \\ \geq \quad \text{doubt threshold} \end{array}$$

Mixed Equilibrium: Type 2

- $\sigma^*(g) = 1$ and $\sigma^*(i) = \sigma$.
- Indifference after signal i implies:

$$\frac{(1-p)\gamma_G^{\alpha n-1}(1-\gamma_G)^{(1-\alpha)n}}{(1-p)\gamma_G^{\alpha n-1}(1-\gamma_G)^{(1-\alpha)n} + p\gamma_I^{\alpha n-1}(1-\gamma_I)^{(1-\alpha)n}} = q$$

- Use

$$\gamma_G = p + (1-p)\sigma; \;\; \gamma_I = p\sigma + (1-p)$$

- On solving:

$$\sigma(i) = \frac{p(1+f)-1}{p-f(1-p)}$$

$$\text{where} \quad f = \left(\frac{1-q}{q}\right)^{\frac{1}{\alpha n-1}} \left(\frac{1-p}{p}\right)^{\frac{(1-\alpha)n+1}{\alpha n-1}}$$

Mixed Equilibrium: Type 3

- $\sigma^*(g) = \sigma$ and $\sigma^*(i) = 0$.
- Indifference after signal g implies:

$$\frac{p\gamma_G^{\alpha n-1}(1-\gamma_G)^{(1-\alpha)n}}{p\gamma_G^{\alpha n-1}(1-\gamma_G)^{(1-\alpha)n} + (1-p)\gamma_I^{\alpha n-1}(1-\gamma_I)^{(1-\alpha)n}} = q$$

- Use

$$\gamma_G = p + (1-p)\sigma; \;\; \gamma_I = p\sigma + (1-p)$$

- On solving:

$$\sigma(g) = \frac{h-1}{p(h+1)-1}$$

$$\text{where } \; h = \left(\frac{1-q}{q}\right)^{\frac{1}{(1-\alpha)n}} \left(\frac{1-p}{p}\right)^{\frac{\alpha n}{(1-\alpha)n}}$$

Interior Rules

Theorem

Suppose $\alpha \in (0,1)$. (1) There is $\overline{n}$ such that for all $n \geq \overline{n}$, there is a symmetric responsive equilibrium. (2) For symmetric, responsive equilibria

$$\lim_{n\to\infty} \Pr(C|I) = \lim_{n\to\infty} \Pr(A|G) = 0$$

- Both error probabilities (convicting the innocent and acquitting the guilty) vanish as the size of the jury becomes very large.
- Note that *any* interior rule has this property.

Interior Rules

- Limit expressions for mixtures:

$$\lim_{n\to\infty} \sigma(i) = \frac{p\left[1+\left(\frac{1-p}{p}\right)^{\frac{1-\alpha}{\alpha}}\right]-1}{p-\left(\frac{1-p}{p}\right)^{\frac{1-\alpha}{\alpha}}(1-p)} \in (0,1)$$

$$\lim_{n\to\infty} \sigma(g) = \frac{\left(\frac{1-p}{p}\right)^{\frac{\alpha}{1-\alpha}}-1}{p\left[\left(\frac{1-p}{p}\right)^{\frac{\alpha}{1-\alpha}}+1\right]-1} \in (0,1)$$

- East to check the theorem holds.

Outline of Proof

- As $n \longrightarrow \infty$, the following holds:

$$\gamma_I < \alpha < \gamma_G$$

- By the Law of Large Numbers, for large n, the proportion of votes for C is γ_G (when guilty) and γ_I (when innocent).
- Hence, almost surely, the decision is C (when guilty) and A (when innocent).
- For most voting rules, Condorcet's conclusions are valid.

Unanimity Rule

Theorem

Under unanimity rule, if the defendant is convicted with strictly positive probability, then $\Pr(I|C)$ *is bounded below by*

$$\min\left\{\frac{1}{2}, \frac{(1-q)(1-p)^2}{(1-p)^2 + q(2p-1)}\right\}$$

Unanimity Rule

Theorem

Assume condition 1 and $q > 1 - p$. Under unanimity rule, there is a unique responsive symmetric equilibrium with the limiting properties:

$$
\begin{aligned}
\lim_{n \to \infty} \sigma(i) &= 1 \\
\lim_{n \to \infty} \Pr(C|I) &> 0 \\
\lim_{n \to \infty} \Pr(A|G) &> 0
\end{aligned}
$$