Entrance Examination for M. A. Economics

Option B

June 23, 2012

200013

Time 3 hours **Maximum marks** 100

Instructions Please read the following instructions carefully.

- Do **not** break the seal on this booklet until instructed to do so by the invigilator. Anyone breaking the seal prematurely will be evicted from the examination hall and his/her candidature will be cancelled.
- Fill in your Name and Roll Number on the detachable slip below.
- When you finish, hand in this examination booklet to the invigilator.
- Use of any electronic device (e.g., telephone, calculator) is **strictly prohibited** during this examination. Please leave these devices in your bag and away from your person.
- Do **not** disturb your neighbours for any reason at any time.
- **Anyone engaging in illegal examination practices will be immediately evicted and that person's candidature will be cancelled.**

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Marks tally

Question	Marks
I.1-10	
II.11	
II.12	
II.13	
II.14	
II.15	
Total	

Part I

Instructions.

- Check that this examination has pages 1 through 22.
- This part of the examination consists of 10 multiple-choice questions. Each question is followed by four possible answers, at least one of which is correct. If more than one choice is correct, choose **only the best one**. Among the correct answers, the best answer is the one that implies (or includes) the other correct answer(s). **Indicate your chosen answer by circling (a), (b), (c) or (d).**
- For each question, you will get 2 marks if you choose only the best answer. If you choose none of the answers, then you will get 0 for that question. **However, if you choose something other than the best answer or multiple answers, then you will get −2/3 mark for that question.**

You may begin now. Good luck!

QUESTION 1. Two women and four men are to be seated randomly around a circular table. Find the probability that the women are **not** seated next to each other.

(a) 1/2

(b) 1/3

(c) 2/5

(d) 3/5

QUESTION 2. A fair coin is tossed until a head comes up for the first time. The probability of this happening on an odd-numbered toss is

(a) 1/2

(b) 1/3

(c) 2/3

(d) 3/4

QUESTION 3. Let $f(x) = x + |x| + (x-1) + |x-1|$ for $x \in \Re$.

(a) f differentiable everywhere except at 0.

(b) f is not continuous at 0.

(c) f is not differentiable at 1.

(d) f is not continuous at 1.

QUESTION 4. What is the total number of local maxima and local minima of the function

$$f(x) = \begin{cases} (2+x)^3, & \text{if } x \in (-3,-1] \\ x^{2/3}, & \text{if } x \in (-1,2] \end{cases}$$

(a) 1
(b) 2
(c) 3
(d) 4

QUESTION 5. Let $f : \Re_{++} \to \Re$ is differentiable and $f(1) = 1$. Moreover, for every x

$$\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$$

Then $f(x)$ is
(a) $1/3x + 2x^2/3$
(b) $-1/5x + 4x^2/5$
(c) $-1/x + 2/x^2$
(d) $1/x$

QUESTION 6. An n-gon is a regular polygon with n equal sides. Find the number of diagonals (edges of an n-gon are not considered as diagonals) of a 10-gon.
(a) 20 diagonals
(b) 25 diagonals
(c) 35 diagonals
(d) 45 diagonals

QUESTION 7. The equation $x^7 = x + 1$
(a) has no real solution.
(b) has a real solution in the interval $(0, 2)$.
(c) has no positive real solution.
(d) has a real solution but not within $(0, 2)$.

QUESTION 8. $\lim_{n \to \infty} \left(\sqrt{n-1} - \sqrt{n}\right)$
(a) equals 1.
(b) equals 0.
(c) does not exist.
(d) depends on n.

QUESTION 9. A rectangle has its lower left hand corner at the origin and its upper right hand corner on the graph of $f(x) = x^2 + x^{-2}$. For which x is the area of the rectangle minimized?

(a) $x = 0$
(b) $x = \infty$
(c) $x = \left(\frac{1}{3}\right)^{1/4}$
(d) $x = 2^{1/3}$

QUESTION 10. Consider the system of equations

$$\alpha x + \beta y = 0$$
$$\mu x + \nu y = 0$$

α, β, μ and ν are i.i.d. random variables, each taking value 1 or 0 with equal probability. Consider the following propositions. (A) The probability that the system of equations has a unique solution is 3/8. (B) The probability that the system of equations has at least one solution is 1.

(a) Proposition A is correct but B is false.
(b) Proposition B is correct but A is false.
(c) Both Propositions are correct.
(d) Both Propositions are false.

Part II

Instructions.

• Answer **any four** of the following five questions in the space following the relevant question. No other paper will be provided for this purpose.

You may use the blank pages at the end of this booklet, marked **Rough work**, to do calculations, drawings, etc. Your "Rough work" will not be read or checked.

• Each question is worth 20 marks.

QUESTION 11. Let l^2 be the set of sequences of real numbers $x = (x_n)_{n \in \mathcal{N}}$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$.

(A) With $\Re$ as the field of scalars, show that l^2 is a vector space.

(B) Given $x \in l^2$, let $\|x\| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$. Verify that $\|.\|$ is a norm on l^2.

(C) Given $x, y \in l^2$, let $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$. Verify that $\langle ., . \rangle$ is an inner product (i.e., dot product) on l^2.

(D) Given $x, y \in l^2$, let $d(x, y) = \|x - y\|$. Verify that d is a distance function on l^2.

(E) Let $B = \{x \in l^2 \mid \|x\| < 1\}$. Show that B is a convex set.

(F) Define $e : \mathcal{N} \times l^2 \to \Re$ by $e(n, x) = x_n$. Show that $e(n, .)$ is continuous for every $n \in \mathcal{N}$.

(G) Show that the normed space $(l^2, \|.\|)$ is complete, i.e., every Cauchy sequence in l^2 converges to a limit point in l^2.

ANSWER.

QUESTION 12. Consider the system of differential equations

$$Dy(t) = By(t) \quad \text{where} \quad B = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

where $\alpha, \beta \in \Re$ and $(\alpha, \beta) \neq (0,0)$.

(A) If the roots of B are purely imaginary, verify that the solution of this system $y = (y_1, y_2)$ is of the form $y_1(t) = c_1 \cos(c_2 + \beta t)$ and $y_2(t) = c_1 \sin(c_2 + \beta t)$.

(B) Characterize the orbit of y, namely $y(\Re_+)$. Graph the orbit in $\Re^2$.

(C) Comment on the stability properties of y when the roots of B are purely imaginary.

(D) If the roots of B are not purely imaginary,verify that the solution of this system $y = (y_1, y_2)$ is of the form $y_1(t) = c_1 e^{\alpha t} \cos(c_2 + \beta t)$ and $y_2(t) = c_1 e^{\alpha t} \sin(c_2 + \beta t)$.

(E) Derive and graph the orbit of y when the roots of B are not purely imaginary.

(F) Comment on the stability properties of y when the roots of B are not purely imaginary.

(G) What determines the direction of rotation of the orbit as $t \uparrow \infty$?

ANSWER.

QUESTION 13. Consider vector spaces V, W and U. Let $A : V \to W$ and $B : W \to U$ be linear mappings. $A^- : W \to V$ is the linear mapping such that $AA^-A = A$ and $B^- : U \to W$ is the linear mapping such that $BB^-B = B$.

Given a linear mapping L, let $\mathcal{R}(L)$ denote its range space, $\rho(L)$ its rank and $\nu(L)$ its nullity.

Prove the following propositions.

(A) $\rho(A^-) \geq \rho(A)$.

(B) $\rho(AA^-) = \rho(A^-A) = \rho(A)$.

(C) $\nu(AA^-) = \nu(A^-A) = \nu(A)$.

(D) AA^- projects W on $\mathcal{R}(A)$ and A^-A projects V on $\mathcal{R}(A^-A)$.

(E) If $\rho(BA) = \rho(B)$, then $A(BA)^- = B^-$.

(F) If $\rho(BA) = \rho(A)$, then $(BA)^-B = A^-$.

(G) If $\rho(BA) = \rho(A)$, then $A(BA)^-B$ projects W on $\mathcal{R}(A)$.

ANSWER.

QUESTION 14. Given $x, y \in \Re^n$, define $(x, y) = \{tx + (1-t)y \mid t \in (0,1)\}$. We say that $C \subset \Re^n$ is a convex set if $x, y \in C$ implies $(x, y) \subset C$.

Let $X \subset \Re^n$ and let $\{C_i \mid i \in I\}$ be the family of *all* convex subsets of $\Re^n$ such that $X \subset C_i$ for every $i \in I$. The convex hull of X is $\operatorname{co} X = \cap_{i \in I} C_i$.

Prove the following propositions.

(A) X is a convex set if and only if $X = \operatorname{co} X$.

(B) Let $\Delta_m = \{p \in \Re_+^m \mid \sum_{i=1}^m p_i = 1\}$ for $m \in \mathcal{N}$. Then, X is convex if and only if for every $m \in \mathcal{N}$, $p \in \Delta_m$ and $\{x^1, \ldots, x^m\} \subset X$, we have $\sum_{i=1}^m p_i x^i \in X$.

(C) $\operatorname{co} X = \cup_{m \in \mathcal{N}} \left\{ \sum_{i=1}^m p_i x^i \;\middle|\; (p_1, \ldots, p_m) \in \Delta_m \;\wedge\; x^1, \ldots, x^m \in X \right\}$

(D) $\operatorname{co} X = \cup_{m=1}^{n+1} \left\{ \sum_{i=1}^m p_i x^i \;\middle|\; (p_1, \ldots, p_m) \in \Delta_m \;\wedge\; x^1, \ldots, x^m \in X \right\}$

(Hint: By (C), every $x \in \operatorname{co} X$ can be represented as the convex combination of a selection $\{x^1, \ldots, x^m\} \subset X$. If $m > n+1$, then the vectors in this selection are linearly dependent. Use this fact to show that x can be represented as a convex combination of $m-1$ vectors from $\{x^1, \ldots, x^m\}$.)

ANSWER.

QUESTION 15. Prove the following propositions, where P[.] denotes the probability of event [.] and E is the expectation operator.

(A) If X is a random variable and $X \geq 0$, then for every $p \in \Re_+$,

$$\mathrm{E}X^p = \int_0^\infty dx\, px^{p-1}\mathrm{P}[X > x]$$

(B) Let X be a real-valued random variable and let $f : \Re \to \Re_+$ be increasing. Then, $f(b)\mathrm{P}[X > b] \leq \mathrm{E}f \circ X$ for every $b \in \Re$.

(C) Let U be a random variable with the uniform distribution on $(0, 1)$ and let $X = -c^{-1} \ln U$, where $c > 0$. Show that X has the exponential distribution with scale parameter c.

(D) Let X and Y be independent standard Gaussian (i.e., Normal) random variables. Show that the distribution μ of $Z = X/Y$ has the form

$$\mu(dz) = dz \frac{1}{\pi(1 + z^2)}$$

for $z \in \Re$.

ANSWER.