APEC 5152 - Handout 1

February 16, 2017

Micro-Preliminaries

Contents

Introduction - microeconomic foundations

Throughout these notes, the following notation denotes factor endowments, factor rental rates and output prices. Sectors are indexed by $j \in \{1, 2, 3\}$, and denote the quantity of sector-j's output by the scalar Y_j. Corresponding output prices are denoted $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3_{++}$, with the scalar p_j representing the per-unit price of sector-j output. We consider three factor endowments: labor, L, capital, K, and land Z. The corresponding factor rental rates are: w is the wage rate, r is the rate of return to capital, and τ is the unit land rental rate. Represent the vector of factor rental rates by $\mathbf{w} = (w, r, \tau)$.

1 Consumer preferences

The economy is composed of a large number of atomistic households. Each household faces the same vector of prices $\mathbf{p}$ and the same vector of factor rental rates $\mathbf{w}$. Let $\upsilon^\eta = (L^\eta, K^\eta, Z^\eta) \in \mathbb{R}^3_{++}$ denote the level of factor endowments held by household-η, with L^η, K^η and Z^η representing the household's endowment of labor, capital and land. The notation $\upsilon^\eta \in \mathbb{R}^3_{++}$ is a shorthand way of saying each household owns a strictly positive amount of each factor. In most applications that follow we suppress the η superscript of υ^η and use instead $\upsilon = (L, K, Z)$. Given factor rental rates $\mathbf{w}$, the household's income is given by

$$\mathbf{w} \cdot \upsilon = wL + rK + \tau Z$$

which is used to purchase q_j units of consumption good j at market price p_j, $j = a, m, s$. Given final good prices $\mathbf{p}$, total expenditure on final goods is equal to

$$\mathbf{p} \cdot \mathbf{q} = p_1 q_1 + p_2 q_2 + p3 q_3$$

where $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3_{++}$. The notation $(q_1, q_2, q_3) \in \mathbb{R}^3_{++}$ means the household consumes a strictly positive amount of each consumption good. Then, the household's budget constraint is given by

$$\mathbf{w} \cdot \upsilon \geq \mathbf{p} \cdot \mathbf{q}$$

Consumer preferences over goods are represented by the utility function $u : \mathbb{R}^M_{++} \to \mathbb{R}_+$, defined as $u(\mathbf{q})$.

Assumption 1: $u(\mathbf{q})$ satisfies the following properties:

1. $u(\mathbf{q})$ is increasing and strictly concave in $\mathbf{q}$,
2. $u(\mathbf{q})$ is everywhere continuous, and everywhere twice differentiable,
3. $u(\mathbf{q})$ is homothetic[1].

Assumption 1.1 yields indifference curves that are convex, Assumption 1.2 ensures Marshallian demands are continuous functions, while Assumption 1.3 yields Marshallian demands that are separable in prices and income (see Cornes or Varian).

Problem 1 *Show that the Cobb-Douglas utility function*

$$u = q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3}, \text{ where } \alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{1}$$

is increasing and twice differentiable in $\mathbf{q}$, *and homothetic.*

The utility function is used to derive two "dual" functions: the indirect utility function and the expenditure function. The first step in deriving the indirect utility function is to choose the level of goods 1, 2 and 3 to maximize utility subject to a budget constraint. The solution to this exercise is a set of utility maximizing Marshallian demand functions. This constrained optimization exercise is standard fare in a typical intermediate microeconomics course. Next, by substituting the optimal Marshallian demands into the utility function one gets the indirect utility function. The expenditure function is also derived in two steps: first, minimize the cost of achieving a given level of utility, yielding cost minimizing Hicksian demand functions – another standard problem solved in a typical intermediate microeconomics course. Then, substitute the optimal Hicksian demand functions back into the original optimization problem (discussed shortly) to get the expenditure function.

Both the indirect utility function and expenditure function have very useful derivative properties, the most famous of which are Roy's identity and Shephard's lemma. By Roy's identity, discussed shortly, one can derive Marshallian demand functions using a ratio of partial derivatives of the

[1] A utility function $u(\mathbf{q})$ is homothetic if

$$u(\mathbf{q}') > u(\mathbf{q}'') \text{ if and only if } u(t\mathbf{q}') > u(t\mathbf{q}'') \text{ for any } t > 0.$$

indirect utility function. Shephard's lemma allows us to derive an optimal Hicksian demand function by taking the partial derivative of the expenditure function with respect to the demand function's price. These results obtain for a general class of utility functions – particularly those that satisfy Assumption 1 – and have proven extremely useful in both conceptual and applied research.

1.1 The indirect utility function

The *indirect utility function* gives the household's maximum attainable utility given final good prices, $\mathbf{p}$, and income $\mathbf{w} \cdot \upsilon$, and is defined as

$$\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \upsilon) \equiv \max_{\mathbf{q}} \{u(\mathbf{q}) : \mathbf{w} \cdot \upsilon \geq \mathbf{p} \cdot \mathbf{q}\} \tag{2}$$

The indirect utility function inherits the following properties from the direct utility function (see Cornes, pp. 67-70):

V1. Homogeneous of degree zero in $\mathbf{p}$ and $\mathbf{w} \cdot \upsilon$; $\mathcal{V}(\theta\mathbf{p}, \theta\mathbf{w} \cdot \upsilon) = \mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \upsilon)$, $\theta > 0$,

V2. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \upsilon)$ is convex in $\mathbf{p}$,

V3. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \upsilon)$ is continuous and differentiable in $\mathbf{p}$ and $\mathbf{w} \cdot \upsilon$,

V4. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \upsilon) = v(\mathbf{p})\, \mathbf{w} \cdot \upsilon$: separable in $\mathbf{p}$ and $\mathbf{w} \cdot \upsilon$,

By V4, the marginal utility of an additional unit of income is $v(\mathbf{p})$.

V5. Given differentiability, Marshallian demands follow from Roy's identity,

$$q^j(\mathbf{p})(\mathbf{w} \cdot \upsilon) = -\frac{v_{p_j}(\mathbf{p})}{v(\mathbf{p})}\mathbf{w} \cdot \upsilon \tag{3}$$

where, throughout the text, the subscript on a function indicates a partial derivative, e.g., $v_{p_j} = \partial v(\mathbf{p}) / \partial p_j$ and $v_{p_1 p_2} = \partial^2 v(\mathbf{p}) / \partial p_1 \partial p_2$.

In applied work, the most important properties of the indirect utility function are V1, V2 and V5. By V1, a doubling of input and output prices leaves the level of utility unchanged. V2 implies consumer demand curves are downward sloping (discussed shortly), and V5 says if you have an indirect utility function, you can recover the consumer's demand function using ratios of first derivatives of the indirect utility function.

To see these properties in action, introduce the Cobb-Douglas utility function (1) into the optimization problem (2). The corresponding Lagrangian is

$$\mathcal{L} = q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3} + \lambda \left[\mathbf{w} \cdot \upsilon - p_1 q_1 - p_2 q_2 - p3 q_3 \right]$$

and, assuming an interior solution, first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial q_i} &= \alpha_i q_1^{\alpha_1} q_2^{\alpha 2} q_3^{\alpha 3} q_i^{-1} + \lambda p_i = 0, \qquad i = 1, 2, 3 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \mathbf{w} \cdot \upsilon - p_1 q_1 - p_2 q_2 - p3 q_3 = 0 \end{aligned} \tag{4}$$

Problem 2 *Show the solution to (4) yields the Marshallian demand curves*

$$q_j^* \left(p_j \right) \left(\mathbf{w} \cdot \upsilon \right) = \frac{\alpha_j \mathbf{w} \cdot \upsilon}{p_j}, \qquad j = 1, 2, 3 \tag{5}$$

where $q_j^* \left(p_j \right) = \frac{\alpha_j}{p_j}$.

1. *Take the Marshallian demand curves and substitute into the original Cobb-Douglas utility function (1) and simplify. The result is the indirect utility function.*
2. *Apply Roy's identity and verify that the resulting demands are identical to (5).*

Problem 3 *Use Mathematica to derive the Marshallian demand functions, and indirect demand function you derived in Problem 1. Using Mathematica, verify Roy's identity works.*

With preferences that are linearly homothetic, the indirect utility function is separable in final good prices and household endowments. Since consumers face the same prices and have identical preferences, we can aggregate across households to get an aggregate – "community" – indirect utility function. Assuming there are N households, the aggregate indirect utility function is given by

$$\mathcal{V} = v \left(\mathbf{p} \right) \sum_{\eta=1}^{N} \left(\mathbf{w} \cdot \upsilon^{\eta} \right)$$

Accordingly, the total Marshallian demand for good j is

$$Q_j = q^j \left(\mathbf{p} \right) \sum_{\eta=1}^{N} \left(\mathbf{w} \cdot \upsilon^{\eta} \right), \ \forall \ j \in \{1, 2, 3\} \tag{6}$$

These functions are the simple aggregation of individual consumer welfare and demands. It also follows from V1 that (6) is homogeneous of degree minus one in prices $\mathbf{p}$ and of degree one in income.

1.2 The expenditure function

The *expenditure function* gives the minimum cost of achieving utility level $q \in \mathbb{R}$ at given prices $\mathbf{p}$, and is defined as

$$E(\mathbf{p}, q) \equiv \min_{\mathbf{q}} \{\mathbf{p} \cdot \mathbf{q} : q \leq u(\mathbf{q})\} \tag{7}$$

The expenditure function inherits from $u(\cdot)$, the following properties:

E1. $E(\mathbf{p}, q) > 0$ for any $\mathbf{p}$ and $q > 0$,

E2. $E(\mathbf{p}, q)$ is non-decreasing in $\mathbf{p}$ and q,

E3. $E(\mathbf{p}, q)$ is concave and continuous in $\mathbf{p}$,

E4. $E(\lambda\mathbf{p}, q) = \lambda E(\mathbf{p}, q)$, $\lambda > 0$*:* homogeneous of degree 1 in $\mathbf{p}$,

E5. $E(\mathbf{p}, q) = \mathcal{E}(\mathbf{p})\, q$*:* separable in $\mathbf{p}$ and q,

E6. Shephard's lemma: If $E(\mathbf{p}, q)$ is differentiable in $\mathbf{p}$, then

$$q_j = E_{p_j}(\mathbf{p}, q) = \mathcal{E}_{p_j}(\mathbf{p})\, q, \;\; j = 1, ..., M$$

E1 says purchasing a strictly positive consumption bundle is costly. **E**2 says, all else equal, (i) if the price of a consumption good increases, then the cost of achieving the same level of utility increases, or (ii) increasing utility requires an increase in expenditures. By **E**3, the expenditure function is continuous, with first (partial) derivatives being positive and second partial derivatives being negative. Condition **E**4 implies demand functions are homogeneous of degree zero in $\mathbf{p}$. **E**5 results from Assumption 1.3 and implies demand functions are separable in $\mathbf{p}$ and q (see Chambers, 1988, chapter 2). Later, we interpret the quantity q to be a composite consumption good, the unit cost of which is $\mathcal{E}(\mathbf{p})$.

From the standpoint of the applied work that follows, the most important of the expenditure properties are **E**3 and **E**6. Together, they imply – if differentiable – Shephard's lemma will yield downward sloping Hicksian demand functions.

To see **E**3 and **E**6 in action, introduce the Cobb-Douglas utility function (1) into the optimization in (7). The corresponding Lagrangian is

$$\mathcal{L} = p_1 q_1 + p_2 q_2 + p3 q_3 + \lambda \left[q - q_1^{\alpha_1} q_2^{\alpha 2} q_3^{\alpha 3} \right]$$

Assuming an interior solution, the first order conditions for the optimal choice of $\mathbf{q}$ is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial q_i} &= p_i - \lambda \alpha_i q_1^{\alpha 1} q_2^{\alpha 2} q_3^{\alpha 3} q_i^{-1} = 0, \quad i = 1,2,3 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \mathbf{w} \cdot \upsilon - p_1 q_1 - p_2 q_2 - p3 q_3 = 0 \end{aligned} \tag{8}$$

Problem 4 *Show the solution to (8) yields the Hicksian demand curves*

$$\tilde{q}_j^* (\mathbf{p})\, q = \frac{\alpha_1^{-\alpha_1} \alpha_2^{-\alpha_2} \alpha_3^{-\alpha_3} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} q}{\alpha_j p_j}, \quad j = 1,2,3 \tag{9}$$

where $\tilde{q}_j^* (\mathbf{p}) = \frac{\alpha_1^{-\alpha_1} \alpha_2^{-\alpha_2} \alpha_3^{-\alpha_3} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}{\alpha_j p_j}$.

1. *Take the Hicksian demand curves and substitute into the expression* $\mathbf{p} \cdot \mathbf{q}$*, and simplify. The result is the expenditure function.*

2. *Apply Shephard's lemma to the expenditure function derived in part 1 of this problem, and verify that the resulting demands are identical to (9).*

Problem 5 *Use Mathematica to derive the Hicksian demand functions, and the expenditure function you derived in Problem 4. Using Mathematica, verify Shephard's lemma gives you the Hicksian demands (9).*

1.3 Aggregation

With homothetic preferences, the expenditure function is separable in final good prices and the utility index. If households are identical, assuming there are N households, the aggregate expenditure function is given by

$$E(\mathbf{p}, q) = \mathcal{E}(\mathbf{p})\, Nq$$

Accordingly, the total Hicksian demand for good j is

$$Q_j = \tilde{q}_j^* (\mathbf{p})\, Nq, \ \forall\ j \in \{1,2,3\}$$

These functions are the simple aggregation of individual consumer welfare and demands. It also follows from **E**4 that the aggregate expenditure function is homogeneous of degree one in prices $\mathbf{p}$.

2 Production technologies

We examine, now, the production side of the economy. Unless stated otherwise, assume each sector j is composed of a large number of identical, atomistic firms, and each firm faces the same vector of input and output prices. Let y_j be the output of each firm in sector j and let $\upsilon^j = (\kappa_j, l_j, z_j) \in \mathbb{R}^3_+$ represent the vector of productive factors used by a sector-j firm, where, κ_j, l_j and z_j are the levels of K, L and Z used by the sector-j firm. Note that the notation $(\kappa_j, l_j, z_j) \in \mathbb{R}^3_+$ implies a sector might not use all factors to produce output. Represent the technology of a sector-j firm by the production function $f^j : \mathbb{R}^3_+ \to \mathbb{R}_+$, defined as $y_j = f^j(\kappa_j, l_j, z_j)$, where, for one or more sectors, $z_j = 0$.

Assumption 2. $f^j(\upsilon^j)$ satisfies the following properties:

1. $f^j(\mathbf{0}) = 0$, and $f^j(\upsilon^j) > 0$ for any $\upsilon^j \gg \mathbf{0}^N$,
2. $f^j(\upsilon^j)$ is linearly homothetic in υ^j,
3. $f^j(\upsilon^j)$ is non-decreasing and strictly concave in υ^j,
4. $f^j(\upsilon^j)$ is everywhere continuous and everywhere twice differentiable in υ^j.

Here $\mathbf{0}^N \in \mathbb{R}^N_+$ is a vector of N zeros and the notation $\upsilon^j \gg \mathbf{0}$ means at least one element of υ^j is strictly positive. Assumption 2.1 ensures it is not possible to produce a positive level of output with no input, and ensures there are no fixed costs. Assumption 2.2 says individual firm technologies satisfy constant returns to scale (CRS). An important implication of Assumption 2.2 is, when all firms face the same output and input prices, sectoral production levels and input demands are simple linear aggregations of individual firm choices. Another implication is the corresponding cost function is separable in input prices and output levels. Assumption 2.3 ensures the production technology is well-behaved and yields the familiar convex isoquants: it imposes diminishing marginal returns on individual input use. Assumptions 2.1 and 2.3 also ensures the existence of a cost and aggregate value-added (GDP) function defined later. Finally, Assumption 2.4 allows the use of differential calculus to derive corresponding cost and GDP functions.

Problem 6 *Show that the Cobb-Douglas production function*

$$y = \Psi K^{\alpha_1} L^{\alpha_2} Z^{\alpha_3}, \ \textit{where } \alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{10}$$

satisfies Assumptions2.2 and 2.2. Does it satisfy Assumption2.1. Is it twice differentiable? Here, the scalar $\Psi \in \mathbb{R}_{++}$ *is a productivity parameter.*

As with the utility function, the production function is also used to create two "dual" functions: the cost function and the value-added function. The cost function is the producer analog to the consumer's expenditure function. It is derived by choosing input levels to minimize the cost of producing a given level of output, yielding cost minimizing factor (input) demand functions. By substituting these optimal factor demands back into the original objective function we get the producer's cost function. The value-added function is the maximum net income a producer can earn, given one or more of her productive factors is fixed. To derive a value-added function, we solve a standard "profit maximization" problem to get optimal, constrained input demand functions that each depend on the output price, factor variable prices, and the fixed factor(s). Substituting the resulting constrained input demands into the original objective function yields the value-added function.

2.1 The cost function

In this section and the next, we suppress all sector-related indices, i.e., the "j" subscripts and superscripts. The *cost function* gives the minimum cost of producing an arbitrary level of output, y, when factor prices are equal to $\mathbf{w}$, and is defined as:

$$c\left(r, w, \tau, y\right) \equiv \min_{\kappa, l, z} \left\{ r\kappa + wl + \tau z : y \leq f\left(\kappa, l, z\right) \right\} \tag{11}$$

As noted above, the cost function is the firm's analog of the household's expenditure function. In what follows, let $\tilde{v} = (\kappa, l, z)$ be an arbitrary input vector. The cost function from Assumption 2, the following properties:

C1. $c\left(\mathbf{w}, y\right) > 0$ for any $\mathbf{w}$ and $y_j > 0$,

C2. $c\left(\mathbf{w}, y\right)$ is non-decreasing in $\mathbf{w}$ and y,

C3. $c\left(\mathbf{w}, y\right)$ is concave and continuous in $\mathbf{w}$,

C4. $c\left(\theta\mathbf{w}, y\right) = \theta c\left(\mathbf{w}, y\right)$: homogeneous of degree one in $\mathbf{w}$,

C5. $c\left(\mathbf{w}, y\right) = C\left(\mathbf{w}\right) y$: separable in $\mathbf{w}$ and y, where $C(\mathbf{w})$ is the unit cost of producing a unit of output. Finally, letting we have

C6. Shephard's lemma: If $c(\mathbf{w}, y)$ is differentiable in $\mathbf{w}$, then

$$\kappa^*(\mathbf{w}, y) = C_r(\mathbf{w})\, y, \quad l^*(\mathbf{w}, y) = C_r(\mathbf{w})\, y \text{ and } z^*(\mathbf{w}, y) = C_r(\mathbf{w})\, y$$

where $C_i(\cdot)\, y$ is the derived unit demand for input i.

C1 says producing a strictly positive level of output is costly. **C**2 says, all else equal, if the price of an input increases, production cost increases; or increasing output increases production costs. By **C**3, the cost function is continuous and yields conditional input demand functions that are decreasing in own prices. By condition **C**4, if all input prices are increased by a factor θ, production costs increase by a factor of θ. Although not necessarily obvious at first glance, condition **C**4 implies input demand functions are homogeneous of degree zero in $\mathbf{w}$. **C**5 results from Assumption 2.2 and implies constant marginal and average costs. Furthermore, given **C**5, the output supply and input demand functions are both separable in $\mathbf{w}$ and y_j (see Chambers, 1988, chapter 2).

More than likely, in your intermediate microeconomics class you solved a constrained optimization problem where the production function was Cobb-Douglas. To solve this problem you would introduce (a two input version of) the Cobb-Douglas production function (10) into the optimization problem (11). The corresponding Lagrangian is

$$\mathcal{L} = r\kappa + wl + \tau z + \lambda \left[y_j - \Psi \kappa^{\alpha_1} l^{\alpha_2} z^{\alpha_3} \right]$$

and, assuming an interior solution, the first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K_j} &= r - \lambda \alpha_1 \Psi \kappa^{\alpha_1} l^{\alpha_2} z^{\alpha_3} \kappa_j^{-1} = 0 \\ \frac{\partial \mathcal{L}}{\partial L_j} &= w - \lambda \alpha_2 \Psi \kappa^{\alpha_1} l^{\alpha_2} z^{\alpha_3} l_j^{-1} = 0 \\ \frac{\partial \mathcal{L}}{\partial Z_j} &= \tau - \lambda \alpha_3 \Psi \kappa^{\alpha_1} l^{\alpha_2} z^{\alpha_3} z_j^{-1} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= y_j - \Psi \kappa^{\alpha_1} l^{\alpha_2} z^{\alpha_3} = 0 \end{aligned} \tag{12}$$

Problem 7 *Show the solution to (12) yields the factor demand curves*

$$\begin{aligned}\kappa^*(r,w,\tau)\,y &= \left(\frac{r}{\alpha_1}\right)^{\alpha_1-1}\left(\frac{w}{\alpha_2}\right)^{\alpha_2}\left(\frac{\tau}{\alpha_3}\right)^{\alpha_3}\frac{y}{\Psi} \\ l^*(r,w,\tau)\,y &= \left(\frac{r}{\alpha_1}\right)^{\alpha_1}\left(\frac{w}{\alpha_2}\right)^{\alpha_2-1}\left(\frac{\tau}{\alpha_3}\right)^{\alpha_3}\frac{y}{\Psi} \\ z^*(r,w,\tau)\,y &= \left(\frac{r}{\alpha_1}\right)^{\alpha_1}\left(\frac{w}{\alpha_2}\right)^{\alpha_2}\left(\frac{\tau}{\alpha_3}\right)^{\alpha_3-1}\frac{y}{\Psi}\end{aligned} \tag{13}$$

1. *Take the factor demand curves in (13) and substitute into the production cost expression, $r\kappa + wl + \tau z$, and simplify. The result is the (three input version of the) producer's cost function.*

2. *Apply Shephard's lemma to the cost function you derived in part 1 of this problem, and verify that the resulting demands are identical to those in (13).*

Problem 8 *Use Mathematica to derive the factor demand functions, and the cost function you derived in Problem 7. Using Mathematica, verify Shephard's lemma works.*

2.2 The value-added function

The value added function gives the maximum rent (profit) a firm can earn on its fixed factors. For example, a rural household could be endowed with land and household labor, and not be able to rent land from another farmer (e.g., a village leader makes land assignments to household members) and unable to afford hiring non-family labor. In our three input case, assume land is a fixed factor, but labor and capital are variable inputs. Ingnoring sector sub- and superscripts, the value added function is defined as:

$$\Pi(p,w,r,z) \equiv \max_{y,\kappa,l}\{py - r\kappa - wl : y \leq f(\kappa,l,z)\} \tag{14}$$

Given Assumption 2, the sectoral value-added function properties include:

Π1. $\Pi(p,w,r,z) \geq 0$ for all p, r, w and z,

Π2. $\Pi(p,\mathbf{w},\mathbf{z})$ is nondecreasing in p and nonincreasing in r and w,

Π3. $\lambda\Pi(p,w,r,z) = \Pi(\lambda p,\lambda w,\lambda r,z)$, $\lambda > 0$: linearly homogeneous in p_j, w and r,

Π4. $\lambda\Pi(p,w,r,z) = \Pi(p,w,r,\lambda z)$, $\lambda > 0$: linearly homogeneous in z

Π5. $\Pi(p,w,r,z) = \pi(p,w,r)\,z$: separable in fixed endowments,

Π6. Hotelling's lemma. If $\Pi(\cdot)$ is everywhere differentiable in p, w, r and z, then firm y and sectoral factor demands are, respectively,

$$y = \Pi_p(p,w,r,z)$$
$$\kappa = -\Pi_r(p,w,r,z)$$
$$l = -\Pi_w(p,w,r,z)$$

The factor rental rate (or shadow price) of the fixed factor is given by

$$\tau = \Pi_z(p,w,r,z) = \pi(p,w,r)$$

In the example, here, where the household has a single fixed, $\pi(p,w,r)$ is the minimum rental rate the household member would take to rent out his or her land to someone else.

Π1 says the optimal rent is non-negative: the household can always choose to not produce anything. By Π2, all else equal, if the output price increases, land rent increases, but if an input price increases, land rent falls. By Π3, increasing the output price and input prices by a (strictly positive) factor increases rent by that factor (i.e., double input and output prices, and rent doubles). By condition Π4, doubling the fixed factor endowment doubles the total rent to that factor. Property Π6 tells us that the first derivative of the value added function with respect to output and factor prices lets us recover the household's output supply and factor demand curves. Also, the partial derivative of $\Pi(\cdot)$ with respect to z yields the inverse (shadow) demand for the fixed factor.

Let $f(\kappa,l,z) = \Psi\kappa^{\alpha_1}l^{\alpha_2}z^{\alpha_3}$. Then the constrained optimization problem associated with (14) yields the following Lagrangian:

$$\mathcal{L} = p\Psi\kappa^{\alpha_1}l^{\alpha_2}z^{\alpha_3} - r\kappa - wl$$

Assuming an interior solution, the first order conditions are:

$$\frac{\partial\mathcal{L}}{\partial\kappa} = p\alpha_1\Psi\kappa^{\alpha_1}l^{\alpha_2}z^{\alpha_3}\kappa^{-1} - r = 0 \qquad (15)$$
$$\frac{\partial\mathcal{L}}{\partial l} = p\alpha_2\Psi\kappa^{\alpha_1}l^{\alpha_2}z^{\alpha_3}l^{-1} - r = 0$$

Problem 9 *Show the solution to (15) yields the factor demand curves*

$$\kappa^* (p, r, w)\, z = \left(\frac{r}{\alpha_1}\right)^{\alpha_1 - 1} \left(\frac{w}{\alpha_2}\right)^{\alpha_2} (p\Psi)^{\alpha_3}\, z \qquad (16)$$
$$l^* (p, r, w)\, z = \left(\frac{r}{\alpha_1}\right)^{\alpha_1} \left(\frac{w}{\alpha_2}\right)^{\alpha_2 - 1} (p\Psi)^{\alpha_3}\, z$$

1. *Take the factor demand curves in (13) and substitute into the net revenue expression,* $p\Psi\kappa^{\alpha_1} l^{\alpha_2} z^{\alpha_3} - r\kappa - wl$*, and simplify. The result is the producer's value added function.*

2. *Apply Hotelling's lemma to the value added function you derived in part 1 of this problem, and verify that the resulting demands are identical to those in (16).*

3. *Use Hotelling's lemma and derive the supply function associated with the value added function you derived in part 1 of this problem.*

Problem 10 *Use Mathematica to derive the factor demand functions, output supply, and value-added function you derived in Problem 9. Using Mathematica, verify Hotelling's lemma works.*

2.3 Aggregation

Later, it will be convenient to aggregate the factor demands and output supplies of individual producers to derive sectoral factor demands and supplies. We first take up aggregating firm level cost and value added functions, and then introduce the concept of a aggregate gross domestic product function.

2.3.1 The aggregate cost function and value added function

Since all firms in a sector employ the same technology and face the same output and input prices, characterizing the aggregate technology for the sector is straightforward. For this section, denote the number of firms sector-j by I_j. While the total number of firms in a sector is indeterminate, their identical nature implies if each firm produces a share, Υ_j^o, of total sectoral output, Y_j, then the representative firm from that sector also employs the same Υ_j^o share of factor inputs. If $y_j^i = y_j$ represents the output produced by a sector-j firm, it follows that $y_j = \Upsilon_j^o Y_j$ and

$$(\kappa_j, l_j, z_j) = \left(\Upsilon^o K_j, \Upsilon_j^o L_j, \Upsilon_j^o Z_j\right)$$

for each firm. Here, (κ_j, l_j, z_j) is the vector of inputs used by the representative sector-j firm.[2] Hence,

$$y_j = f^j\left(\Upsilon_j^o K_j, \Upsilon_j^o L_j, \Upsilon_j^o Z_j\right)$$

In such a case, the sector level production function is a linear expansion of individual firm production functions. That is, with any constant returns to scale technology

$$Y_j = I_j f^j\left(\Upsilon_j^o K_j, \Upsilon_j^o L_j, \Upsilon_j^o Z_j\right) = f^j\left(I_j \Upsilon_j^o K_j, I_j \Upsilon_j^o L_j, I_j \Upsilon_j^o Z_j\right)$$

which implies the sector level production function is

$$Y_j = f^j\left(K_j, L_j, Z_j\right)$$

To distinguish between firm level and aggregate sectoral production however, it is convenient to represent the aggregate technology for sector j by the production function $\mathcal{F}^j : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$, defined as

$$Y_j = \mathcal{F}^j\left(K_j, L_j, Z_j\right) \tag{17}$$

Then, the corresponding sectoral cost function, denoted TC_j, is given by

$$TC_j = C^j\left(\mathbf{w}\right) Y_j \tag{18}$$

where

$$C^j\left(\mathbf{w}\right) Y_j \equiv \min_{K_j, L_j, Z_j} \left\{rK_j + wL_j + \tau Z_j : Y_j \leq \mathcal{F}^j\left(K_j, L_j, Z_j\right)\right\}$$

Consider now, the case where a sector has a factor only it employs: for example, land might be viewed as factor having value only in producing agricultural products. Farmers can rent land in and out among themselves at some market determined land rental rate, but they do not rent land to producers in other sectors of the economy. If each farmer's production function satisfies Assumption 2, there exists a corresponding sectoral agricultural production and cost function (17) and (18). However, in the case of the sectoral production function, the sector specific factor is pre-determined or fixed, and the sectoral level production function exhibits decreasing returns to scale in all the other factors employed in other sectors of the economy. This property gives rise to a sector level value-added function.

[2] Note, given Υ_j^o is the share of output produced by the representative firm in sector-j, with I_j firms, $\sum_{i=1}^{I_j} \Upsilon_j^o = I_j \Upsilon_j^o = 1$.

In the three factor example, again let z be the fixed factor, and let $Z_j = I_j z$. With the fixed factor Z_j, the j^{th} sector value-added function is defined as:

$$\Pi^j(p_j, r, w, Z_j) \equiv \max_{Y_j, K_j, L_j} \left\{ p_j Y_j - rK_j - wL_j : Y_j \leq \mathcal{F}^j(K_j, L_j, Z_j) \right\} \tag{19}$$

where $\mathbf{0}^{N-\zeta_j} \in \mathbb{R}_+^{N-\zeta_j}$ is a vector of zeros. Given Assumption 2, the sectoral value-added function properties include:

Π1. $\Pi^j(p_j, r, w, Z_j) \geq 0$ for all p_j, r, w, and Z_j,

Π2. $\Pi^j(p_j, r, w, Z_j)$ is nondecreasing in p_j and nonincreasing in (r, w),

Π3. $\lambda \Pi^j(p_j, r, w, Z_j) = \Pi^j(\lambda p_j, \lambda r, \lambda w, Z_j)$, $\lambda > 0$: linearly homogeneous in p_j and (r, w),

Π4. $\lambda \Pi^j(p_j, r, w, Z_j) = \Pi^j(p_j, r, w, \lambda Z_j)$, $\lambda > 0$: linearly homogeneous in Z_j

Π5. $\Pi^j(p_j, r, w, Z_j) = \pi^j(p_j, r, w) Z_j$: separable in fixed endowments,

Π6. Hotelling's lemma. If $\Pi^j(\cdot)$ is everywhere differentiable in p, r, w and Z_j, then sectoral supply and sectoral factor demand are, respectively,

$$\begin{aligned} Y_j &= \Pi^j_{p_j}(p_j, r, w, Z_j) \\ K_j &= -\Pi^j_r(p_j, r, w, Z_j) \\ L_j &= -\Pi^j_w(p_j, r, w, Z_j) \end{aligned}$$

The factor rental rate (or shadow price) of the sector specific factors is given by

$$\tau_j = \Pi^j_{Z_j}(p_j, r, w, Z_j) = \pi^j(p_j, r, w)$$

For the case of a single sector specific factor that is rented in or out among producers, $\pi^j(p_j, \mathbf{w})$ is the rental rate that clears the fixed factor (e.g., agricultural land) rental market.

2.3.2 The aggregate gross national product function

The economy-wide gross national product (GNP) function is obtained by maximizing aggregate sectoral income subject to the technology (17) and the endowment constraints. The economy-wide GNP function is defined as:

$$G(\mathbf{p}, K, L, Z) \equiv \max_{K_j, L_j, Z_j} \left\{ \sum_{j=1}^{3} p_j \mathcal{F}^j(K_j, L_j, Z_j) : K \geq \sum_{j=1}^{3} K_j, L \geq \sum_{j=1}^{3} L_j, Z \geq \sum_{j=1}^{3} Z_j, \right\} \tag{20}$$

Woodland (1982, pp. 123) shows the function $G(\cdot)$ satisfies the following properties:[3]

G1. $G(\mathbf{p}, K, L, Z) \geq 0$ for all $\mathbf{p}$ and $(K, L, Z) \gg 0$,

G2. $G(\lambda \mathbf{p}, K, L, Z) = \lambda G(\mathbf{p}, K, L, Z)$, $\lambda > 0$: linearly homogeneous in $\mathbf{p}$,

G3. $G(\mathbf{p}, \lambda K, \lambda L, \lambda Z) = \lambda G(\mathbf{p}, K, L, Z)$, $\lambda > 0$: linearly homogeneous in (K, L, Z),

G4. $G(\mathbf{p}, K, L, Z)$ is continuous, non-decreasing, and convex in $\mathbf{p}$,

G5. $G(\mathbf{p}, K, L, Z)$ is continuous, non-decreasing, and concave in (K, L, Z),

G6. Hotelling's lemma. If $G(\cdot)$ is everywhere differentiable in $\mathbf{p}$ and (K, L, Z), then

$$Y_j = G_{p_j}(\mathbf{p}, K, L, Z)$$

and

$$r = G_K(\mathbf{p}, K, L, Z), \; w = G_L(\mathbf{p}, K, L, Z) \text{ and } \tau = G_Z(\mathbf{p}, K, L, Z)$$

The major implication of conditions G1 – G6 is that the partial derivatives of $G(\cdot)$ yield aggregate sectoral supply functions, $G_{p_j}(\mathbf{p}, K, L, Z)$, that are non-decreasing in own-prices, homogeneous of degree zero in prices $\mathbf{p}$, and homogeneous of degree one in factor endowments K, L and Z. Also, the inverse factor demand functions, e.g., $G_K(\mathbf{p}, K, L, Z)$ are downward sloping in own factor levels, homogeneous of degree one in prices and homogeneous of degree zero in endowments. The Hessian matrix of $G(\mathbf{p}, K, L, Z)$ is positive semi-definite.[4]

3 Appendix

3.1 The Primal-Dual Problem (Envelope Theorem)

Define the *indirect* function

$$\phi(\alpha) \equiv f(x^*(\alpha), \alpha) = \max_x \{f(x, \alpha)\}$$

[3] $G(\cdot)$ is analogous to a constrained revenue function.

[4] Young's theorem implies that the second derivative matrix of $G(\mathbf{p}, K, L, Z)$ is symmetric, i.e., $G_{p,K}(\cdot) = G_{K,,p}(\cdot)$. Thus, an increase in a factor price due to a unit increase in p_j is equal to the increase in Y_j due to an increase in a factor endowment. See Diewert (1973, 1974).

where α are parameters, (which you might interpret to be the price of output and/or inputs) and $f(\cdot)$ is a continuous, increasing and quasi-concave function in x. The primal-dual specification

$$L(x,\alpha) = \phi(\alpha) - f(x,\alpha)$$

evaluated at $x^*(\alpha)$ suggests

$$\frac{\partial}{\partial\alpha}\phi(\alpha) - \frac{\partial}{\partial\alpha}f(x^*(\alpha),\alpha) \equiv 0 \Rightarrow \frac{\partial}{\partial\alpha}\phi(\alpha) = \frac{\partial}{\partial\alpha}f(x^*(\alpha),\alpha)$$

Thus, a finite change in the "parameter" α, which implies x changing according to the rule $x^*(\alpha)$, equals the rate of change of the direct objective function $f(\cdot)$ holding x constant at x^*. In the case of the "typical" profit function, this result is known as Hotelling's lemma.

In the case of constrained optimization problems, e.g.

$$\Omega(\alpha) \equiv \min_x \; \{f(x,\alpha) \text{ s.t. } g_1(x,\alpha) = 0, \cdots, g_N(x,\alpha) = 0\}$$

where the constraints, $g_n(\cdot), n = 1, ..., N$, are continuous and concave in x (thus, the choice set $x \in X$ is convex), and at the solution (x^*) hold with equality. Consider the primal-dual problem

$$L = \Omega(\alpha) - \left\{ f(x,\alpha) \;\; + \sum_{n=1}^{N} \lambda_n g_n(x,\alpha) \right\}$$

evaluated at $\{x^*\}$

$$\frac{\partial}{\partial\alpha}\Omega(\alpha) - \frac{\partial}{\partial\alpha}f(x,\alpha) - \sum_{n=1}^{N} \lambda_n \frac{\partial}{\partial\alpha} g_n(x,\alpha) \equiv 0$$

If α did not appear in the constraint $g_n(x,\alpha)$ set, $n \in N$ (as in the constrained cost minimization problem where α are prices of inputs and, for example, a constraint $g(x,\cdot)$, characterizes the technology,), then we obtain

$$\frac{\partial}{\partial\alpha}\Omega(\alpha) = \frac{\partial}{\partial\alpha}f(x,\alpha)$$

In this case $\Omega(\alpha)$ is the total cost *function*, and the result is Shephard's lemma

$$\frac{\partial}{\partial\alpha}\Omega(\alpha) = x$$

where $\frac{\partial}{\partial\alpha}\Omega(\alpha)$ is conditional factor demand. In the case of constrained utility maximization, $\Omega(\alpha)$ is the indirect utility function, α are prices of goods, and a single *linear* equation $g_n(x,\alpha)$,characterizes the budget constraint. The direct utility function is $f(x,\cdot)$ where α does not appear.

Problem 11 *Let* $\phi(\alpha) = \pi(p,w,r)\, Z \equiv \max_{K,L}\{pf(K,L,Z) - wL - rK\}$. *In this problem setup, what is* α*? Use the Envelope Theorem to establish Hotelling's Lemma.*

3.2 Elasticities and homogenous functions

Definition The twice continuously differentiable function $f : R_+^2 \to R$, defined as $z = f(x, y)$, is homogeneous of degree n, if for all possible real values of λ

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

Differentiable homogeneous functions satisfy Euler's theorem:

$$\frac{\partial f(x,y)}{\partial x}x + \frac{\partial f(x,y)}{\partial y}y = nf(x,y).$$

The two special cases are when $n = 1$ and when $n = 0$. If $n = 1$, then we have

$$\frac{\partial f(x,y)}{\partial x}x + \frac{\partial f(x,y)}{\partial y}y = z \tag{21}$$

and if $n = 0$ we have

$$\frac{\partial f(x,y)}{\partial x}x + \frac{\partial f(x,y)}{\partial y}y = 0. \tag{22}$$

If we want to convert these functions into elasticities, in the case of (21), dividing by z we have:

$$\frac{\partial f(x,y)}{\partial x}\frac{x}{z} + \frac{\partial f(x,y)}{\partial y}\frac{y}{z} = 1$$

Now, for all real values of x, y, z, the elasticities

$$\underbrace{\xi^x}_{\frac{\partial f(x,y)}{\partial x}\frac{x}{z}} + \underbrace{\xi^y}_{\frac{\partial f(x,y)}{\partial y}\frac{y}{z}} = 1$$

sum to unity. For the case of $n = 0$, dividing (22) by z we have

$$\underbrace{\xi^x}_{\frac{\partial f(x,y)}{\partial x}\frac{x}{z}} + \underbrace{\xi^y}_{\frac{\partial f(x,y)}{\partial y}\frac{y}{z}} = 0$$

If the homothetic function $f(x, y)$ is homogeneous of degree $n \geq 1$, then the partial derivatives

$$\frac{\partial f(x,y)}{\partial x}, \text{ or } \frac{\partial f(x,y)}{\partial y}$$

are homogeneous of degree $n - 1$ (e.g., Varian, p 482).

Suppose we ask, how does z change with changes in x and y? Mathematically, the question is

$$dz = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

Multiply the first and second term by x/x and y/y, respectively

$$dz = \frac{\partial f(x,y)}{\partial x} x \frac{dx}{x} + \frac{\partial f(x,y)}{\partial y} y \frac{dy}{y}$$

Divide the entire expression by z

$$\frac{dz}{z} = \frac{\partial f(x,y)}{\partial x} \frac{x}{z} \frac{dx}{x} + \frac{\partial f(x,y)}{\partial y} \frac{y}{z} \frac{dy}{y}$$

Finally, define the **rate** of change $\hat{z} \equiv dz/z$, and use the elasticity formulas above to obtain

$$\hat{z} = \xi^x \hat{x} + \xi^y \hat{y}$$

Knowing whether $f(x,y)$ is homogeneous of degree 1 or 0, helps to explain the change in $\hat{z}$ given a change in the exogenous variables $\hat{x}, \hat{y}$. Suppose $f(x,y)$ is a production function that is homogeneous of degree one in the inputs (x,y). Then, the term $\partial f(x,y)/\partial x$ is the marginal physical product of x. If a firm maximizes profits, we have

$$p\frac{\partial f(x,y)}{\partial x} x + p\frac{\partial f(x,y)}{\partial y} y = pz \tag{23}$$

where we simply multiply each term by output price p. Profit maximization implies the marginal value product of x should equal the unit cost of x, say w_x, and likewise for the input y, say w_y, i.e.,

$$p\frac{\partial f(x,y)}{\partial x} = w_x; \text{ and } p\frac{\partial f(x,y)}{\partial y} = w_y \tag{24}$$

Thus, we have

$$w_x \cdot x + w_y \cdot y = pz$$

That is, when the production function exhibits constant returns to scale, the value of output pz is just sufficient to pay for the factors of production $w_x x + w_y y$

Suppose $f(x,y)$ is Cobb-Douglas, i.e.,

$$z = a x^{\alpha} y^{1-\alpha}$$

What kind of data do we need to estimate the parameters a, and α? First, suppose we have data on the **values** $w_x x$, $w_y y$, and pz. Then, (24) becomes

$$p\frac{\alpha z}{x}\alpha = w_x, \text{ and } p\frac{(1-\alpha)z}{y} = w_y \Rightarrow \alpha = \frac{w_x x}{pz}, \text{ and } (1-\alpha) = \frac{w_y y}{pz}$$

Thus, the coefficients, or production elasticities equal their respective cost (revenue) shares $w_x x/pz$

References

[1] Mas-Colell, Whinston and Green (Section 3.D, pp:51-63; Section 5.C, pp:135-142, The Envelope Theorem 964-966)

[2] Roe, T., R.B.W. Smith and S. Saraçoglu, Chapter 2

[3] Varian, 3^{rd} edition, pages: 81, 90, 91, 106, 129, 294

[4] Silberberg 2^{nd} edition, pages:195-207, 281-287