APEC 5152
Utility, Indirect Utility and Expenditure

Consumption

We'll use Cobb-Douglas preferences, \( u = q_1^{\gamma_1} q_2^{\gamma_2} q_3^{\gamma_3} \), where \( \gamma_1 + \gamma_2 + \gamma_3 = 1 \). It turns out that using the Cobb-Douglas function doesn't work with the Solve function. Taking a monotonic transformation of the function - a log transformation - however, yields a solution.

The Indirect Utility Function

The lagrangian

\[
\begin{align*}
U &= \log[q_1^{\gamma_1} q_2^{\gamma_2} q_3^{\gamma_3}] \quad \text{// PowerExpand} \\
L_0 &= U + \lambda \left( wv - p_1 q_1 - p_2 q_2 - p_3 q_3 \right) \\
\end{align*}
\]

The first-order conditions

\[
\begin{align*}
\text{FOC}_1 &= \frac{\partial L_0}{\partial q_1} = \gamma_1 q_1 - p_1 \lambda \\
\text{FOC}_2 &= \frac{\partial L_0}{\partial q_2} = \gamma_2 q_2 - p_2 \lambda \\
\text{FOC}_3 &= \frac{\partial L_0}{\partial q_3} = \gamma_3 q_3 - p_3 \lambda \\
\end{align*}
\]
\textbf{Expenditure Function}

The optimization problem is \( L = p_1 q_1 + p_2 q_2 + p_3 q_3 + \lambda (u - q_1^{\gamma_1} q_2^{\gamma_2} q_3^{\gamma_3}) \)

The solution gives the system of demands
Applying Shepard’s lemma to \( p1 \) gives

\[
\partial_{p1} \text{CC} = \left( \frac{p1}{\gamma1} \right)^{1-\gamma1} \left( \frac{p2}{\gamma2} \right)^{2} \left( \frac{p3}{\gamma3} \right)^{\gamma3}
\]

Since \(-1 + \gamma1 = \gamma2 + \gamma3\), this is the same as \( q1\text{demand} \).
1. Cobb-Douglas production function, cost minimization problem, setup and first-order conditions

Introduce a three-input (k, l, z) Cobb-Douglas production function, and use it to derive a cost function. We do this by solving a cost minimization problem, with the objective function defined by “cst” and setting up the typical lagrangian, and then taking the first-order conditions.

\[ y = \psi k^{\alpha_1} l^{\alpha_2} z^{\alpha_3} \]
\[ \text{cst} = r k + w l + \tau z \]
\[ \text{Lagrangian} = \text{cst} + \lambda (y - y) \]
\[ \text{fc} = \frac{\partial}{\partial k} \text{Lagrangian} = \lambda \alpha_1 \]
\[ \text{fd} = \frac{\partial}{\partial l} \text{Lagrangian} = \lambda \alpha_2 \]
\[ \text{fz} = \frac{\partial}{\partial z} \text{Lagrangian} = \lambda \alpha_3 \]

Now solve the system of equations for optimal levels of all inputs (k, l, z) and for the Lagrangian multiplier, \( \lambda \). This yields the factor demand curves, \( \text{fdk} \), \( \text{fdl} \), \( \text{fdz} \). Using // ExpandAll eliminates the exponential type answers.

\[ \text{cf} = \text{Solve}[\{\text{fc} = 0, \text{fd} = 0, \text{fz} = 0, \text{fc4} = 0\}, \{k, l, z, \lambda\}] \]

\[ \text{fdk} = \text{cf}[1, 1, 2] \]
\[ \text{fdl} = \text{cf}[1, 2] \]
\[ \text{fdz} = \text{cf}[1, 3] \]

1.1 Substitute factor demand curves into cost function to derive producer's cost function

Here, we substitute the factor demand curves derived above into the original objective function - the result is the parameterized version of the cost function associated with the Cobb-Douglas production function.
1.2 Apply Shepard’s lemma to the producer’s cost function to derive factor demand curves

\[ \psi = \frac{\alpha}{\partial w} + \text{Rent} \]

\[ \frac{\partial}{\partial \alpha} \left( \frac{1}{\psi} \right) = \frac{1}{\psi} \frac{\partial \psi}{\partial \alpha} = \frac{1}{\partial \alpha} \]

\[ \alpha \]

Now solve for the factor demand curves.
factdem = Solve[{FOCl = 0, FOCl = 0}, {k, l}] // ExpandAll // FullSimplify;
FDk = factdem[[1, 1]]
FDl = factdem[[1, 2]]

k \rightarrow p^{-1} r z \alpha_1 \alpha_2 \psi \alpha_1 \alpha_2
l \rightarrow p^{-1} r z \alpha_1 \alpha_2 \psi \alpha_1 \alpha_2

2.1 Substitute factor demand curves into net revenue function to derive producer's value-added function

vaf = Rent /. factdem[[1]] // PowerExpand // FullSimplify

- p^{-1} r z \alpha_1 \alpha_2 \psi \alpha_1 \alpha_2

2.2 Apply Hotelling's lemma to the value-added function to derive factor demand curves

khotel = -\partial_r vaf
lhotel = -\partial_w vaf

Now test to see if Hotelling's lemma holds for factor demands

khotel - FDk // Simplify
lhotel - FDl // Simplify

0
0

2.3 Apply Hotelling's lemma to derive the supply function for this single-sector model

\partial_p vaf

D[vaf, w] // {\alpha_1 \rightarrow 0.2, \alpha_2 \rightarrow 0.3, \alpha_3 \rightarrow 0.5}
D[%, w] // {\alpha_1 \rightarrow 0.2, \alpha_2 \rightarrow 0.3, \alpha_3 \rightarrow 0.5}
D[vaf, p] // {\alpha_1 \rightarrow 0.2, \alpha_2 \rightarrow 0.3, \alpha_3 \rightarrow 0.5}
D[%, p] // {\alpha_1 \rightarrow 0.2, \alpha_2 \rightarrow 0.3, \alpha_3 \rightarrow 0.5}

0.0765255 p^2 z^1 \psi^2
r^0.4 w^{1.6}
0.122441 p^2 z^1 \psi^2
r^0.4 w^{2.6}
0.255085 p^1 z^1 \psi^2
r^0.4 w^{0.6}
0.255085 z^1 \psi^2
r^0.4 w^{0.6}
\[ rK = 26460.4 + 51410.2 + 9930.2 + 151297.3 \]
\[ wL = 6902.7 + 44125.4 + 7798.6 + 242620.1 \]
\[ \text{Solve}\left[\frac{rK}{K} == 0.1, K\right] \]
\[ \text{Solve}\left[\frac{wL}{L} == 3257, L\right] \]
239098.
301447.
\{\{K \to 2.39098 \times 10^6\}\}\]
\{\{L \to 92.5535\}\}\]
\[ \frac{rK}{2390981} // N \]
0.1