Handout 9 - The Three Sector Ramsey Model

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AP 5152 Applied Macro-Economics: Income and Employment

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Contents

The Three Sector Ramsey Model

1 Introduction

Handout 8 introduced a closed economy, two sector Ramsey model - a neoclassical growth model with endogenous savings. This handout extends that two-sector model by: (i) opening the economy and adding a third sector, and (ii) adding labor force growth and technical change. For the sake of discussion, call the three sectors, agriculture, manufacturing and services. The agricultural and manufacturing goods are traded, while service sector output is non-traded. Although the notation below changes slightly, the manner in which the model is developed follows closely, the structure and discussion style in handout 8. Here, we can gloss over much of the model, as aside from adding the agricultural sector, it is very similar to the two sector model. Again, we begin with the household's intertemporal optimization problem and then introduce the production side of the model. We then lay out the intratemporal equilibrium conditions and end the conceptual discussion with intertemporal equilibrium. You will develop an empirical model.

The three sector dynamic model is a convenient point of departure for the development of policy models with more sectoral detail, and for the study of various other aspects of economic growth that have received attention at least from the time of Arthur Lewis. The seminal work of Lewis (1954), further developed by Fei and Ranis (1961) emphasize the supply of surplus labor from the farm sector to the rest of the economy as an essential part of the growth process. This theme was also emphasized in the work of Jorgenson (1967). In spite of the renewed interest in growth theory in the 1980s, Matsuyama (1992) was among the first to develop a model of endogenous growth with two distinct sectors, agriculture and manufacturing. In a series of papers, Echivarria (1995, 1997, 2000), and more recently, Gollin et al (2004) develop neoclassical growth models in which agriculture and a home good are used to show how the sectoral composition of an economy explains an important part of the variation in growth rates across countries.

2 The model environment

Consider a small open, and competitive economy which, at time $t = 0$, is endowed with $L(0)$ units of labor, $K(0)$ units of capital and Z units of land. The economy has three productive sectors, indexed by $j = a, m, s$. Here, a represents agriculture, m represents manufacturing and s represents the service sector. The manufacturing sector produces a capital good, some of which is directly consumed by households and the rest reinvested to increase the economy's stock of capital. The agricultural and service sectors each produce a pure consumption good. The agricultural and manufacturing goods are traded internationally at (exogenous) fixed prices p_a and p_m, respectively - with the manufactured good price serving as numeraire.

The time-t service good price, denoted $p_s(t)$, is the price that clears the service good market - i.e., p_s is an endogenous variable whose value is determined by market forces within the country.

The underlying technology for each sector satisfies constant returns to scale, and each sector requires both labor services L and physical capital K as inputs. In addition to capital and labor, agriculture requires the land resource. Both capital and labor are mobile across the three sectors, while land is used only by agriculture - such a resource is often referred to as a *sector specific resource.* Households provide the flow of services from these resources to firms in exchange for wages $w(t)$, returns to capital $r(t)$, and land rent, $\Pi(t)$. Households accumulate assets through savings, denoted $\dot{A}(t)$. We assume no foreign ownership of assets, i.e., all capital stock is owned by domestic households.

Assume the labor force grows at rate n, and that Harrod-neutral technological change augments labor at the rate x. Let $\mathcal{A}(t)$ represent time-t labor productivity, with $\mathcal{A}(0) = 1$. Then the stock of labor in efficiency units (LEU) is $\mathcal{A}(t) L(t) = L(0) e^{(x+n)t}$. Although not necessary, we typically normalize the initial stock of labor, $L(0)$, to unity. Harrod-neutral technical change is often referred to as *exogenous technical change.* We also introduce exogenous technological change in agriculture, denoted $\mathcal{B}(t) = e^{\eta t}$, where $\eta > 0$. For fixed Z, the effective units of land at time-t is $\mathcal{B}(t) Z$. Unless indicated otherwise, we assume the *sustainability condition*

$$\eta = x + n \tag{1}$$

As noted in class, although the sustainability has an economic interpretation, it is primarily introduced to allow us to use the time elimination numerical method.

Disallowing the foreign ownership of assets, total asset holdings can be expressed as

$$A(t) = K(t) + P_H(t) H$$

where the price p_m of capital is normalized to unity, and P_H is the unit price of land.

2.1 No-arbitrage between capital and land assets

At each t, in a risk free setting, asset markets should equate the rate of return to capital and land, otherwise arbitrage can occur. Express the return to assets $r(t) A(t)$ as the the sum of the returns to capital plus the returns to total land rent, that is

$$r(t) A(t) = r(t) [K(t) + P_H(t) Z] = r(t) K(t) + \Pi(t) Z$$

For this to hold, a no-arbitrage condition between the two assets is implied. To see this, express the flow budget constraint in terms of assets as,

$$\dot{A} = wL + rA - E \tag{2}$$

and in terms capital and land as

$$\dot{K} = wL + rK + \Pi Z - E \tag{3}$$

where $E = \sum_{j=a,m,s} p_j Q_j$ represents total expenditure. In other words,

$$\dot{A} - rA = \dot{K} - rK - \Pi Z$$

Substituting $A = K + P_H H$ into this equation and rearranging terms gives

$$r = \frac{\Pi}{P_Z} + \frac{\dot{P}_Z}{P_Z} \tag{4}$$

Consequently, using the flow budget constraint (3) presumes an environment which guarantees that the returns to the two assets are equalized at each instant of time[1]. The left hand-side of (4) represents the return to the household from one unit of income invested in physical capital. This same unit of income can also buy $1/P_H$ units of land, generating, at time $t + dt$, rental income equal to Π/P_H plus the rate of change in the price of land. If this condition did not hold, optimizing investors could exploit the arbitrage opportunity and move investments out of land and into capital. Hence, (4) is referred to as the no-arbitrage condition.

The no-arbitrage condition (4) is actually a differential equation in P_Z, whose solution is

$$P_{KZ}(t) = \int_t^\infty e^{-\int_t^\tau r(v)dv} \Pi(\tau)\, d\tau$$

where $P_{KZ} = P_Z/P_K$. Hence, when $P_K(t) = 1$ for all t, we have

$$P_Z(t) = \int_t^\infty e^{-\int_t^\tau r(v)dv} \Pi(\tau)\, d\tau \tag{5}$$

2.2 Household behavior

Current generation households behave as though they take into account the welfare and resources of their descendants. The extended immortal family structure is appropriate if parents are altruistic in providing transfers to their children who in turn provide transfers to their children[2]. Thus, the representative household of the current generation is presumed to maximize the present value of discounted inter-temporal utility $\mathcal{U}$ subject to a budget constraint defined over an infinite horizon.

Let the quantities $Q_a(t), Q_m(t)$ and $Q_s(t)$ denote the household's time t level of agricultural, manufacturing and service good consumption. Assume the number of workers is proportional to total population, and define consumption per worker as $q_j(t) = Q_j(t)/L(t), j = a, m, s$. Represent household preferences by the function

$$\mathcal{U} = \int_0^\infty \frac{u(q_a(t), q_m(t), q_s(t))^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \tag{6}$$

[1] In terms of units per effective worker, the result is $r = \hat{\pi}/\hat{p}_H + \dot{\hat{p}}_H/\hat{p}_H + (x+n)$ where $\hat{\pi} = \Pi/\mathcal{A}(t)L(t)$ and $\hat{p}_H = P_H/\mathcal{A}(t)L(t)$.

[2] Barro (1974) shows that this specification is equivalent to a setting where individuals are connected via a pattern of intergenerational transfers motivated by altruism.

where $u(\cdot)$ is presumed linearly homogeneous, twice continuously differentiable, and non-decreasing and strictly concave in all arguments. As in handout 8, the time argument$t$, is typically suppressed. The time preference rate $\rho > 0$, and the inter-temporal elasticity of substitution $1/\theta$, with $\theta > 0$, have the same interepretation as in handout 8 - i.e., the influence the households consumption-savings choices, with $1/\theta$ are less than one in low income countries.

2.2.1 Household expenditures and final good demand

Given q and prices p_a, p_m and p_s, the intra-temporal consumption problem is to choose q_a, q_m and q_s to solve the following cost minimization problem:

$$E(p_a, p_m, p_s, q) \equiv \min_{(q_a, q_m, q_s)} \left\{ p_a q_a + p_m q_m + p_s q_s : q \leq \Lambda q_a^{\lambda_a} q_m^{\lambda_m} q_s^{\lambda_s},\ \ (q_a, q_m, q_s)\ \in \mathbb{R}^3_{++} \right\} \tag{7}$$

Here, $\lambda_a + \lambda_m + \lambda_s = 1$, and $\Lambda = \lambda_a^{-\lambda_a} \lambda_m^{-\lambda_m} \lambda_s^{-\lambda_s}$. Given the Cobb-Douglas utility function, $E(\cdot)$ is continuously differentiable in all arguments, separable in prices and q, and satisfies Shepard's lemma.

With separability we can write $E(p_a, p_m, p_s, q) = \mathcal{E}(p_a, p_m, p_s)\, q$, where at each instant in time, the differentiable function $\mathcal{E}(p_a, p_m, p_s)$ represents the price (cost) index of aggregate consumption q. Shephard's lemma gives the Hicksian demand function for each consumption good: i.e., for $j = a, m, s$

$$q_j = \mathcal{E}_{p_j}(p_a, p_m, p_s)\, q \tag{8}$$

where $\mathcal{E}_{p_j}(p_a, p_m, p_s) = \frac{\partial \mathcal{E}(p_a, p_m, p_s)}{\partial p_j}$.

At each point in time t, the representative household provides labor services in exchange for wages $w(t)$. The household owns assets $K(t)$ that can be rented out to firms as capital or loaned to other households. In return, households receive interest income $r(t)$ per unit of asset if lending to other households, and receive $r^k(t) = r(t) + \delta$ when lending to firms, where δ is the rate at which physical capital depreciates. The household also earns rent $\Pi(t)$ on each unit of land. The household allocates income to purchase Q_a, Q_m and Q_s for consumption, and saves by accumulating additional assets $\dot{A}(t)$, where the "dot" signifies a time derivative, i.e., $\dot{K} = dK/dt$.

2.2.2 Intertemporal behavior of the household

The household's problem is still analogous to that in Handout 8, however, it must now account for the net change in the number of working household members. To account for this change, normalize the household's budget constraint (**??**) by the number of household members, $L(t) = L(0)\, e^{nt} = e^{nt}$. This yields

$$\dot{A}(t)\, e^{-nt} = \left[w(t)\, e^{nt} + r(t)\, A(t) - \mathcal{E}(p(t))\, Q(t) \right] e^{-nt}$$

Recognizing that

$$\dot{a(t)} \equiv \frac{\partial A(t)\, e^{-nt}}{\partial t} = \dot{A}(t)\, e^{-nt} - n a(t)$$

the budget constraint in per labor terms now becomes

$$\dot{a} = w + (r - n)\, a - \mathcal{E}\,(p)\, q \tag{9}$$

where, in per worker terms, $a = Ae^{-nt}$, $\epsilon = E\,(p)\, q$, and $q = Qe^{-nt}$ Redefining the quantities q_a, q_m and q_s similarly, the households utility function is

$$\mathcal{U} = \max_{q_a, q_m, q_s} \left\{ \int_0^\infty \frac{u\,(q_a, q_m, q_s)^{1-\theta} - 1}{1-\theta} e^{(n-\rho)t} dt \right\}$$

or, equivalently

$$\mathcal{U} = \max_{q} \left\{ \int_0^\infty \frac{q^{1-\theta} - 1}{1-\theta} e^{(n-\rho)t} dt \right\} \tag{10}$$

Then the household's problem is to solve (10) subject to (9), given initial savings $a\,(0)$, initial non-traded good price p_s, and the limitation on borrowing

$$\lim_{t\to\infty} \left\{ a(t) \cdot \exp\left[- \int_0^t \left[r(v) - n \right] dv \right] \right\} \geq 0.$$

The corresponding present value Hamiltonian is

$$J = \frac{q^{1-\theta} - 1}{1-\theta} e^{(n-\rho)t} + \mu \left[w + (r-n)a - \mathcal{E}\,(p)\, q \right]$$

Assuming away corner solutions, the first-order necessary conditions for maximizing $\mathcal{U}$ are

$$\frac{\partial J}{\partial q} = q^{-\theta} e^{(n-\rho)t} - \mu \mathcal{E}\,(p) = 0 \tag{11}$$

$$\dot{\mu} = -\frac{\partial J}{\partial a} \Rightarrow \frac{\dot{\mu}}{\mu} = -\,(r-n) \tag{12}$$

and the transversality condition is

$$\lim_{t\longrightarrow\infty} \{\mu\,(t) \cdot a\,(t)\} = 0.$$

As in the two sector Ramsey problem, rearrange (11) to obtain

$$\mu = \frac{q^{-\theta} e^{(n-\rho)t}}{\mathcal{E}(p)}. \tag{13}$$

Then, log-differentiate (13) and use (12) to eliminate $\dot{\mu}/\mu$ to obtain the Euler condition

$$\frac{\dot{q}}{q} = \frac{1}{\theta} \left[r - \rho - \lambda_s \frac{\dot{p}}{p} \right] \tag{14}$$

where we make use of the property of Cobb-Douglas preferences, i.e.,

$$\frac{p_s \mathcal{E}_{p_s}\,(p_a, p_m, p_s)\, q}{\mathcal{E}\,(p_a, p_m, p_s)\, q} = \lambda_s$$

Since the production relationships in the following section are expressed in units per effective worker, it is useful to also express the budget constraint and the Euler equation in these terms. Normalizing the budget constraint

$$\dot{A}\,(t) = w\,(t)\, e^{nt} + r\,(t)\, A\,(t) - \mathcal{E}\,(p\,(t))\, Q\,(t)$$

by $e^{(x+n)t}$ implies

$$\dot{A}e^{-(x+n)t} = \left[we^{nt} + rA - \mathcal{E}(p)\,Q\right] e^{-(x+n)t}$$

Using the notation $\hat{a} \equiv Ae^{-(x+n)t}$, $\hat{w} \equiv we^{-xt}$, and $\hat{q} \equiv Qe^{-(x+n)t}$, recognize that

$$\dot{\hat{a}} = \frac{\partial Ae^{-(x+n)t}}{\partial t} = \dot{A}e^{-(x+n)t} - \hat{a}\,(-x-n)$$

which leads to the normalized budget constraint

$$\dot{\hat{a}} = \hat{w} + \hat{a}\,(r-x-n) - \mathcal{E}(p)\,\hat{q},$$

or, given the stock of land remains constant over time

$$\dot{\hat{k}} = \hat{w} + \hat{k}\,(r-x-n) - \mathcal{E}(p)\,\hat{q}, \tag{15}$$

expressed in units per effective worker.

To express the Euler condition in expenditure per effective worker terms, log-differentiate the expenditure equation, $E(p)\,\hat{q}$,

$$\frac{\dot{\hat{\epsilon}}}{\hat{\epsilon}} = \frac{\mathcal{E}_p(p)\,p}{\mathcal{E}(p)}\frac{\dot{p}}{p} + \frac{\dot{q}}{q} - x$$

where $\dot{\hat{q}}/\hat{q} = \dot{q}/q - x$, and substitute this result for $\dot{q}/q$ in (14) to obtain

$$\frac{\dot{\hat{\epsilon}}}{\hat{\epsilon}} = \frac{1}{\theta}\left[r - \rho - \theta x - \lambda_s\,(1-\theta)\,\frac{\dot{p}}{p}\right] \tag{16}$$

For the case of unitary elasticity of intertemporal substitution, (**??**), we obtain

$$\frac{\dot{\hat{\epsilon}}}{\hat{\epsilon}} = r - \rho - x \tag{17}$$

In the long-run, if a steady state exists, it must be the case that $\dot{\hat{\epsilon}}/\hat{\epsilon} = 0$. This result implies the rate of expenditure per worker in the long run is positive forever, and

$$\frac{\dot{\epsilon}}{\epsilon} = x$$

Technological progress precludes diminishing returns to capital in the long-run. In contrast to the Euler conditions from the two sector Ramsey (with no labor force growth or technical change), the presence of exogenous growth in effective labor supply at the rate x causes the household's steady state rate of return r^{ss} to exceed the rate of time preference ρ.

2.3 Production

The manufacturing and home-good sectors employ technologies

$$Y_j = \mathcal{F}^j\left(K_j, \mathcal{A}(t)\,L_j\right), j = m, s \tag{18}$$

which we will assume are constant returns to scale Cobb-Douglas functions. Expressed in intensive from, we have

$$\hat{y}_j = \frac{Y_j}{\mathcal{A}L} = f^j\left(l_j, \hat{k}_j\right). \tag{19}$$

Here $\widehat{y}_j = Y_j e^{-(x+n)t}$ is sector j output per effective worker, l_j is the share of workers employed in the sector, and $\hat{k}_j = K_j / A(t)\, L$ is the amount of capital stock per effective economy-wide worker employed in sector j. The corresponding cost functions are given by

$$C^j\left(\hat{w}, r^k\right)\hat{y}_j \equiv \min_{l_j, \hat{k}_j} \left\{l_j \hat{w} + r^k \hat{k}_j \;:\; \hat{y}_j \le f^j(l_j, \hat{k}_j)\right\},\ j = m, s$$

where $r^k = r + \delta$. As with the Heckscher-Ohlin and two-sector Ransey model, $C^j(\cdot)$ is continuous, twice differentiable, convex and linearly homogeneous in prices, separable in prices and output and accommodates Shepard's lemma.

Agricultural production is governed by the technology

$$Y_a = \mathcal{F}^a\left(K_a, \mathcal{A}(t)\, L_a, \mathcal{B}(t)\, Z\right) \tag{20}$$

where $\mathcal{F}^a(\cdot)$ is a constant returns to scale Cobb-Douglas function. Given the sustainability condition associated with the rate of land augmentation, (1), the technology expressed in intensive form is

$$\hat{y}_a = \frac{Y_a}{\mathcal{A}L} = f^a\left(\hat{k}_a, l_a, Z\right)$$

where $\hat{k}_a = K_a / \mathcal{A}L$. The value added by land is defined as

$$\pi^a\left(p_a, r^k, \hat{w}\right) Z \equiv \max_{l_a, \hat{k}_a} \left\{p_a f^a\left(\hat{k}_a, l_a, Z\right) - \hat{w} l_a - r^k \hat{k}_a\right\} \tag{21}$$

With an underlying Cobb-Douglas technology, $\pi^a(\cdot)$ is continuously differentiable, separable in prices and the land endowment, linearly homogeneous in prices. Here, $\pi^a(p_a, \hat{w}, r^k)$ is the rental rate per unit of land per effective worker required for the rental market among farmers to clear. Competition among agricultural firms ensures zero profits for the sector,

$$p_a \hat{y}_a - \hat{w} l_a - r^k \hat{k}_a - \hat{\pi} Z = 0$$

Given differentiability, by Hotelling's lemma the gradients of $\pi^a\left(p_a, r^k, \hat{w}\right) Z$ yield – what can be referred to as – the partial equilibrium agricultural supply and derived capital and labor demand (per economy-wide effective labor), e.g.,

$$y^a\left(p_a, r^k, \hat{w}\right) Z = \pi^a_{p_a}\left(p_a, r^k, \hat{w}\right) Z \tag{22}$$

2.4 Equilibrium

The equilibrium conditions closely parallel the conditions derived in the previous chapter. We nevertheless repeat the derivations here, and then draw upon these results in the next chapter.

2.4.1 Definition

Given an initial home good price, $p_s(0)$, initial resource endowments $\{K(0), L(0), H\}$ and constant world market prices, p_m and p_a, a competitive equilibrium for this economy is a sequence of positive home good prices and capital stock levels $\left\{p_s(t), \hat{k}(t)\right\}_{t\in[0,\infty)}$, household consumption plans $\{\hat{q}_a(t), \hat{q}_m(t), \hat{q}_s(t)\}_{t\in[0,\infty)}$, factor rental prices $\{r(t), \hat{w}(t), \hat{\pi}(t)\}_{t\in[0,\infty)}$ for labor, capital and land, and production plans

$$\left\{\hat{y}_a(t), \hat{y}_m(t), \hat{y}_s(t), \hat{k}_a(t), \hat{k}_m(t), \hat{k}_s(t), l_a(t), l_m(t), l_s(t)\right\}_{t\in[0,\infty)}$$

such that at each instant of time t,

1. The representative household solves its utility maximization problem,
2. Firms maximize profits subject to their technologies, yielding zero profits
3. Markets clear for
   (a) commodities

$$\hat{y}_m - \hat{q}_m - \dot{\hat{k}} - \hat{k}(x+n+\delta) > 0 \;\; \text{(Export)} \;\; < 0 \;\; \text{(Import)} \tag{23}$$

$$\hat{y}_s - \hat{q}_s = 0$$

$$\hat{y}_a - \hat{q}_a > 0 \;\; \text{(Export)} \;\; < 0 \;\; \text{(Import)}$$

   (b) labor

$$\sum_{j=a,m,s} l_j = 1$$

   (c) capital

$$\sum_{j=a,m,s} \hat{k}_j = \hat{k}$$

4. And the no arbitrage condition between the assets of capital and land to assure the optimal allocation of savings (i.e., that the returns to the two types of investment are equalized)

$$r = \frac{\pi^a(p_a, \hat{w}, r^k)}{\hat{P}_Z} + \frac{\dot{\hat{P}}_Z}{\hat{P}_Z} + (x+n) \tag{24}$$

2.4.2 Characterization

The following conditions characterize the equilibrium. Given the endogenous sequence $\left\{\hat{k}, \hat{\epsilon}\right\}_{t\in[0,\infty)}$, of values, the five-tuple sequence of positive values $\left\{r^k, \hat{w}, \hat{y}_m, \hat{y}_s, p_s\right\}_{t\in[0,\infty)}$ satisfies the five intratemporal conditions for each t:

1. zero profits in production of the manufactured and the home good

$$C^j\left(r^k \hat{w},\right) = p_j, \;\; j = m, s \tag{25}$$

2. labor market clearing

$$\sum_{j=m,s} \frac{\partial}{\partial \hat{w}} C^j \left(r^k \hat{w}, \right) \hat{y}_j - \frac{\partial}{\partial \hat{w}} \pi^a \left(p_a, r^k, \hat{w}\right) Z = 1 \tag{26}$$

3. capital market clearing

$$\sum_{j=m,s} \frac{\partial}{\partial r^k} C^j \left(r^k \hat{w}, \right) \hat{y}_j - \frac{\partial}{\partial r^k} \pi^a \left(p_a, r^k, \hat{w}\right) Z = \hat{k} \tag{27}$$

4. and home good market clearing

$$\frac{\partial \mathcal{E}\left(p_a, p_s\right) \hat{q}}{\partial p_s} = \hat{y}_s \tag{28}$$

Note the similarity of this intratemporal characterization to that of the static two sector model of handout 8.

The system of equations (25) - (28) can, in principle, be solved to express the endogenous variables $\left\{r^k, \hat{w}, \hat{y}_m, \hat{y}_s, p_s\right\}$ as a function of the exogenous variables $\left(p_a, p_m, Z\right)$, and the remaining endogenous variables $\left(\hat{k}, \hat{\epsilon}\right)$. Thus, a solution $\left\{\hat{k}^*, \hat{\epsilon}^*\right\}_{t\in[0,\infty)}$ is sufficient to find a solution for the remaining variables based upon the intratemoral conditions.

2.4.3 Intratemporal equilibrium conditions

As in the static H-O model, use the zero profit condition (25) to express $\hat{w}$ and r^k as an implicit function of $p_m = 1$ and p_s. Exporess to the resulting solution as

$$\hat{w} = W\left(p_s\right) \tag{29}$$

$$r^k = R\left(p_s\right) \tag{30}$$

The economy's GDP function in effective worker terms is given by

$$\hat{w} + r^k \hat{k} + \pi^a \left(p_a, \hat{w}, r\right) Z$$

Substituting (29) and (30) into the above expression yields

$$G\left(p_a, p_s, \hat{k}, Z\right) = W\left(p_s\right) + R\left(p_s\right) \hat{k} + \pi^a \left(p_a, R\left(p_s\right), W\left(p_s\right)\right) Z \tag{31}$$

Next, substitute (29) and (30) into the factor market clearing conditions (26) and (27) and solve for $\hat{y}_m$ and $\hat{y}_s$ as a function of the endogenous variables p_s, and $\hat{k}$.[3] Express the resulting solution for $\hat{y}_s$ as

$$\hat{y}_s = \tilde{y}^s \left(p_s, \hat{k}\right) \equiv y^s \left(p_a, p_s, \hat{k}, H\right). \tag{32}$$

[3] An alternative derivation is to simply derive the supply functions from the respective output price gradient of the GDP function.

and

$$\hat{y}_m = \tilde{y}^m \left(p_s, \hat{k}\right) \equiv y^m \left(p_a, p_s, \hat{k}, H\right) \tag{33}$$

In an attempt to decrease notational clutter, we adopt the following convention. A function accented with " $\sim$ " denotes a function for which all exogenous variables are suppressed, e.g., $\tilde{y}^s \left(p_s, \hat{k}\right)$ suppresses the p_a in $y^s(\cdot)$ as does the expenditure function $\tilde{E}(p_s) = E(p_a, p_m, p_s)$.

The supply function for agriculture is obtained by substituting (29) and (30) for $\hat{w}$ and r^k in the partial equilibrium supply function (22). The next section derives the requisite equations of motion and steady-state expressions as functions of $\hat{k}$ and p_s.

2.4.4 Intertemporal equilibrium conditions

The equations of motion

As in the two sector Ramsey model, we now derive two differential equations in $\hat{k}$ and p_s. As you might guess, (15) is one candidate equation. A with the two-sector model, we need to eliminate the $\hat{\epsilon}$ in the budget constraint (15). Given Cobb-Douglas preference, the home good market clearing condition (28) can be expressed as

$$\hat{\epsilon} = \frac{p_s}{\lambda_s} \tilde{y}^s \left(p_s, \hat{k}\right). \tag{34}$$

Substitute (34), along with (29) and (30) into (15) to obtain

$$\dot{\hat{k}} = \tilde{K}\left(p_s, \hat{k}\right) \equiv W(p_s) + \hat{k}\left(R(p_s) - x - n - \delta\right) + \tilde{\pi}^a(p_s) - \frac{p_s}{\lambda_s}\tilde{y}^s\left(p_s, \hat{k}\right) \tag{35}$$

where for notational convenience

$$\tilde{\pi}^a(p_s) = \pi^a\left(p_a, W(p_s), R(p_s)\right) Z$$

To derive the differential equation for p_s, differentiate the home good market clearing condition (34 with respect to time to get

$$\dot{\hat{\epsilon}} = \frac{1}{\lambda_s}\left[\left(\tilde{y}^s\left(p_s, \hat{k}\right) + p_s \tilde{y}^s_{p_s}\left(p_s, \hat{k}\right)\right)\dot{p}_s + p_s \tilde{y}^s_{\hat{k}}\left(p_s, \hat{k}\right)\dot{\hat{k}}_s\right] \tag{36}$$

Next, replace $\dot{\hat{\epsilon}}$ by first expressing the Euler condition (16) in expenditure terms as

$$\dot{\hat{\epsilon}} = \hat{\epsilon}\frac{1}{\theta}\left[R(p_s) - \rho - \theta x - \delta + \lambda_s \frac{\dot{p}_s}{p_s}(\theta - 1)\right] \tag{37}$$

where, as inHandout 8, we differentiate the expenditure function, use $r = R(p_s) - \delta$, and rearrange terms to obtain

$$\frac{\dot{\hat{q}}}{\hat{q}} = \frac{\dot{\hat{\epsilon}}}{\hat{\epsilon}} - \lambda_s \frac{\dot{p}_s}{p_s}.$$

The final steps are to substitute the home good market clearing condition (34) for $\hat{\epsilon}$ in (37), and then use this result to substitute for $\dot{\hat{\epsilon}}$ in (36). The resulting equation is linear in $\dot{p}_s$. Solving for $\dot{p}_s$ yields

$$\dot{p}_s = p_s \frac{\left[R\left(p_s\right) - \rho - \theta x - \delta\right] \tilde{y}^s\left(p_s, \hat{k}\right) - \theta \tilde{y}^s_{\hat{k}}\left(p_s, \hat{k}\right) \dot{\hat{k}}_s}{\theta\left[\tilde{y}^s\left(p_s, \hat{k}\right) + p_s \tilde{y}^s_{p_s}\left(p_s, \hat{k}\right)\right] + \tilde{y}^s\left(p_s, \hat{k}\right) \lambda_s \left(1 - \theta\right)} \tag{38}$$

Replacing $\dot{\hat{k}}_s$ by (35) completes the equation. For the case of unitary intertemporal elasticity of substitution, $\theta \to 1$, we have

$$\dot{p}_s = p_s \frac{\left[R\left(p_s\right) - \rho - x - \delta\right] \tilde{y}^s\left(p_s, \hat{k}\right) - \tilde{y}^s_{\hat{k}}\left(p_s, \hat{k}\right) \dot{\hat{k}}}{\tilde{y}^s\left(p_s, \hat{k}\right) + p_s \tilde{y}^s_{p_s}\left(p_s, \hat{k}\right)} \tag{39}$$

This result is virtually identical in structure to the corresponding differential equation of previous chapter.

If a steady state exists such that $r^k = \rho + \theta x + \delta$ for $\theta > 0$ and $\dot{\hat{k}} = 0$, then (38) suggests $\dot{p}_s = 0$. These two differential equations are autonomous because of the restriction (1). Otherwise, t can appear as a separate argument in the differential equations. As in the two sector model, these differential equations cannot be solved analytically. The Time-Elimination Method developed by Mulligan and Sala-i-Martin (1991) is used to empirically solve the system[4].

The steady state

Assume an interior solution to the steady state exists. Then, deriving the steady state values is virtually identical to that of the two sector model. The first step is to obtain the steady-state values for r^k, p_s,and $\hat{w}$, and then substitute these values into the budget constraint and solve for $\hat{k}$.

If a steady state exists, the Euler condition (16) implies

$$r^{ss} = \rho + x. \tag{40}$$

Combining the above expression with (30) gives,

$$\rho + x + \delta = R\left(p_s\right) - \delta$$

Assuming $R\left(\cdot\right)$ is invertible, the steady-state home good price, denoted p_s^{ss}, satisfies

$$p_s^{ss} = R^{-1}\left(\rho + x + \delta\right) \tag{41}$$

and the effective steady-state wage rate is

$$\hat{w}^{ss} = W\left(p_s^{ss}\right) \tag{42}$$

Knowing p^{ss} from (41), (35) is a single equation that is linear in $\hat{k}$. The root k^{ss} satisfying (35) for $\dot{\hat{k}} = 0$ and $p_s = p^{ss}$ is the steady-state level of capital stock per effective worker. Given the steady state

[4] See also Barro and Sala-i-Martin (1995), pp. 488-491 for a discussion of this method.

values $\left(r^{ss}, \hat{w}^{ss}, p_s^{ss}, \hat{k}^{ss}\right)$, we can calculate the remaining endogenous variables using Shepard's or Hotelling's lemma.

3 Conclusion

This handout extends the basic two-sector growth model in two ways. First, we add a third sector, and reinterpret the model as a small open economy model having an agricultural, manufacturing and service sector: the agricultural and manufacturing goods are traded, while the service sector is non-traded. We also introduce labor force growth and exogenous technical change - two features consistent with the features of most country's economic growth data. As with the two sector Ramsey model, laying out the model involved three basic steps : (i) state the model's primitives, (ii) define the equilibrium and (iii) characterize the equilibrium in a way that emphasizes the intratemporal and intertemporal features of the model. Both the intratemporal and intertemporal characterizations closely resemble that of the two-sector static Ramsey model.

A final observation is, although the intratemporal equilibrium conditions are consistent with what one would find in a static computable general equilibrium (CGE) model, the dynamic model can be validated - i.e., you can compare the model results with actual economic data over time and compute statistics like an error sum of squares or Theil statistic. One cannot validate a static CGE model.

The dynamic three sector model is a great point of departure for examining a wide variety of economic growth questions, and a great tool for policy analysis. It has been used to examine the economics of land grabs (Choi), competition between traditional and commercial agriculture (Larson), and natural resource valuation (Smith; Smith and Gemma; Smith, Nelson and Roe).

4 Problems

Problem 1 *Choose a country (except the U.S. or the U.K.) and construct a three sector Ramsey model of the country. This means:*

- *Find a social accounting matrix or input-output (IO) table for the country (I can likely provide you with an IO table)*
- *Perform a growth accounting exercise for the country to get an average Harrod neutral rate of technical change and average labor force growth rate, and a capital stock series*
- *Write the Mathematica code for the model. See me if you need help with the dynamic section, but at*

least get the code written through the differential equations corresponding to equations (35) and (38) above.

A. *Once you have constructed, coded and run the three sector model, create a set of tables for each of the following variable combinations - each table should include the predicted values for 60 years, e.g., from 2010 through 2070. It is okay to report the results for every two, five or ten years.*

1. *Aggregate and sector value added levels*
2. *Unit land rental rates, rates of return to capital, wages and non-traded good prices*
3. *Sector derived demand for capital, and sector derived demand for labor*
4. *The unit stock price of land,* $P_Z(t)$

B. *Create a 60 year graph of:*

1. *Sector value added shares*[5] *(all sector shares in the same graph)*
2. *Aggregate and sectoral, capital to labor ratios (all ratios in the same graph)*

C. *Describe what you think are the important patterns in the data. Do you think the patterns are related to Stolper-Samuelson or Rybzhinski type effects - i.e., could you use relative factor intensities to explain any of the patterns?*

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[5] A sector share is the ratio of that sector's value added to aggregate GDP.

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