Interim Bayesian Nash Equilibrium on Universal Type Spaces for Supermodular Games

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Main result

Existence of pure-strategy Bayesian Nash equilibrium with:

- **interim** formulation of a Bayesian game and no common prior.
- **interim** definition of a BNE.

Assumptions: Supermodular payoffs but otherwise general:

- **Type spaces:** any.
- **Actions:** compact metric lattice.
- **Payoffs:** measurable in types, continuous in actions, bounded.
- **Interim beliefs:** measurable in own type.

Interim formulation of a Bayesian game

Players: $N = \{1, \ldots, n\}$, indexed by i.

For each player i:

1. **Type space:** $(T_i, \mathscr{F}_i)$.
2. **Interim beliefs:** $p_i : T_i \to \mathscr{M}_{-i}$,

   where $\mathscr{M}_{-i}$ is the set of probability measures on $(T_{-i}, \mathscr{F}_{-i})$.
3. **Action set:** A_i.
4. **Payoff function** $u_i : A \times T \to \mathbb{R}$.

Interim Bayesian Nash equilibrium

Strategy of player i:

$$\text{measurable } \sigma_i : T_i \to A_i\,.$$

Let Σ_i be set of strategies (**NOT equivalence classes**).

BNE in words: Each type of each player chooses action to maximize expected utility given beliefs for that type.

For each i and each t_i, $\sigma_i(t_i)$ is best response to σ_{-i}:

$$\sigma_i(t_i) \;\in\; \arg\max_{a_i} \int_{T_{-i}} u_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i})\, dp_i(t_{-i} \mid t_i)$$

Ex ante Bayesian Game and BNE

- Belief mappings replaced by a common prior.
- Strategies are equivalences classes.

BNE in words: Player chooses a strategy before observing his type in order to maximize **unconditional** expected utility.

$\Rightarrow$ interim optimality for **almost every** type, rather than every type.

Ex ante BNE in ex ante Bayesian Game

Balder (1988) (improving on Milgrom and Weber (1985):

- ex ante formulation of game and BNE;
- assumes independent types (or equivalent to such a game).

Results for games with order structure

Supermodular games

Vives (1990) and Milgrom and Roberts (1990):

- ex ante formulation of game and BNE;
- action sets are Euclidean.

Monotone strategies

Athey (2001), McAdams (2003), Reny (2006):

1. ex ante formulation of game and BNE;
2. types are Euclidean cube;
3. atomless prior.
4. slightly more restrictive action sets.

Who cares about interim vs. ex post

From Myerson (2002):

> "Harsanyi's point here is that the type represents what the player knows at the beginning of the game, and so calculations of the player's expected payoff before this type is learned cannot have any decision-theoretic significance in the game."
>
> "For example, if a player's type includes a specification of his or her gender (about which some other players are uncertain), then the normal-form analysis would require us to imagine the player choosing a contingent plan of what to do if male and what to do if female, maximizing the average of male and female payoffs."

Main result, restated

Consider any interim Bayesian game with …

1. No restriction on T_i.
2. A_i is compact* metric lattice.
3. u_i is supermodular in a_i and has increasing differences in (a_i, a_{-i}).
4. $t_i \mapsto p_i(F_{-i} \mid t_i)$ is measurable for $F_{-i} \in \mathscr{F}_{-i}$.
5. u_i is bounded, measurable in t, and continuous* in a.

Then the game has greatest and least pure-strategy **interim** BNE.

*Needed? Not usually in supermodular games. For measurability here. Can be weakened?

Genesis

From project with Xavier Vives:

"Monotone Equilibrium in Bayesian Games of Strategic Complementarities"

Adds these assumptions for each player

- payoff has increasing differences in own action and profile of types;
- interim beliefs are increasing in type with respect to first-order stochastic dominance.

Obtains also these results

- Extremal equilibria are in strategies that are increasing in type.
- Comparative statics: Shift interim beliefs up by first-order stochastic dominance (type-by-type). Then extremal equilibria increase.

Example: Local network externalities (Sundararajan, 2004)

- Players choose between adopting ($a_i = 1$) or not ($a_i = 0$).
- **Local** network externalities on a graph (externality only between neighbors). Let G_i be neighbors of player i.
- Player i's valuation is increasing in adoption decisions of neighbors.

Then complete information game has strategic complementarities.

Incomplete-information version

Captures idea that players have only local knowledge about the structure of the network:

- the graph is drawn randomly with a known distribution ρ; and
- each player observes only who her neighbors are.

Type of player i is G_i. (Can also introduce valuation parameters that are private information; suppressed for this presentation.)

The partial order on G_i is set inclusion. Having more neighbors increases network externality $\Rightarrow$ increases valuation. Then i's payoff has increasing differences in (a_i, G_i) (does not depend directly on G_j for $j \neq i$).

Increasing beliefs condition

Need the distribution of the neighborhood sets to have property that, if $G_i' \subset G_i''$ then, for any $\{G_j\}_{j \neq i}$, probability that all players $j \neq i$ have neighborhoods that include at least G_j should be weakly higher conditional on G_i'' compared to conditional on G_i'.

Loosely, in words: having more neighbors makes player believe that other players have more neighbors, i.e., that network is more connected.

Satisfied for a random graph in which the existence of an edge between any pair of agents is independent of the existence of other edges (for example, ρ is the uniform distribution on Γ).

Some properties of this example

1. Types are inherently correlated: each player, by learning who her neighbors are, learns something about who the other players' neighbors are.

2. Types are inherently discrete.

3. Types are inherently multidimensional (no natural linear order).

Because of the discreteness, this game is not covered by Athey (2001) or McAdams (2003). Furthermore, the increasing beliefs condition is easier to check than affiliation.

Implications of our main result for this example

1. Game has a greatest and a least pure-strategy equilibrium, increasing in type: with more neighbors, player may switch from not-adopt to adopt, but not vice-versa.

2. If the network becomes "probabilistically more dense", then greatest and least equilibria are higher.

3. Game has positive externalities: each player's payoff is increasing in the actions of the other players.
   $\Rightarrow$ greatest equilibrium Pareto dominates all other equilibria.

4. If we have an equilibrium selection of the greatest or the least equilibrium, then each player's interim payoff would increase as a consequence of the shift described in item 2.

Back to this paper

Consider any interim Bayesian game with …

1. No restriction on T_i.
2. A_i is compact* metric lattice.
3. u_i is supermodular in a_i and has increasing differences in (a_i, a_{-i}).
4. $t_i \mapsto p_i(F_{-i} \mid t_i)$ is measurable for $F_{-i} \in \mathscr{F}_{-i}$.
5. u_i is bounded, measurable in t, and continuous* in a.

Then the game has greatest and least pure-strategy **interim** BNE.

*Needed? Not usually in supermodular games. For measurability here. Can be weakened?

Main steps

Step 1

Show that each player has a greatest best reply (GBR) $\bar{\beta}_i(\sigma_{-i})$, which is increasing in σ_{-i}.

Step 2

Apply a lattice fixed-point theorem to the profile of GBR mappings

$$\bar{\beta}(\sigma) = \left(\bar{\beta}_1(\sigma_{-1}), \ldots, \bar{\beta}_n(\sigma_{-n})\right) .$$

(First step 2, then step 1.)

Step 2 for ex ante model

Assumption. A_i is a compact sublattice of Euclidean space.

Then: Σ_i (set of equivalence classes) is a complete lattice.

So we can apply Tarski's fixed-point theorem to $\bar{\beta} \colon \Sigma \to \Sigma$:

> **Suppose**
>
> - X **is a complete lattice,**
> - $f \colon X \to X$ **is an increasing function.**
>
> **Then** f **has a fixed point.**

Why we can't do the same thing in interim model

Σ_i is set of **functions**, not equivalence classes.

Consider partial order

$$\sigma_i' \geq \sigma_i \iff \sigma_i'(t_i) \geq \sigma_i(t_i) \ \forall t_i$$

$A_i^{T_i}$ (set of ALL functions $T_i \to A_i$) is complete lattice.

And Σ_i is a sublattice of $A_i^{T_i}$.

But Σ_i is **not complete** (typically): pointwise sup of an uncountable set of measurable functions may not be measurable.

Example: Suppose $G_i \subset T_i$ is not measurable but all singletons are measurable. Then $\left\{\mathbf{1}_{\{t_i\}} \mid t_i \in G_i\right\} \subset \Sigma_i$ has no supremum in Σ_i.

Fixed-point theorem for partially ordered set $(X, \geq)$

Definition. $(X, \geq)$ is **downward sequentially complete** if every decreasing sequence has a greatest lower bound.

Definition. Suppose $(X, \geq)$ is downward sequentially complete. A functional $f : X \to \mathbb{R}$ is **downward sequentially continuous** if, for every decreasing sequence $\{x_1, x_2, \ldots\}$, $\lim f(x_n) = f(\lim x_n)$.

Theorem. Suppose

- $(X, \geq)$ is downward sequentially complete and has greatest element,
- $f : X \to X$ is increasing and downward sequentially continuous.

Then f has a greatest fixed point.

This works because Σ_i is sequentially complete.

How does this fixed-point theorem work?

It is just "packaged" Cournot tatônnement, as used by Vives (1990).

Proof:

- Let x_0 be greatest element of X.
- For $k \geq 1$, define $x_k = f(x_{k-1})$.
- Then $\{x_k\}$ is a decreasing sequence, ...
- which converges to the greatest fixed point.

Main steps

Step 1 **Now on to this part**

Show that each player has a greatest best reply (GBR) $\bar{\beta}_i(\sigma_{-i})$, which is increasing in σ_{-i}.

Step 2 **We just finished this**

Apply a lattice fixed-point theorem to the profile of GBR mappings

$$\bar{\beta}(\sigma) = \big(\bar{\beta}_1(\sigma_{-1}), \ldots, \bar{\beta}_n(\sigma_{-n})\big) .$$

Characterizing the GBR mapping: ex ante model

Easy:

- Σ_i are complete lattices.
- Induced ex ante utility functions are continuous.
- Apply "optimization on complete lattices" (e.g., Milgrom and Roberts (1990).

Characterizing the GBR mapping: interim model

Easy part …

… that $\bar{\beta}_i$ is an increasing function

Follows straight from the complementarity assumptions.

Also pretty easy …

… that $\bar{\beta}_i$ is sequentially order continuous.

From continuity of u_i in actions and dominated convergence.

Hard part …

…that $\bar{\beta}_i$ is well-defined.

Measurability problems!!

Focusing on the hard part

Fix σ_{-i} .
(So we can suppress it as an argument.)

Objective function $\pi_i : A_i \times T_i \to \mathbb{R}$:

$$\pi_i(a_i, t_i) \ := \int_{T_{-i}} u_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i}) \, dp_i(t_{-i} \mid t_i) \, .$$

Solution correspondence $\phi_i : T_i \to A_i$:

$$\phi_i(t_i) \ := \ \arg\max_{a_i \in A_i} \pi_i(a_i, t_i)$$

Greatest solution $\bar{\sigma}_i : T_i \to A_i$:

$$\bar{\sigma}_i(t_i) \ := \ \max \ \phi_i(t_i)$$

Is $\bar{\sigma}_i(t_i)$ well defined for all t_i ? (Yes, optimization on lattices ...)

Is $\bar{\sigma}_i : T_i \to A_i$ measurable ?? (If yes, then $\bar{\sigma}_i$ is the GBR.)

First 2 steps

Step 1. π_i is continuous and supermodular in a_i and bounded.

Easy: continuity and supermodularity are preserved by integration.

Step 2. π_i is measurable in t_i.

Coming up …

Step 2: π_i is measurable in t_i

Fix $a_i \in A_i$. Define

$$U_i(t_i, t_{-i}) \;:=\; u_i(a_i, \sigma_{-i}(t_i), t_i, t_{-i})$$

Then

$$\pi_i(a_i, t_i) \;=\; \int_{T_{-1}} U_i(t_i, t_{-i})\, dp_i(t_{-i} \mid t_i)\,.$$

When is

$$t_i \;\to\; \int_{T_{-1}} U_i(t_i, t_{-i})\, dp_i(t_{-i} \mid t_i)$$

measurable?

Abstract version

1. $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ are measurable spaces;
2. $\mathscr{M}$ is the set of probability measures on $(Y, \mathscr{G})$;
3. $p \colon X \to \mathscr{M}$;
4. $U \colon X \times Y \to \mathbb{R}$;
5. $\pi(x) \;:= \displaystyle\int_Y U(x, y)\, dp(y \mid x)$.

When is $\pi \colon X \to \mathbb{R}$ $\mathscr{F}$-measurable?

Answer

- $U \colon X \times Y \to \mathbb{R}$ is $\mathscr{F} \otimes \mathscr{G}$-measurable and bounded;
- For $G \in \mathscr{G}$, $x \mapsto p(G \mid x)$ is $\mathscr{F}$-measurable.

(Generalizes a result by Ely and Peski (2006).)

So our problem reduces to …

[Suppress subscript i: $\pi(a,t)$, $\phi(t)$, $\bar{\sigma}(t)$.]

Given $\pi\colon A \times T \to \mathbb{R}$, that is

$$\left.\begin{array}{l}\bullet\ \text{continuous in } a;\\[2ex] \bullet\ \text{measurable in } t;\end{array}\right\}\ \pi \text{ is a Carathéodory function}$$

- supermodular in a.

When is $t \mapsto \max\left\{\arg\max_{a\in A} \pi(a,t)\right\}$ measurable?

A key tool

Definition. A correspondence $\zeta\colon X \twoheadrightarrow Y$ from a measurable space $(X, \mathscr{F})$ to a topological space Y is **$\mathscr{F}$-measurable** if

$$\zeta^{w}(D) := \left\{x \in X \mid \zeta(x) \cap D \neq \varnothing\right\} \in \mathscr{F}$$

for every closed $D \subset Y$.

(This is stronger than "graph of ζ is measurable".)

Theorem. [Castaing & Valadier]
Let $(X, \mathscr{F})$ be a measurable space and let Y be a complete separable metric space. Let $\zeta\colon X \twoheadrightarrow Y$ be a measurable correspondence with non-empty and closed values. Then there is a countable family $\{f_k \mid k \in \mathbb{N}\}$ of measurable selections of ζ such that $\zeta(x) = \mathrm{cl}\{f_k(x) \mid k \in \mathbb{N}\}$ for all $x \in X$.

Brief summary of remaining steps

- Show that solution correspondence $\phi : T \to A$ is measurable. **(From Measurable Maximum Theorem)**
- Let $\{\sigma_k \mid k \in \mathbb{N}\}$ be the countable collection of measurable selections.
- Define recursively $\bar{\sigma}_k(t) = \sup\{\sigma_k(t), \bar{\sigma}_{k-1}(t)\}$.
- Each $\bar{\sigma}_k$ is measurable because lattice operation $\sup(\cdot, \cdot)$ is measurable.
- $\{\bar{\sigma}_k\}$ is increasing sequence of measurable functions; converges pointwise to measurable function.
- Can show that limit is $\bar{\sigma}$.

Back to case of monotone strategies

Add:

- complementarities between action and types;
- interim beliefs are increasing in type with respect to FOSD.

Then greatest best reply to monotone-in-type strategies is monotone in type.

Cournot tatônnement, starting at the greatest strategy profile and using greatest best replies, starts with monotone strategies, stays with monotone strategies, and converges to monotone strategies.