EVOLUTIONARY GAME THEORY
Nash's two interpretations of equilibrium, evolutionary stability, and the replicator dynamic

Jörgen Weibull

Delhi Winter School, December 2017

1 Nash equilibrium

- In Nash (PNAS, 1950) the 22 years old John Nash defined an *equilibrium point* in a finite normal-form game as a (pure or mixed) strategy profile that is a best reply to itself

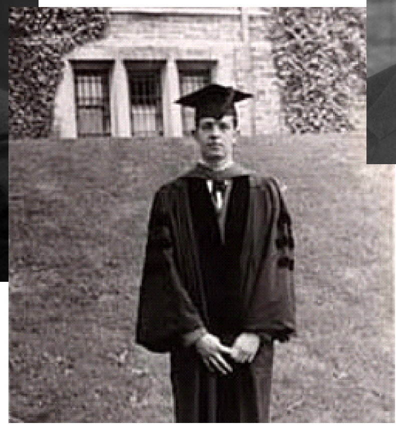
  – He established existence of such points in all finite games by way of Kakutani's fixed-point theorem

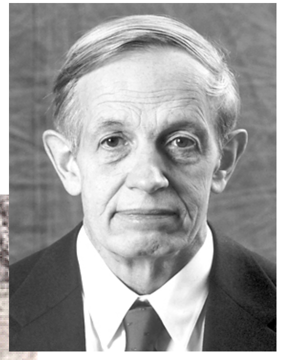
  – A year later he gave an ingenious proof, based on Brouwer's fixed point theorem. A proof that today is used for proving the existence of Walrasian equilibrium in exchange economies.

- But to what question is "Nash equilibrium" the answer?

John Nash

(born 1928, PhD 1950)





2 John Nash's two interpretations

- In his Ph. D. thesis (Princeton, 1950), Nash sketches informally two interpretations:
  - The "rationalistic", or "epistemic" interpretation
  - The "mass action", or "evolutionary" interpretation

2.1 The rationalistic interpretation

1. The players have never interacted before and will never interact in the future

2. The players are *rational* in the sense of Savage, *The Foundations of Statistics* (1954); they play optimally under *some* probabilistic belief about what other players do

3. Each player *knows* the game (all players' strategy sets and preferences)

- However, this clearly does not imply that they must play a Nash equilibrium, not even if they know that all players know the game and are rational

- In fact not even if the game and all players' rationality is *common knowledge* (CK) among them (Lewis, 1969, Aumann, 1976)

Counter-examples:

Coordination game:

	A	B
A	$2,2$	$0,0$
B	$0,0$	$1,1$

Zero-sum game:

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

Game with a unique NE:

	L	C	R
T	$7,0$	$2,5$	$0,7$
M	$5,2$	$3,3$	$5,2$
B	$0,7$	$2,5$	$7,0$

- Question: What *is* then implied by CK[game&rationality]?

- Answer: That some rationalizable strategy profile will be played

Definition 2.1 (Bernheim, 1984, Pearce, 1984) *A pure strategy is* **rationalizable** *if it survives the iterated elimination of pure strategies that are not best replies to any (pure or mixed) strategy profile.*

- Notation: Write $R_i \subseteq S_i$ for player i's set of rationalizable pure strategies, and $R = \times_i R_i$ for the set of rationalizable pure-strategy profiles. Write $Q = \times_i Q_i$ for the set of pure-strategy profiles that survive the iterated elimination of strictly dominated strategies

- Facts:

  - For any finite game: $\varnothing \neq R \subseteq Q$

  - $n = 2 \quad \Rightarrow \quad R = Q$

  - All Nash equilibria only use rationalizable strategies

2.2 The mass-action interpretation

1. For each *player role* $i \in I = \{1, 2, ..., n\}$ in the game there is a large population of *ex ante* identical individuals

2. The game is recurrently played, in time periods $t = 0, 1, 2, 3, ...$ by randomly drawn individuals, one from each player population

3. Individuals always play pure strategies

- A *mixed strategy* for a player role is a statistical distribution over the pure strategies used in that role

- Nash (1950, Ph.D. thesis) argues informally that if all individuals avoid suboptimal pure strategies, and the population distribution of strategy profiles is stationary, then it constitutes a Nash equilibrium

Reconsider the above examples in this interpretation!

Coordination game:

	A	B
A	$2,2$	$0,0$
B	$0,0$	$1,1$

Zero-sum game:

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

Game with a unique NE:

	L	C	R
T	$7,0$	$2,5$	$0,7$
M	$5,2$	$3,3$	$5,2$
B	$0,7$	$2,5$	$7,0$

2.3 Another example

- A large population of individuals who are now and then randomly matched into pairs to together run a business partnership, or write a joint paper for their master's degree, or, more generally, solve some work task together

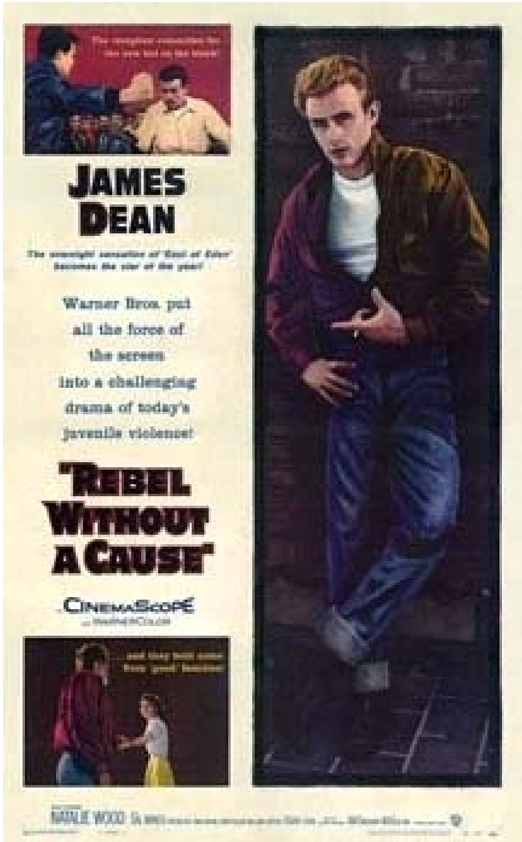
- To *work* or *shirk*?

	W	S
W	$3,3$	$0,4$
S	$4,0$	$-1,-1$

- Their roles are symmetric and they are not told that one is "player 1" and the other "player 2" (so they cannot condition on their player role)

- What will happen? In the rationalistic interpretation? In the mass-action interpretation for a single population?

- Such a game is usually called ”Hawk-Dove”, sometimes ”Chicken” (Film: ”Rebel without a Cause”), or ”Brinkmanship” (Bertrand Russell about the cold war),



- When illustrating their concept of *evolutionary stability*, Maynard Smith and Price (1973) illustrated it with precisely such a game, which they then called *Mouse and Dove*

- Heuristically, an *evolutionarily stable strategy* (*ESS*) is a (pure or mixed) strategy that 'cannot be overturned' once it has become the 'convention' in a population

- The mixed strategy to randomize 50/50 between "work" and "shirk" in the above example turns out to be its unique ESS

3 Evolutionary game theory

- Evolutionary process =

  = mutation process + selection process

- The unit of selection: usually strategies (”strategy evolution”), sometimes utility functions (”preference evolution”)

1. **Evolutionary stability**: focus on mutations

2. **Replicator dynamic**: focus on selection

3. **Stochastic evolutionary processes**: both selection and mutations

4 Evolutionary stability

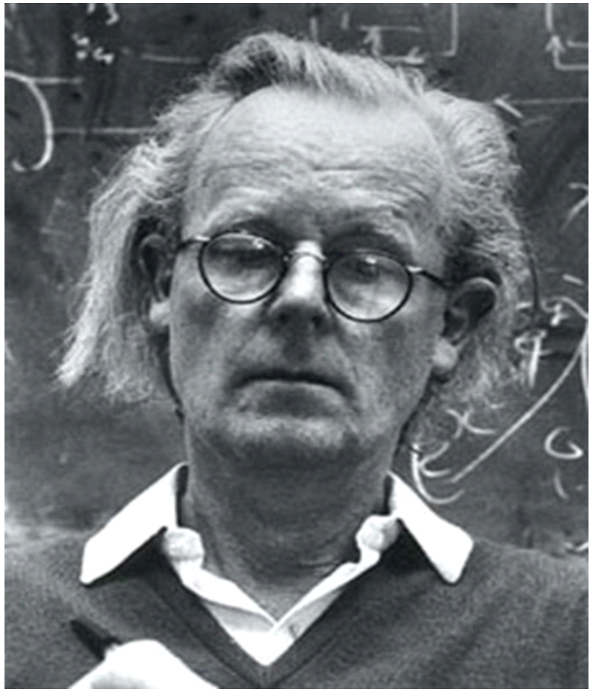
- Consider a large population of individuals who are recurrently and (uniformly) randomly matched in pairs to play a finite and symmetric game

  - Initially, all individuals use the same pure or mixed strategy, x, the *incumbent*, or *resident*, strategy

  - Suddenly, a small population share switches to another pure or mixed strategy, y, the *mutant* strategy

- If the residents on average *fare better* (in terms of payoff) than the mutants, then x is *evolutionarily stable against* y

- Any strategy x is *evolutionarily stable* if it is evolutionarily stable against *all* mutants $y \neq x$



John Maynard Smith
1920 - 2004
.

George R. Price
1922 - 1975

4.1 Domain

- *Symmetric finite two-player games in normal form*

Definition 4.1 *A two-player game is* **symmetric** *if* $S_1 = S_2$ *and* $u_2(s_1, s_2) = u_1(s_2, s_1)$ $\forall s_1, s_2 \in S_1 = S_2$.

- Write S for the common pure-strategy set, and let $m = \#S$

- Written as a *payoff bimatrix* (A, B), where $A = (a_{hk})$, $B = (b_{hk})$, for $h, k \in S$, the game is symmetric iff $B = A^T$

Example 4.1 (Prisoners' dilemma)

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 1,1 \end{array}$$

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}$$

Symmetric since $B = A^T$.

Example 4.2 (Matching Pennies)

$$\begin{array}{ccc} & H & T \\ H & 1,-1 & -1,1 \\ T & -1,1 & 1,-1 \end{array}$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here $B^T \neq A$. Not a symmetric game.

- Thus, matching pennies games fall outside the domain of evolutionary stability analysis. (But if player roles are randomly assigned, with equal chance to be in each player role, then this "metagame" is symmetric.)

Example 4.3 (Coordination game) *Payoff bimatrix:*

$$\begin{array}{ccc} & L & R \\ L & 2,2 & 0,0 \\ R & 0,0 & 1,1 \end{array}$$

$$A = B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

A **doubly** *symmetric game:* $B = A^T = A$, *an example of a* **potential game** *[Rosenthal (1974), Monderer and Shapley (1996)]*

4.2 Notation

- Write Δ or $\Delta(S)$ for the mixed-strategy simplex:

$$\Delta = \{x \in \mathbb{R}_{+}^{m} : \sum_{i \in S} x_i = 1\}$$

- Write the payoff to any strategy $x \in \Delta$, when used against any strategy $y \in \Delta$ as

$$\pi(x, y) = x \cdot Ay$$

  Note that the first argument, x, is *own* strategy, and the second argument, y, the *other* party's strategy

- Mixed best replies to $x \in \Delta$:

$$\beta^{*}(x) = \{x^{*} \in \Delta : \pi(x^{*}, x) \geq \pi\left(x', x\right) \ \forall x' \in \Delta\}$$

- This defines a *correspondence* from $\boldsymbol{\Delta}$ to itself: $\beta^* : \boldsymbol{\Delta} \rightrightarrows \boldsymbol{\Delta}$

- Let

$$\boldsymbol{\Delta}^{NE} = \{x \in \boldsymbol{\Delta} : x \in \beta^* (x)\}$$

  Hence, $x \in \boldsymbol{\Delta}^{NE}$ iff the strategy profile (x, x) is a Nash equilibrium

Proposition 4.1 *$\boldsymbol{\Delta}^{NE} \neq \varnothing$.*

[**Proof**: Application of Kakutani's Fixed-Point Theorem.]

4.3 Definition

Definition 4.2 $x \in \Delta$ *is an* **evolutionarily stable strategy (ESS)** *if for each strategy* $y \neq x$ $\exists$ $\bar{\varepsilon}_y \in (0,1)$ *such that, for all* $\varepsilon \in (0, \bar{\varepsilon}_y)$,

$$\pi\left[x, \varepsilon y + (1-\varepsilon)x\right] > \pi\left[y, \varepsilon y + (1-\varepsilon)x\right].$$

- "Post-entry population mixture":

  $$p = \varepsilon y + (1-\varepsilon)x \in \Delta,$$

  a convex combination of x and y, a point on the straight line between them

- Note that $\bar{\varepsilon}_y$ may depend on the particular mutant y at hand

- Let $\Delta^{ESS} \subset \Delta$ denote the set of ESSs

- Note that an ESS has to be a *best* reply to itself: if $x \in \Delta^{ESS}$ then $\pi(y, x) \leq \pi(x, x)$ for all $y \in \Delta$

- Hence $\Delta^{ESS} \subset \Delta^{NE}$

- Note also that an ESS has to be a *better* reply to its alternative best replies than they are to themselves: if $x \in \Delta^{ESS}$, $y \in \beta^*(x)$ and $y \neq x$, then $\pi(x, y) > \pi(y, y)$

Proposition 4.2 $x \in \Delta^{ESS}$ *if and only if for all* $y \neq x$*:*

$$\pi(y, x) \leq \pi(x, x)$$

and

$$\pi(y, x) = \pi(x, x) \Rightarrow \pi(y, y) < \pi(x, y)$$

4.4 Examples

4.4.1 A prisoners' dilemma

	C	D
C	3, 3	0, 4
D	4, 0	2, 2

$$\Delta^{ESS} = \Delta^{NE} = \{D\}$$

4.4.2 A coordination game

$$\begin{array}{ccc} & L & R \\ L & 2,2 & 0,0 \\ R & 0,0 & 1,1 \end{array}$$

$$\Delta^{NE} = \left\{ L, R, \frac{1}{3}L + \frac{2}{3}R \right\}$$

$$\Delta^{ESS} = \{L, R\}$$

The mixed NE is *perfect* (and even *proper*), but not evolutionarily stable!

4.4.3 A Hawk-dove game

	W	S
W	3, 3	0, 4
S	4, 0	−1, −1

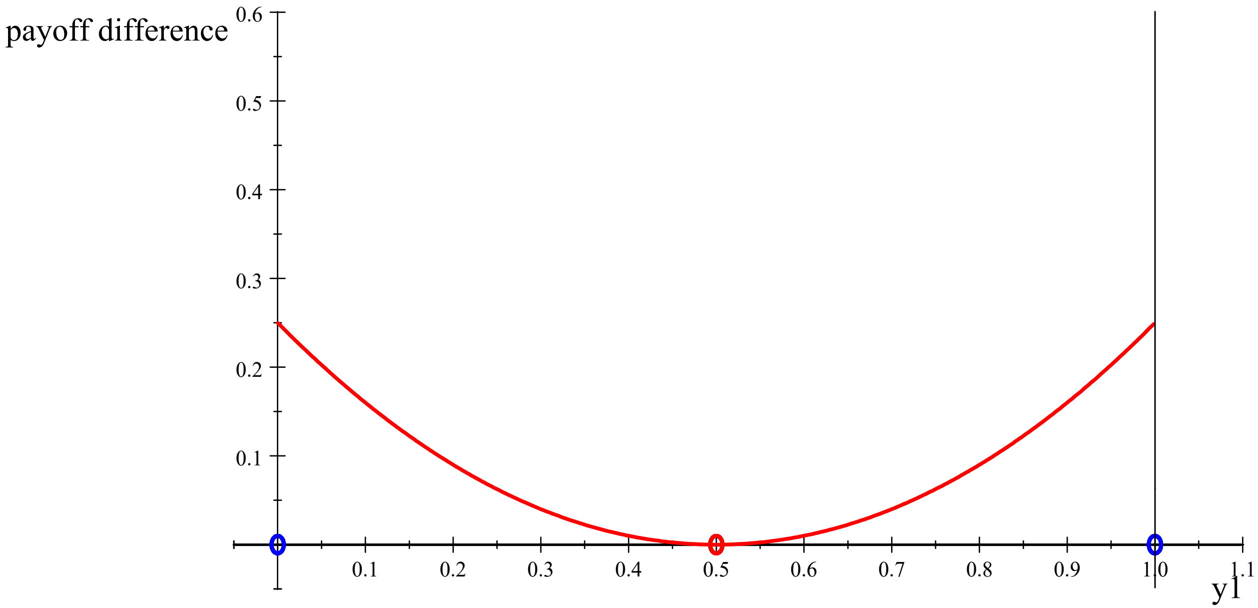
- Unique symmetric NE: randomize uniformly, $x^* = (1/2, 1/2)$ and

$$\Delta^{NE} = \{x^*\}$$

- Hence $\Delta^{ESS} \subseteq \{x^*\}$, and x^* an ESS iff

$$\pi(x^*, y) > \pi(y, y) \qquad \forall y \neq x^*$$

- Payoff difference $\pi(x^*, y) - \pi(y, y)$:



- Some games have no ESS. For instance, when all payoffs are the same. But also in more interesting games such as

Example 4.4 (Rock-scissors-paper) *Rock beats Scissors, Scissors beat Paper, and Paper beats Rock:*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Unique Nash equilibrium: $x^* = (1/3, 1/3, 1/3)$. *Hence,* $\Delta^{NE} = \{x^*\}$. *All pure strategies are best replies and do just as well against themselves as* x^* *does against them:* $\Delta^{ESS} = \varnothing$. *However,* x^* *is* neutrally stable, *in the sense that it meets the definition with weak inequality.*

4.5 Local superiority

- We note that an *interior* ESS necessarily earns a higher payoff against all mutants than these earn against *themselves*

- A form of "global superiority"

Definition 4.3 $x \in \Delta$ *is* **locally superior** *if it has a neighborhood U such that $u(x, y) > u(y, y)$ for all $y \neq x$ in U.*

Proposition 4.3 (Hofbauer, Schuster & Sigmund, 1979) $x \in \Delta^{ESS}$ *if and only if x is locally superior.*

- This is important for Darwin's "gradualism" and for the replicator dynamics

4.6 Uniform invasion barrier

- Each ESS has a *uniform invasion barrier*:

Proposition 4.4 *$x \in \Delta$ is an ESS $\Rightarrow \exists\ \bar{\varepsilon} \in (0,1)$ such that for all $y \neq x$ and $\varepsilon \in (0, \bar{\varepsilon})$:*

$$\pi\left[x, \varepsilon y + (1-\varepsilon)x\right] > \pi\left[y, \varepsilon y + (1-\varepsilon)x\right].$$

- This is important because in a finite population, say of size N, the smallest mutant population share ε is $1/N$

4.7 Perfection

Definition 4.4 (Selten, 1975) *A* **perfect equilibrium** *of any finite normal-form game is any Nash equilibrium that is the limit of ε-perfect equilibria, as $\varepsilon \downarrow 0$, where ε-perfection requires that all pure strategies are used and non-optimal pure strategies have probabilities $\leq \varepsilon$.*

Proposition 4.5 $x \in \Delta^{ESS} \Rightarrow (x, x)$ *is a perfect equilibrium.*

Lemma: Every ESS is undominated (that is, not weakly dominated by any pure or mixed strategy).

5 The replicator dynamic

[Taylor and Jonker, 1978]

- Domain of analysis the same as for ESS: finite and symmetric two-player games

- However, now a mixed strategy is interpreted as a population state in a population where all individuals only use pure strategies when called to play (like in Nash's mass-action interpretation)

Heuristically:

1. A population of individuals who are recurrently and (uniformly) randomly matched in pairs to play the game

2. Individuals use only *pure strategies*

3. A mixed strategy is now interpreted as a *population state*, a vector of populations shares

4. Population shares change, depending on the *current average payoff* to each pure strategy

5. The changes are described by a *system of ordinary differential equations*

Formally:

- Again a large (continuum) population playing a symmetric finite game

- But now each individual always plays a pure strategy

- At each time $t \in \mathbb{R}$, and for each $h \in S$, let $x_h(t)$ be the population share of h-*strategists* (individuals who use pure strategy h)

- *Population state*: $x(t) = (x_1(t), ..., x_m(t)) \in \Delta$

- Expected payoff to pure strategy h at a random match (with $\mathbf{1}_h \in \boldsymbol{\Delta}$ denoting the h^{th} unit vector):

$$\pi(\mathbf{1}_h, x) = \mathbf{1}_h \cdot Ax$$

- *Population average payoff* :

$$\pi(x, x) = \sum_{h \in S} x_h \pi(\mathbf{1}_h, x)$$

The replicator dynamic:

$$\dot{x}_h = [\pi(1_h, x) - \pi(x, x)] \cdot x_h \quad \forall h \in S$$

- *Growth rate* of positive population shares:

$$\frac{\dot{x}_h}{x_h} = \pi(1_h, x) - \pi(x, x)$$

- Better (worse) than-average strategies grow (decline) and *best* replies have the highest growth rate

5.1 Solving the replicator dynamic

- Polynomial *vector field*

$$f_h(x) = [\pi(1_h, x) - \pi(x, x)]\, x_h$$

- Picard-Lindelöf Theorem: $\exists!$ (global) *solution* $\xi : \mathbb{R} \times \mathbf{\Delta} \to \mathbf{\Delta}$

  through any initial state $x^o \in \mathbf{\Delta}$

- Here $x = \xi(t, x^o)$ is the population state at time t if the initial state was x^o

Dynamic stability concepts

- A population state x is *Lyapunov stable* if small perturbations does not initiate a movement away from x. [Formally: for every neighborhood B of x there should exist a sub-neighborhood $B^o \subset B$ of x such that if $x^o \in B^o$ then $\xi(t, x^o) \in B$ for all $t > 0$.]

- A population state is *asymptotically stable* if it is Lyapunov stable and, moreover, the population returns asymptotically (over time) towards x after any sufficiently small perturbation. [Formally: in addition to Lyapunov stability, x should have a neighborhood A such that $x^o \in A \Rightarrow \xi(t, x^o) \to x$ as $t \to +\infty$.]

- Consider the replicator dynamic in PD, CO, HD, RSP!

5.2 Connection to ESS

Proposition 5.1 *If $x \in \Delta^{ESS}$, then x is an asymptotically stable population state in the replicator dynamic.*

- The converse holds for 2×2 games, but not in general

- Counter-example in class

Proposition 5.2 *If $x \in \Delta$ is asymptotically stable in the replicator dynamic, then (x, x) is a perfect equilibrium.*

5.3 Connections to non-cooperative solutions

- Every $x \in \Delta^{NE}$ is stationary, but not necessarily stable, in the replicator dynamic. Examples!

Proposition 5.3 *(a)* $x \in \Delta$ *Lyapunov stable* $\Rightarrow x \in \Delta^{NE}$

(b) $x^o \in int\,(\Delta) \wedge \lim_{t\to+\infty} \xi\,(t, x^o) = x^* \quad \Rightarrow \quad x^* \in \Delta^{NE}$

(c) $x^o \in int\,(\Delta)$ *and* $k \in S$ *strictly dominated* $\Rightarrow\ \lim_{t\to+\infty} \xi_k\,(t, x^o) = 0$

- Note that (c) does not presume that the solution trajectory converges, and that it can be strengthened to all non-rationalizable pure strategies $h \in S$

Hence, it is *as if* in the long run

CK[rationality+game]

holds (at least approximately)!