EVOLUTIONARY GAME THEORY
Population dynamics, social learning, and conventions

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1 Multi-population selection dynamics

- The replicator dynamic readily generalizes to arbitrary finite $n$-player games

- Domain now: finite normal-form games $G = \langle I, S, u \rangle$
  - $I = \{1, \ldots, n\}$ the set of player roles
  - $S = \times_{i \in I} S_i$ the set of pure-strategy profiles, $s = (s_1, s_2, \ldots, s_n)$
  - $u : S \to \mathbb{R}^n$ the combined payoff function, $u_i (s) \in \mathbb{R}$ being the payoff to the individual in player role $i \in I$

- A continuum "player population" of individuals for each player role, and all individuals play pure strategies
• Let $\Delta (S_i)$ be the mixed-strategy simplex for player role $i$

• Let $\square (S)$ be the polyhedron of mixed-strategy profiles,
  
  $\square (S) = \times_{i \in I} \Delta (S_i)$

• Extend $u$ from $S$ to $\square (S)$ in the usual way, so that $u_i (x)$ is player $i$’s expected payoff when mixed-strategy profile $x = (x_1, .., x_n)$ is played
1.1 Examples

1. Taylor (1979):

\[ \dot{x}_{ih} = [u_i(1_{ih}, x_{-i}) - u_i(x)] \cdot x_{ih} \quad \forall i \in I, h \in S_i \]

(a) The growth rate of the subpopulation of \( h \)-strategists within player population \( i \) equals the absolute payoff advantage of its strategy:

\[ g_{ih}(x) = u_i(1_{ih}, x_{-i}) - u_i(x) \]
2. Maynard-Smith (1982) suggested instead population growth rates based on \textit{relative} payoff advantage (assuming $u_i(x) > 0$ for all $i$ and $x$):

\[
\dot{x}_{ih} = \left[ \frac{u_i(1_{ih}, x_{-i})}{u_i(x)} - 1 \right] \cdot x_{ih}
\]

\[
g_{ih}(x) = \frac{u_i(1_{ih}, x_{-i})}{u_i(x)} - 1
\]
1.2 Two-player games

Example 1.1 (Coordination) In a $2 \times 2$-coordination game each strict equilibrium is asymptotically stable and the mixed equilibrium is unstable:
Example 1.2 (Hawk-Dove) Again the interior NE is unstable while both strict equilibria are asymptotically stable. The long-run outcome is again “history dependent”. Generic polarization between player. Explain the difference from the single-population dynamics!
Example 1.3 (Entry-deterrence) *Unique SPE but infinitely many other NE.*

**Extensive form:**

![Game Tree](image)

**Normal form:**

\[
\begin{array}{ccc}
\text{C} & \text{F} \\
\text{A} & 1,3 & 1,3 \\
\text{E} & 2,2 & 0,0 \\
\end{array}
\]
Solution orbits of the Taylor replicator dynamic:
2 General selection dynamics

- Arbitrary finite games $G = \langle I, S, u \rangle$

- Generalize the dynamics to allow for a wide range of imitation behaviors and social learning:

$$\dot{x}_{ih} = g_{ih}(x)x_{ih}$$

where $g$ is regular (Lipschitz continuous and $x_i \cdot g_i(x) \equiv 0$ for all $i$)
Definition 2.1 A growth-rate function \( g \) is payoff monotonic if

\[
u_i(1_{ih}, x_{-i}) > u_i(1_{ik}, x_{-i}) \Rightarrow g_{ih}(x) > g_{ik}(x)\]

Definition 2.2 A growth-rate function \( g \) is payoff positive if

\[
u_i(1_{ih}, x_{-i}) > u_i(x) \Rightarrow g_{ih}(x) > 0\]
\[
u_i(1_{ih}, x_{-i}) < u_i(x) \Rightarrow g_{ih}(x) < 0\]
• For any mixed-strategy profile $x$ and any player $i$, let

$$B_i(x) = \{ h \in S_i : u_i(1_{ih}, x_{-i}) > u_i(x) \}$$

— these are the pure strategies that yield payoffs above average

**Definition 2.3** A growth-rate function $g$ is **weakly payoff-positive** if:

$$B_i(x) \neq \emptyset \Rightarrow g_{ih}(x) > 0 \text{ for some } h \in B_i(x)$$
Proposition 2.1 (Weibull, 1995)  For any regular weakly payoff-positive dynamic:

(a) \( x \in \Box \text{ Lyapunov stable} \Rightarrow x \in \Box^{NE} \)

(b) \( x^o \in \text{int} (\Box) \land \lim_{t \to +\infty} \dot{x} (x^o, t) = x^* \Rightarrow x^* \in \Box^{NE} \)

- Property (c) of the single-population replicator dynamics in symmetric two-player games can be showed to hold for arbitrary finite games if the growth rate has a certain monotonicity property with respect to payoffs:
Definition 2.4 (Hofbauer and Weibull, 1996) A growth-rate function $g$ is convex monotonic if:

$$ u_i(y_i, x_{-i}) > u_i(1_{ih}, x_{-i}) \Rightarrow y_i \cdot g_i(x) > g_{ih}(x) $$

Proposition 2.2 (Hofbauer and Weibull, 1996) For any regular convex-monotonic selection dynamic: if $x^o \in \text{int}(\square)$ and $k \in S_i$ is iteratively strictly dominated for player $i$, then

$$ \lim_{t \to +\infty} \xi_{ik}(t, x^o) = 0 $$
Example 2.1 (outside-option game)

Three subgame-perfect equilibria, but only one, $s^* = (Ra, A)$, is compatible with “forward-induction”.

The purely reduced normal form:

$$
\begin{array}{ccc}
A & B \\
L & 2, v & 2, v \\
Ra & 3, 1 & 0, 0 \\
Rb & 0, 0 & 1, 3 \\
\end{array}
$$
3 Stochastic evolution in finite populations

- We recall that any evolutionary process has two parts: *mutation* (creation of varieties) and *selection*

- However, all the above dynamics are pure selection dynamics. No mutations occur. Robustness against mutations was studied by considering dynamic stability (Lyapunov and asymptotic).

- Moreover, the above dynamics are all deterministic and treat each player population as a continuum.

- What can be said if the player-populations are *finite* and subject to *perpetually occurring random mutations* alongside selection?

- This is the research task we now tackle
3.1 Stochastic population processes

[Benaïm & Weibull, 2003]

- Consider any finite $n$-player game

- For each player role $i \in I$ there is a player population of constant and finite size $N$

- Individuals always play pure strategies (just as in the deterministic selection dynamics and in Nash’s mass-action interpretation)

- A individual in player population $i$ who plays a pure strategy $h \in S_i$ will be called an $h$-strategist
• At discrete times $t = 0, \delta, 2\delta, 3\delta, \ldots$, for $\delta = 1/N$, one individual is randomly drawn for "strategy review"

• Equal probability for all individuals in the whole population to be drawn (i.i.d.)

• For each individual, the average time between two successive opportunities for strategy review is independent of $N$
Definition 3.1 A population state is a mixed-strategy profile \( x = (x_1, \ldots, x_n) \in \square(S) \) where each mixed strategy \( x_i \in \Delta(S_i) \) is such that \( N x_{ih} \) is a non-negative integer. Denote the (finite) set of population states \( \Theta^N \subset \square(S) \).

Definition 3.2 At times \( t \in D^N = \{0, 1/N, 2/N, \ldots\} \), let the vector \( X^N(t) = (X^N_1(t), \ldots, X^N_n(t)) \in \Theta^N \) be the current population shares of \( h \)-strategists, for all pure strategies \( h \in S_i \) and all player populations \( i \in I \).

- We analyze population processes \( \left\langle X^N(t) \right\rangle_{t \in D^N} \) that are (time homogeneous) Markov chains with transition probabilities

\[
P_{ikh}^N(x) = \Pr \left[ X^N_i(t + \frac{1}{N}) = x_i + \frac{1}{N} (1_{ih} - 1_{ik}) \mid X^N(t) = x \right]
\]
3.2 The mean-field equations

- The expected net increase in the population share of $h$-strategists in player population $i$, in population state $x$:
  $$ F_{ih}^N(x) = \sum_{k \neq h} p_{ikh}^N(x) - \sum_{k \neq h} p_{ihk}^N(x) $$

- Assume $F^N \to F$ as $N \to \infty$ (which we will see is the case in canonical examples)

- The mean-field equations, the flow approximation of the stochastic process (for $N$ large) is
  $$ \dot{x}_{ih} = F_{ih}(x) \quad \forall i \in I, \ h \in S_i, \ x \in \Box(S) $$

Q: Are the solution to these mean-field equations "good approximations" of the movements of the stochastic population process?
3.3 Why care?

- Because if, for some stochastic population process, the associated mean-field $F$ is of the form

$$F_{ih}(x) = g_{ih}(x)x_{ih}$$

for some growth-rate function $g$, then we can use the game-theoretic results for deterministic selection dynamics to obtain game-theoretic results for stochastic population processes.
3.3.1 Example: aspiration and imitation (or Herbert Simon meets John Nash)

- Herbert Simon (1916-2001, pioneer in behavioral economics, coining the term "bounded rationality", and 1978 economics Nobel laureate) suggested that real economic agents do not maximize, they "satisfice" (seek to meet aspiration levels)

- Suppose individuals use the rule of thumb: “If I am dissatisfied with the performance of my current strategy, then I will imitate a randomly drawn individual in my own player population.” [Gale, Binmore and Samuelson (1995), Björnerstedt and Weibull (1996), and Binmore and Samuelson (1997)]

- Suppose that the aspiration levels within each player population $i$ are statistically independent and uniformly distributed on an interval $[a_i(x), b_i(x)]$ that contains the range of the payoff function $\pi_i$
• Then

\[ F_{ih}(x) = \frac{1}{n} \frac{u_i(1_{ih}, x_{-i}) - u_i(x)}{b_i(x) - a_i(x)} \cdot x_{ih}. \]

• If \( a_i(x) \equiv \alpha_i \) and \( b_i(x) \equiv \beta_i \) for some \( \alpha_i < \beta_i \), then this is but a player-specific time rescaling of the Taylor (1979) multi-population replicator dynamic.

• If instead \( a_i(x) \equiv \alpha_i u_i(x) \) and \( b_i(x) \equiv \beta_i u_i(x) \) for some \( \alpha_i < \beta_i \) (and all payoffs are positive), then we obtain a player-specific time rescaling of the Maynard Smith (1982) multi-population replicator dynamics.

• If the mean field is a good approximation of these stochastic processes, then we have established that ”Herbert Simon asymptotically meets John Nash”: If a mean-field solution converges from some interior initial state, then the limit point is a Nash equilibrium, and then also
the stochastic population process will probabilistically converge to Nash equilibrium (in a precise sense)

3.4 The key lemma

- The answer to the question Q raised before is a qualified "yes". It hinges on the fact that the stochastic evolutionary process is exponentially well approximated by its mean field over bounded time intervals:

**Proposition 3.1 (Benaïm & Weibull, 2003)** *For every* \( T > 0 \) *there exist constants* \( c, K > 0 \) *such that for all* \( \varepsilon > 0 \) *and all* \( N \) *large enough:

\[
\Pr \left[ \max_{0 \leq t \leq T} \| X^N(t) - \xi(t, x^o) \| \geq \varepsilon \mid X^N(0) = x_0 \right] \leq Ke^{-c\varepsilon^2N}
\]
• In other words: the probability for any $\varepsilon$-deviation tends exponentially to zero as population size $N \to \infty$

• Using this, also asymptotic results can be obtain (see paper)
4 Social learning and stochastic stability

- Models of best replies to "recent history of play" with perpetual perturbations:
  
  
  
  
4.1 Young’s model in a nutshell

- Arbitrary finite game $G = \langle I, S, u \rangle$

- For each player role $i$ a finite population of size $N$

- Each round $t = 0, 1, 2, \ldots$ one individual is randomly drawn from each player population, and each of these $n$ individuals draws a sample of size $k$ (without replacement) from the $m \geq k$ last rounds of play, and these individuals play the game (see below)

- A state of this Markov chain is the $m$-history of recent play; the $m$ most recent pure-strategy profiles

- A successor history, after any history, is obtained by adding the new strategy profile and deleting the oldest
• The *unperturbed process*: each individual called to play always plays a best reply against its $k$-sample from the current $m$-history

- statistically independent samples across individuals in the same round and across periods

• The *perturbed process*:

  with probability $1 - \varepsilon$: an individual called to play plays a best reply against its $k$-sample from the current $m$-history

  with probability $\varepsilon$: the individual instead draws a pure strategy at random (uniformly)

• The perturbed process is *irreducible* (it can with positive probability get from any given state to any other given state in a finite number of rounds)
• If $k/m$ is sufficiently small and $m$ sufficiently large, then the perturbed process is also *aperiodic* (there exist no cycles that it cannot leave)

• A classical theorem in the theory of Markov chains is that any irreducible and aperiodic Markov chain with finite state space has a unique invariant probability distribution over its state space, a distribution to which it converges from any initial state

• Applied to the present class of perturbed Markov chains: There exists a unique invariant probability distribution $\mu^\varepsilon$ over $m$–histories of play, to which the process converges from any initial $m$-history.

• Next step: Take the limit of $\mu^\varepsilon$ as the probability for mistakes tends to zero, $\varepsilon \to 0$, this leads to $\mu^\varepsilon \to \mu^*$
• The so obtained limiting distribution $\mu^*$ is an invariant distribution under the unperturbed process (which may admit multiple invariant distributions)

• Any strategy profile that appears in a history in the support of the limiting distribution $\mu^*$ is called stochastically stable

• If a game has a unique stochastically stable strategy profile that repeats itself in every period, then it is a social convention
Proposition 4.1 (Young, 1993) Let $G$ be a $2 \times 2$-coordination game. If $k/m \leq 1/2$ then the unperturbed process ($\varepsilon = 0$) converges from any initial state with probability one to one of the two strict equilibria. If $k/m \leq 1/2$ and $m$ is sufficiently large, then the perturbed process ($\varepsilon > 0$) has a unique invariant distribution and, in the limit as $\varepsilon \to 0$, this places probability one on (repeated play of) the risk dominant equilibrium.

- Risk dominance (Harsanyi & Selten, 1988)

- The mixed NE, although perfect, is never selected (in line with ESS and the replicator dynamic)
Example 4.1 Consider the coordination game

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>4,4</td>
<td>0,2</td>
</tr>
<tr>
<td>b</td>
<td>2,0</td>
<td>3,3</td>
</tr>
</tbody>
</table>

The strict equilibrium \((a, a)\) Pareto dominates the strict equilibrium \((b, b)\), but \((b, b)\) risk dominates \((a, a)\).
• A generalization to arbitrary finite games:

**Definition 4.1 (Young, 1998)** A finite normal-form game $G = (I, S, u)$ has non-degenerate best replies (NDBR) if, every pure strategy is either not a best reply to any mixed-strategy profile, or it is a best reply to a set of mixed-strategy profiles with non-empty interior.

• This is a generic property of finite normal-form games.
• For each player role $i$ and any nonempty $T_i \subseteq S_i$, let $T = \times_{i \in N} T_i$ and let $\square(T) = \times_{i \in I} \Delta(T_i)$.

• Such block may be "closed under rational behavior" in a precise sense:

**Definition 4.2 (Basu and Weibull, 1991)** $T$ is a CURB set if

$$\beta[\square(T)] \subseteq T.$$ 

• Examples: the entry-deterrence game, the outside-option game, coordination games. They all have (unique) social conventions.
Theorem 4.2 (Young, 1998) Let $G$ be a finite game with the NDBR property. The unperturbed process converges with probability one to a minimal CURB set if $k/m$ is sufficiently small. In the limit as $\varepsilon \to 0$, the limit invariant distribution $\mu^*$ places unit probability on the strategy profiles on the minimal CURB set that have minimal stochastic potential.

- Hence, set-valued conventions exist and can be mathematically identified.