

## CHAPTER I

### INTRODUCTION

#### I. THE TYPES OF SOCIAL CHOICE

In a capitalist democracy there are essentially two methods by which social choices can be made: voting, typically used to make "political" decisions, and the market mechanism, typically used to make "economic" decisions. In the emerging democracies with mixed economic systems, Great Britain, France, and Scandinavia, the same two modes of making social choices prevail, though more scope is given to the method of voting and decisions based directly or indirectly on it and less to the rule of the price mechanism. Elsewhere in the world, and even in smaller social units within the democracies, social decisions are sometimes made by single individuals or small groups and sometimes (more and more rarely in this modern world) by a widely encompassing set of traditional rules for making the social choice in any given situation, e.g., a religious code.<sup>1</sup>

<sup>1</sup> The last two methods of making social choices are in a sense extreme opposites, developments of conflicting tendencies in a democracy. The rule of the single individual is the extreme of administrative discretion, the rule of a sacred code the extreme of rule by law. But in dynamic situations the rule of a sacred code leads by insensible steps to dictatorship. The code needs interpretation, for conditions change, and, no matter how explicit the code may have been in the first place in determining how society shall act in different circumstances, its meaning becomes ambiguous with the passage of time. It might conceivably happen that the job of interpretation passes to society as a whole, acting through some democratic process—"vox populi, vox dei." Or it can happen that interpretation passes to the hands of the people individually and not collectively; in this case, as soon as differences of opinion arise, the religious code loses all its force as a guide to social action. See, for example, the ultimate consequences in the field of economic ethics of the Protestant insistence on the right of each individual to interpret the Bible himself (R. H. Tawney, *Religion and the Rise of Capitalism*, London: J. Murray, 1926, pp. 97-100). But more likely, in view of the authoritarian character of the sacred code, the interpretation will pass into the hands of a single individual or a small group alone deemed qualified.

The classification of methods of social choice given here corresponds to Professor Knight's distinction among custom, authority, and consensus, except that I have subdivided consensus into the two categories of voting and the market (F. H. Knight, "Human Nature and World Democracy," in *Freedom and Reform*, New York: Harper and Bros., 1947, pp. 308-310).

The last two methods of social choice, dictatorship and convention, have in their formal structure a certain definiteness absent from voting or the market mechanism. In ideal dictatorship there is but one will involved in choice, in an ideal society ruled by convention there is but the divine will or perhaps, by assumption, a common will of all individuals concerning social decisions, so in either case no conflict of individual wills is involved.<sup>2</sup> The methods of voting and the market, on the other hand, are methods of amalgamating the tastes of many individuals in the making of social choices. The methods of dictatorship and convention are, or can be, rational in the sense that any individual can be rational in his choices. Can such consistency be attributed to collective modes of choice, where the wills of many people are involved?

It should be emphasized here that the present study is concerned only with the formal aspects of the above question. That is, we ask if it is formally possible to construct a procedure for passing from a set of known individual tastes to a pattern of social decision-making, the procedure in question being required to satisfy certain natural conditions. An illustration of the problem is the following well-known "paradox of voting." Suppose there is a community consisting of three voters, and this community must choose among three alternative modes of social action (e.g., disarmament, cold war, or hot war). It is expected that choices of this type have to be made repeatedly, but sometimes not all of the three alternatives will be available. In analogy with the usual utility analysis of the individual consumer under conditions of constant wants and variable price-income situations, rational behavior on the part of the community would mean that the community orders the three alternatives according to its collective preferences once for all, and then chooses in any given case that alternative among those actually available which stands highest on this list. A natural way of arriving at the collective preference scale would be to say that one alternative is preferred to another if a majority of the community prefer the first

<sup>2</sup> It is assumed, of course, that the dictator, like the usual economic man, can always make a decision when confronted with a range of alternatives and that he will make the same decision each time he is faced with the same range of alternatives. The ability to make consistent decisions is one of the symptoms of an integrated personality. When we pass to social decision methods involving many individuals (voting or the market), the problem of arriving at consistent decisions might analogously be referred to as that of the existence of an integrated society. Whether or not this psychiatric analogy is useful remains to be seen. The formal existence of methods of aggregating individual choices, the problem posed in this study, is certainly a necessary condition for an integrated society in the above sense; but whether the existence of such methods is sufficient or even forms an important part of the sufficient condition for integration is dubious.

Voting  
paradox.

alternative to the second, i.e., would choose the first over the second if those were the only two alternatives. Let  $A$ ,  $B$ , and  $C$  be the three alternatives, and 1, 2, and 3 the three individuals. Suppose individual 1 prefers  $A$  to  $B$  and  $B$  to  $C$  (and therefore  $A$  to  $C$ ), individual 2 prefers  $B$  to  $C$  and  $C$  to  $A$  (and therefore  $B$  to  $A$ ), and individual 3 prefers  $C$  to  $A$  and  $A$  to  $B$  (and therefore  $C$  to  $B$ ). Then a majority prefer  $A$  to  $B$ , and a majority prefer  $B$  to  $C$ . We may therefore say that the community prefers  $A$  to  $B$  and  $B$  to  $C$ . If the community is to be regarded as behaving rationally, we are forced to say that  $A$  is preferred to  $C$ . But in fact a majority of the community prefer  $C$  to  $A$ .<sup>3</sup> So the method just outlined for passing from individual to collective tastes fails to satisfy the condition of rationality, as we ordinarily understand it. Can we find other methods of aggregating individual tastes which imply rational behavior on the part of the community and which will be satisfactory in other ways?<sup>4</sup>

If we continue the traditional identification of rationality with maximization of some sort (to be discussed at greater length below), then the problem of achieving a social maximum derived from individual desires is precisely the problem which has been central to the field of welfare economics. There is no need to review the history of this subject in detail.<sup>5</sup> There has been controversy as to whether or not the econo-

<sup>3</sup> It may be added that the method of decision sketched above is essentially that used in deliberative bodies, where a whole range of alternatives usually comes up for decision in the form of successive pair-wise comparisons. The phenomenon described in the text can be seen in a pure form in the disposition of the proposals before recent Congresses for federal aid to state education, the three alternatives being no federal aid, federal aid to public schools only, and federal aid to both public and parochial schools. The "paradox of voting" seems to have been first pointed out by E. J. Nanson (*Transactions and Proceedings of the Royal Society of Victoria*, Vol. 19, 1882, pp. 197-240). I am indebted for this reference to C. P. Wright, University of New Brunswick.

<sup>4</sup> The problem of collective rationality has been discussed by Knight, but chiefly in terms of the socio-psychological prerequisites. See "The Plausible Act: The Possibilities and Limitations of Collective Rationality," in *Freedom and Reform*, *op. cit.*, pp. 335-369, especially pp. 346-365.

<sup>5</sup> Good sketches will be found in P. A. Samuelson's *Foundations of Economic Analysis*, Cambridge, Massachusetts: Harvard University Press, 1947, Chapter VIII; and A. Bergson (Burk), "A Reformulation of Certain Aspects of Welfare Economics," *Quarterly Journal of Economics*, Vol. 52, February, 1938, pp. 310-331. A summary of recent developments will be found in the article, "Socialist Economics," by A. Bergson, in *A Survey of Contemporary Economics*, H. S. Ellis, ed., Philadelphia: The Blakiston Co., 1948, Chapter XII. In addition to the above, restatements of the present state of the field will be found in O. Lange, "The Foundations of Welfare Economics," *Econometrica*, Vol. 10, July-October, 1942, pp. 215-228; and M. W. Reder, *Studies in the Theory of Welfare Economics*, New York: Columbia University Press, 1947, Chapters I-V.

mist *qua* economist could make statements saying that one social state is better than another. If we admit meaning to interpersonal comparisons of utility, then presumably we could order social states according to the sum of the utilities of individuals under each, and this is the solution of Jeremy Bentham, accepted by Edgeworth and Marshall.<sup>6</sup> Even in this case we have a choice of different mathematical forms of the social utility function in terms of individual utilities; thus, the social utility might be the sum of the individual utilities or their product or the product of their logarithms or the sum of their products taken two at a time. So, as Professor Bergson has pointed out, there are value judgments implicit even at this level.<sup>7</sup> The case is clearly much worse if we deny the possibility of making interpersonal comparisons of utility. It was on the latter grounds that Professor Robbins so strongly attacked the concept that economists could make any policy recommendations,<sup>8</sup> at least without losing their status as economists and passing over into the realm of ethics. On the other hand, Mr. Kaldor and, following him, Professor Hicks have argued that there is a meaningful sense in which we can say that one state is better than another from an economic point of view,<sup>9</sup> even without assuming the reality of interpersonal comparison of utilities. The particular mechanism by which they propose to accomplish the comparison of different social states, the compensation principle, will be examined in more detail in Chapter IV.

The controversy involves a certain confusion between two levels of argument. There can be no doubt that, even if interpersonal comparison is assumed, a value judgment is implied in any given way of making social choices based on individual utilities; so much Bergson has shown clearly. But, given these basic value judgments as to the mode of aggregating individual desires, the economist should investigate those

<sup>6</sup> F. Y. Edgeworth, *Mathematical Psychics*, London: C. Kegan Paul and Co., 1881, pp. 56-82, especially p. 57; "The Pure Theory of Taxation," in *Papers Relating to Political Economy*, London: Macmillan and Co., 1925, Vol. II, pp. 63-125, especially pp. 100-122. The interpretation of social utility as the sum of individual utilities is implicit in Marshall's use of the doctrine of consumers' surplus, though other assumptions are also involved. (A. Marshall, *Principles of Economics*, New York: The Macmillan Co., eighth edition, 1949, pp. 130-134, 467-476.)

<sup>7</sup> Bergson, "A Reformulation . . .," *op. cit.*, *passim*. See also Samuelson, *op. cit.*, pp. 219-252.

<sup>8</sup> L. Robbins, *An Essay on the Nature and Significance of Economic Science*, second edition, London: Macmillan and Co., 1935, Chapter VI; "Interpersonal Comparisons of Utility: A Comment," *Economic Journal*, Vol. 43, December, 1938, pp. 635-641.

<sup>9</sup> N. Kaldor, "Welfare Propositions of Economics and Interpersonal Comparisons of Utility," *Economic Journal*, Vol. 49, September, 1939, pp. 549-552; J. R. Hicks, "The Foundations of Welfare Economics," *Economic Journal*, Vol. 49, December, 1939, pp. 696-700, 711-712.

mechanisms for social choice which satisfy the value judgments and should check their consequences to see if still other value judgments might be violated. In particular, he should ask the question whether or not the value judgments are consistent with each other, i.e., do there exist any mechanisms of social choice which will in fact satisfy the value judgments made? For example, in the voting paradox discussed above, if the method of majority choice is regarded as itself a value judgment, then we are forced to the conclusion that the value judgment in question, applied to the particular situation indicated, is self-contradictory.

In the matter of consistency, the question of interpersonal comparison of utilities becomes important. Bergson considers it possible to establish an ordering of social states which is based on indifference maps of individuals, and Samuelson has agreed.<sup>10</sup> On the other hand, Professor Lange, in his discussion of the social welfare function, has assumed the interpersonal measurability of utility,<sup>11</sup> and elsewhere he has insisted on the absolute necessity of measurable utility for normative social judgments.<sup>12</sup> Professor Lerner similarly has assumed the meaningfulness of an interpersonal comparison of intensities of utility in his recent work on welfare economics.<sup>13</sup>

In the following discussion of the consistency of various value judgments as to the mode of social choice, the distinction between voting and the market mechanism will be disregarded, both being regarded as special cases of the more general category of collective social choice. The analogy between economic choice and political choice has been pointed out a number of times. For example, Professor Zassenhaus considered the structure of a planned economy by considering the free market replaced by influence conceived generally as a means of distributing the social product.<sup>14</sup> He argued that, under conditions analogous to free competition, the market for exchanging influence for goods would come to equilibrium in a manner analogous to that of the ordinary market, political influence taking the place of initial distribution of goods. His model, however, is expressed only in very general terms, and it is not easy to see how it would operate in a socialist democracy, for example.

<sup>10</sup> See the discussion of the Fundamental Value Propositions of Individual Preference in Bergson, "A Reformulation . . .," *op. cit.*, pp. 318-320; Samuelson, *op. cit.*, p. 228.

<sup>11</sup> Lange, *op. cit.*, pp. 219-224, especially top of p. 222; but there are contradictory statements on p. 223 and at the top of p. 224.

<sup>12</sup> O. Lange, "The Determinateness of the Utility Function," *Review of Economic Studies*, Vol. 1, June, 1934, pp. 224-225.

<sup>13</sup> A. P. Lerner, *Economics of Control*, New York: The Macmillan Co., 1944, Chapter III.

<sup>14</sup> H. Zassenhaus, "Über die ökonomische Theorie der Planwirtschaft," *Zeitschrift für Nationalökonomie*, Vol. 5, 1934, pp. 507-532.

Dr. Howard Bowen has considered voting as the demand for collective consumption.<sup>15</sup> In his treatment he regards distribution of income and costs as given, and other simplifying assumptions are made. Close analogies are found with the ordinary market demand curve.

Knight has also stressed the analogy between voting and the market in that both involve collective choice among a limited range of alternatives.<sup>16</sup> He has also stressed certain differences, particularly that there is likely to be a greater tendency toward inequality under voting than under the market; these differences are, however, largely of a socio-psychological type rather than of the formal type which alone is relevant here.

More recently, there has been a series of papers by Professor Duncan Black, dealing with various aspects of the theory of political choice under certain special assumptions and emphasizing the close similarity between the problems of market and electoral choice.<sup>17</sup> His work will be dealt with in greater detail in Chapter VII, Section 2. There is also a literature on the technical problems of election. The chief relevant point here is that virtually every particular scheme proposed for election from single-member constituencies has been shown to have certain arbitrary features. The problem of choosing by election one among a number of candidates for a single position, such as the Presidency of the United States or membership in a legislative body when each district returns only a single member, is clearly of the same character as choosing one out of a number of alternative social policies; indeed, selection among candidates is presumably a device for achieving selection among policies.

## 2. SOME LIMITATIONS OF THE ANALYSIS

It has been stated above that the present study confines itself to the formal aspects of collective social choice. The aspects not discussed may be conveniently described as the game aspects, especially since that term has acquired a double meaning. In the first place, no con-

<sup>15</sup> H. R. Bowen, "The Interpretation of Voting in the Allocation of Economic Resources," *Quarterly Journal of Economics*, Vol. 58, November, 1943, pp. 27-48.

<sup>16</sup> F. H. Knight, "Economic Theory and Nationalism," in *The Ethics of Competition and Other Essays*, New York: Harper and Bros., 1931, pp. 294-305.

<sup>17</sup> D. Black, "On the Rationale of Group Decision-Making," *Journal of Political Economy*, Vol. 56, February, 1948, pp. 23-34; "The Decisions of a Committee Using a Special Majority," *Econometrica*, Vol. 16, July, 1948, pp. 245-261; "The Elasticity of Committee Decisions with an Altering Size of Majority," *ibid.*, pp. 262-270; and "Un approccio alla teoria delle decisioni di comitato," *Giornale degli economisti e annali di economia*, Vol. 7, Nuova Serie, 1948, pp. 262-284. For the analogy between voting and the market, see especially "The Elasticity of Committee Decisions . . .," pp. 262, 270; and "Un approccio . . .," pp. 262-269.

## CHAPTER II

### THE NATURE OF PREFERENCE AND CHOICE

#### I. MEASURABILITY AND INTERPERSONAL COMPARABILITY OF UTILITY

The viewpoint will be taken here that interpersonal comparison of utilities has no meaning and, in fact, that there is no meaning relevant to welfare comparisons in the measurability of individual utility. The controversy is well-known and hardly need be recited here. During the entire controversy, the proponents of measurable utility have been unable to produce any proposition of economic behavior which could be explained by their hypothesis and not by those of the indifference-curve theorists.<sup>1</sup> Indeed, the only meaning the concepts of utility can be said to have is their indications of actual behavior, and, if any course of behavior can be explained by a given utility function, it has been amply demonstrated that such a course of behavior can be equally well explained by any other utility function which is a strictly increasing function of the first. If we cannot have measurable utility, in this sense, we cannot have interpersonal comparability of utilities a fortiori.

Recently, the issue of measurable utility has been reopened by the results of Professors von Neumann and Morgenstern.<sup>2</sup> These results have been widely misunderstood. They consider a preference pattern not only among certain alternatives but also among alternative probability distributions. Making certain plausible assumptions as to the relations among preferences for related probability distributions, they

<sup>1</sup> Classical demand theory leaves ambiguous the relation between the indifference map of a household and the indifference maps of the individual members thereof. It is the former which is relevant for the behavior of the market. The passage from individual to household maps is a special case of the passage from individual to social orderings; if the present thesis is accepted, household indifference maps can, indeed, only arise from the presence of common standards of value of some sort. But these are, as will be seen, empirically determinable by examination of the individual indifference maps and are not based on some type of intrinsic comparison of intensities of feeling. In what follows we shall ignore the distinction between individual and household indifference maps; this action may be regarded as meaning either that the intra-household aggregation is somehow solved or that that problem is being considered simultaneously with the general problem.

<sup>2</sup> *Op. cit.*, pp. 15-31, 617-632. See also W. S. Vickrey, "Measuring Marginal Utility by Reactions to Risk," *Econometrica*, Vol. 13, October, 1945, pp. 319-333.

find) that there is a utility indicator (unique up to a linear transformation) which has the property that the value of the utility function for any probability distribution of certain alternatives is the mathematical expectation of the utility. Put otherwise, there is one way (unique up to a linear transformation) of assigning utilities to probability distributions such that behavior is described by saying that the individual seeks to maximize his expected utility.

This theorem does not, as far as I can see, give any special ethical significance to the particular utility scale found. For instead of using the utility scale found by von Neumann and Morgenstern, we could use the square of that scale; then behavior is described by saying that the individual seeks to maximize the expected value of the square root of his utility. This is not to deny the usefulness of the von Neumann-Morgenstern theorem; what it does say is that among the many different ways of assigning a utility indicator to the preferences among alternative probability distributions, there is one method (more precisely, a whole set of methods which are linear transforms of each other) which has the property of stating the laws of rational behavior in a particularly convenient way. This is a very useful matter from the point of view of developing the descriptive economic theory of behavior in the presence of random events, but it has nothing to do with welfare considerations, particularly if we are interested primarily in making a social choice among alternative policies in which no random elements enter. (To say otherwise would be to assert that the distribution of the social income is to be governed by the tastes of individuals for gambling.

The problem of measuring utility has frequently been compared with the problem of measuring temperature. This comparison is very apt. Operationally, the temperature of a body is the volume of a unit mass of a perfect gas placed in contact with it (provided the mass of the gas is small compared with the mass of the body). Why, it might be asked, was not the logarithm of the volume or perhaps the cube root of the volume of the gas used instead? The reason is simply that the general gas equation assumes a particularly simple form when temperature is defined in the way indicated. But there is no deeper significance. Does it make any sense to say that an increase of temperature from  $0^{\circ}$  to  $1^{\circ}$  is just as intense as an increase of temperature from  $100^{\circ}$  to  $101^{\circ}$ ? No more can it be said that there is any meaning in comparing marginal utilities at different levels of well-being.

Even if, for some reason, we should admit the measurability of utility for an individual, there still remains the question of aggregating the individual utilities. At best, it is contended that, for an individual, his utility function is uniquely determined up to a linear transformation;



we must still choose one out of the infinite family of indicators to represent the individual, and the values of the aggregate (say a sum) are dependent on how the choice is made for each individual. In general, there seems to be no method intrinsic to utility measurement which will make the choices compatible.<sup>3</sup> It requires a definite value judgment not derivable from individual sensations to make the utilities of different individuals dimensionally compatible and still a further value judgment to aggregate them according to any particular mathematical formula. (If we look away from the mathematical aspects of the matter, it seems to make no sense to add the utility of one individual, a psychic magnitude in his mind, with the utility of another individual. Even Bentham had his doubts on this point.<sup>4</sup>)

We will therefore assume throughout this book that the behavior of an individual in making choices is describable by means of a preference scale without any cardinal significance, either individual or interpersonal.

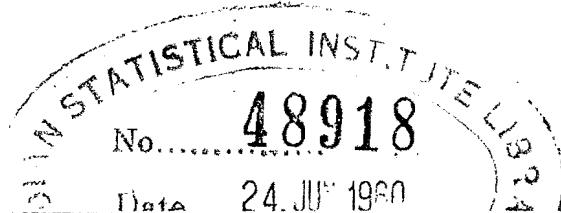
## 2. A NOTATION FOR PREFERENCES AND CHOICE

(In this study it is found convenient to represent preference by a notation not customarily employed in economics, though familiar in mathematics and particularly in symbolic logic) We assume that there is a basic set of alternatives which could conceivably be presented to the chooser. In the theory of consumer's choice, each alternative would be a commodity bundle; in the theory of the firm, each alternative would be a complete decision on all inputs and outputs; in welfare economics, each alternative would be a distribution of commodities and labor re-

<sup>3</sup> It must be granted, though, that, if it is assumed to begin with that all preference scales for individuals are the same (all individuals have the same tastes), then we could choose the utility function the same for all. However, if we take seriously the idea of interpersonal comparison of utilities, we must allow for the possibility that, of two individuals with the same indifference map, one is twice as sensitive as the other, and so the proper utility function for one should be just double that for another. It would be interesting, indeed, to see an operational significance attached to this concept of differing sensitivity.

Von Neumann and Morgenstern (*op. cit.*, pp. 608-616) have considered a case where two individuals have differing powers of discernment, but they have not represented this case by assuming different utilities for the same bundle of goods. Instead, they assume both utility scales can take on only discrete values, though one can take on more such values than the other.

<sup>4</sup> "Tis in vain to talk of adding quantities which after the addition will continue distinct as they were before, one man's happiness will never be another man's happiness: a gain to one man is no gain to another: you might as well pretend to add 20 apples to 20 pears. . . ." (Quoted by W. C. Mitchell in "Bentham's Felicific Calculus," in *The Backward Art of Spending Money and Other Essays*, New York: McGraw-Hill Book Co., 1937, p. 184.)



quirements. In general, an alternative is a vector; however, in the theory of elections, the alternatives are candidates. These alternatives are mutually exclusive; they are denoted by the small letters  $x, y, z, \dots$ . On any given occasion, the chooser has available to him a subset  $S$  of all possible alternatives, and he is required to choose one out of this set. The set  $S$  is a generalization of the well-known opportunity curve; thus, in the theory of consumer's choice under perfect competition it would be the budget plane. It is assumed further that the choice is made in this way: Before knowing the set  $S$ , the chooser considers in turn all possible pairs of alternatives, say  $x$  and  $y$ , and for each such pair he makes one and only one of three decisions:  $x$  is preferred to  $y$ ,  $x$  is indifferent to  $y$ , or  $y$  is preferred to  $x$ . The decisions made for different pairs are assumed to be consistent with each other, so, for example, if  $x$  is preferred to  $y$  and  $y$  to  $z$ , then  $x$  is preferred to  $z$ ; similarly, if  $x$  is indifferent to  $y$  and  $y$  to  $z$ , then  $x$  is indifferent to  $z$ . Having this ordering of all possible alternatives, the chooser is now confronted with a particular opportunity set  $S$ . If there is one alternative in  $S$  which is preferred to all others in  $S$ , the chooser selects that one alternative. Suppose, however, there is a subset of alternatives in  $S$  such that the alternatives in the subset are each preferred to every alternative not in the subset, while the alternatives in the subset are indifferent to each other. This case would be one in which the highest indifference curve that has a point in common with a given opportunity curve has at least two points in common with it. In this case, the best thing to say is that the choice made in  $S$  is the whole subset; the first case discussed is one in which the subset in question, the choice, contains a single element.

Since we have not restricted the type of sets allowed, a third possibility presents itself; there may be no alternative in  $S$  which is preferred or indifferent to all others. That is, for every alternative in  $S$ , there is another which is preferred to it. (For example, suppose that an individual prefers more money to less and that the alternatives in  $S$  include every integral number of dollars. Or, if we wish to require that  $S$  is in some sense bounded, consider the sequence of alternatives  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1 - (1/n), \dots$  dollars.) There cannot really be said to be any rational choice in this case. However, this mathematical point will not play any part in the present work.

(Preference and indifference are relations between alternatives. Instead of working with two relations, it will be slightly more convenient to use a single relation, "preferred or indifferent." The statement " $x$  is preferred or indifferent to  $y$ " will be symbolized by  $x R y$ . The letter  $R$ , by itself, will be the name of the relation and will stand for a knowledge of all pairs such that  $x R y$ . From our previous discussion, we

have that, for any pair of alternatives  $x$  and  $y$ , either  $x$  is preferred to  $y$  or  $y$  to  $x$ , or the two are indifferent. That is, we have assumed that any two alternatives are comparable.<sup>5</sup> But this assumption may be written symbolically, as

AXIOM I: *For all  $x$  and  $y$ , either  $x R y$  or  $y R x$ .* ✓

A relation  $R$  which satisfies Axiom I will be said to be connected. Note that Axiom I is presumed to hold when  $x = y$ , as well as when  $x$  is distinct from  $y$ , for we ordinarily say that  $x$  is indifferent to itself for any  $x$ , and this implies  $x R x$ .<sup>6</sup> Note also that the word "or" in the statement of Axiom I does not exclude the possibility of both  $x R y$  and  $y R x$ . That word merely asserts that at least one of the two events must occur; both may.

The property mentioned above of consistency in the preferences between different pairs of alternatives may be stated more precisely, as follows: If  $x$  is preferred or indifferent to  $y$  and  $y$  is preferred or indifferent to  $z$ , then  $x$  must be either preferred or indifferent to  $z$ . In symbols,

AXIOM II: *For all  $x$ ,  $y$ , and  $z$ ,  $x R y$  and  $y R z$  imply  $x R z$ .* ✓

A relation satisfying Axiom II is said to be transitive.<sup>7</sup> A relation satisfying both Axioms I and II is termed a weak ordering or sometimes simply an ordering. It is clear that a relation having these two properties taken together does create a ranking of the various alternatives. The adjective "weak" refers to the fact that the ordering does not exclude indifference, i.e., Axioms I and II do not exclude the possibility that for some distinct  $x$  and  $y$ , both  $x R y$  and  $y R x$ . A strong ordering, on

<sup>5</sup> The assumption of comparability of all alternatives is the heart of the integrability controversy in the theory of consumer's choice. See V. Pareto, *Manuel d'économie politique*, deuxième édition, Paris: M. Giard, 1927, pp. 546-569. For some of the paradoxical consequences of nonintegrability (which is equivalent to noncomparability of alternatives not infinitesimally close together), see N. Georgescu-Roegen, "The Pure Theory of Consumer's Behavior," *Quarterly Journal of Economics*, Vol. 50, August, 1936, pp. 545-569. Professor Ville has derived the integrability condition, and therewith the comparability of all alternatives, from some plausible hypotheses on the nature of demand functions (J. Ville, "Sur les conditions d'existence d'une ophélimité totale et d'un indice du niveau des prix," *Annales de l'Université de Lyon*, Section A, Vol. 3, No. 9, 1946, pp. 32-39).

<sup>6</sup> Strictly speaking, a relation is said to be connected if Axiom I holds for  $x \neq y$ . A relation  $R$  is said to be reflexive if, for all  $x$ ,  $x R x$ . (See A. Tarski, *Introduction to Logic*, New York: Oxford University Press, 1941, pp. 93-94.) Thus a relation satisfying Axiom I is both connected and reflexive. However, for convenience, we will use the slightly inaccurate terminology in the text, that is, we will use the word "connected" for the longer expression "connected and reflexive."

<sup>7</sup> Tarski, *ibid.*, p. 94.

the other hand, is a ranking in which no ties are possible.<sup>8</sup> A weak ordering is a generalization of the concept "greater than or equal to" applied to real numbers; a strong ordering generalizes the concept "greater than" applied to the same realm.<sup>9</sup>

It might be felt that the two axioms in question do not completely characterize the concept of a preference pattern. For example, we ordinarily feel that not only the relation  $R$  but also the relations of (strict) preference and of indifference are transitive. We shall show that, by defining preference and indifference suitably in terms of  $R$ , it will follow that all the usually desired properties of preference patterns obtain.

DEFINITION 1:  $x P y$  is defined to mean not  $y R x$ .

The statement " $x P y$ " is read " $x$  is preferred to  $y$ ."

DEFINITION 2:  $x I y$  means  $x R y$  and  $y R x$ .

The statement " $x I y$ " is read " $x$  is indifferent to  $y$ ." It is clear that  $P$  and  $I$ , so defined, correspond to the ordinary notions of preference and indifference, respectively.

LEMMA 1: (a) For all  $x$ ,  $x R x$ .

(b) If  $x P y$ , then  $x R y$ .

(c) If  $x P y$  and  $y P z$ , then  $x P z$ .

(d) If  $x I y$  and  $y I z$ , then  $x I z$ .

(e) For all  $x$  and  $y$ , either  $x R y$  or  $y P x$ .

(f) If  $x P y$  and  $y R z$ , then  $x P z$ .

All these statements are intuitively self-evident from the interpretations placed on the symbols. However, it may be as well to give sketches of the proofs, both to show that Axioms I and II really imply all that we wish to imply about the nature of orderings of alternatives and to illustrate the type of reasoning to be used subsequently.

PROOF: (a) In Axiom I, let  $y = x$ ; then for all  $x$ , either  $x R x$  or  $x R x$ , which is to say,  $x R x$ .

(b) Directly from Definition 1 and Axiom I.

(c) From  $x P y$  and  $y P z$ , we can, by (b), deduce  $x R y$ . Suppose  $z R x$ . Then, from  $z R x$  and  $x R y$ , we could deduce  $z R y$  by Axiom II. However, from  $y P z$ , we have, by Definition 1, not  $z R y$ . Hence the

<sup>8</sup> Frequently, indeed, the term "ordering relation" is reserved for strong orderings (Tarski, *ibid.*, pp. 96-97). However, in the present book the unmodified term "ordering" or "ordering relation" will be reserved for weak orderings.

<sup>9</sup> A formal characterization of strong ordering relations will be given later, in discussing the recent work of Professor Duncan Black on the theory of elections; see Chapter VII, Section 2.

supposition  $z R x$  leads to a contradiction, so that we may assert not  $z R x$ , or  $x P z$ , by Definition 1.

(d) From  $x I y$  and  $y I z$ , we can, by Definition 2, deduce  $x R y$  and  $y R z$ . From Axiom II, then,  $x R z$ . Also from  $x I y$  and  $y I z$ , by Definition 2, we have  $z R y$  and  $y R x$ , which imply  $z R x$ , by Axiom II. Since both  $x R z$  and  $z R x$ ,  $x I z$  by Definition 2.

(e) Directly from Definition 1.

(f) Suppose  $z R x$ . From  $z R x$  and  $y R z$  follows  $y R x$ , by Axiom II. But, by Definition 1,  $x P y$  implies not  $y R x$ . Hence the supposition  $z R x$  leads to a contradiction. Therefore, not  $z R x$ , or  $x P z$ .

For clarity, we will avoid the use of the terms "preference scale" or "preference pattern" when referring to  $R$ , since we wish to avoid confusion with the concept of preference proper, denoted by  $P$ . We will refer to  $R$  as an "ordering relation" or "weak ordering relation," or, more simply, as an "ordering" or "weak ordering." The term "preference relation" will refer to the relation  $P$ .

In terms of the relation  $R$ , we may now define the concept of choice, recalling that in general we must regard the choice from a given set of alternatives as itself a set. If  $S$  is the set of alternatives available, which we will term the *environment*,<sup>10</sup> let  $C(S)$  be the alternative or alternatives chosen out of  $S$ .  $C(S)$  is, of course, a subset of  $S$ . Each element of  $C(S)$  is to be preferred to all elements of  $S$  not in  $C(S)$  and indifferent to all elements of  $C(S)$ ; and, therefore, if  $x$  belongs to  $C(S)$ ,  $x R y$  for all  $y$  in  $S$ . On the other hand, if in fact  $x R y$  for all  $y$  in  $S$  and if  $x$  belongs to  $S$ , then, by Definition 1, there is no element  $z$  in  $S$  such that  $z P x$ . Hence, we may define  $C(S)$  formally as follows:

**DEFINITION 3:**  $C(S)$  is the set of all alternatives  $x$  in  $S$  such that, for every  $y$  in  $S$ ,  $x R y$ .

$C(S)$ , it is to be noted, describes a functional relationship in that it assigns a choice to each possible environment. We may call it the choice function; it is a straightforward generalization of the demand function as it appears in the theory of consumer's choice under perfect competition, the sets  $S$  there being budget planes.

Let  $[x, y]$  be the set composed of the two alternatives  $x$  and  $y$ . Suppose  $x P y$ . Then  $x R y$ , by Lemma 1(b), and  $x R x$ , by Lemma 1(a), so that  $x$  belongs to  $C([x, y])$ ; but, again by Definition 1, since  $x P y$ , not  $y R x$ , so that  $y$  does not belong to  $C([x, y])$ , i.e.,  $C([x, y])$  contains the single element  $x$ .

Conversely, suppose  $C([x, y])$  contains the single element  $x$ . Since  $y$  does not belong to  $C([x, y])$ , not  $y R x$ ; by Definition 1,  $x P y$ .

<sup>10</sup> This term is J. Marschak's.

LEMMA 2: A necessary and sufficient condition that  $x P y$  is that  $x$  be the sole element of  $C([x, y])$ .

In case neither  $x P y$  nor  $y P x$ , we have, clearly,  $x I y$ , and this is equivalent to saying that  $C([x, y])$  contains both  $x$  and  $y$ . If, then, we know  $C([x, y])$  for all two-element sets, we have completely defined the relations  $P$  and  $I$  and therefore the relation  $R$ ; but, by Definition 3, knowing the relation  $R$  completely determines the choice function  $C(S)$  for all sets of alternatives. Hence, one of the consequences of the assumptions of rational choice is that the choice in any environment can be determined by a knowledge of the choices in two-element environments.<sup>11)</sup>

The representation of the choice mechanism by ordering relations, as suggested above, has certain advantages for the present analysis over the more conventional representations in terms of indifference maps or utility functions. In regard to indifference maps, there is first the obvious advantage of being able to consider alternatives which are represented by vectors with more than two components. Second, the usefulness of an indifference map usually rests to a large measure on the assumption that the chooser desires more of each component of the alternative to less, all other components remaining the same; this assumption serves to orient the chart.<sup>12</sup> Since the present study is concerned with the choice of a social state, each alternative has many components which may be desirable under certain circumstances and undesirable under others. Third, the use of an indifference map involves assumptions of continuity which are unnecessarily restrictive for the

<sup>11</sup> Instead of starting, as here, with a weak ordering relation  $R$  satisfying certain axioms and then obtaining a choice function, it is possible to impose certain axioms directly on the choice function. It is not hard, in fact, to construct a set of plausible axioms concerning the choice function from which it is possible to deduce that there exists a weak ordering relation which could have generated the choice function, so that the two approaches are logically equivalent. Starting with the choice function instead of the ordering relation is analogous to the approach of Cournot, who started with demand functions having postulated properties instead of deriving those properties from a consideration of indifference maps or utility functions. (A. Cournot, *Mathematical Principles of the Theory of Wealth*, English translation, New York: The Macmillan Co., 1897, pp. 49-50.) The assumptions made by Cournot about the demand function were not very restrictive. More sophisticated treatment of demand from this point of view is to be found in the work of Ville, *op. cit.*, and Samuelson, *op. cit.*, pp. 111-117. Both treatments concern only the case of consumer's choice under perfectly competitive conditions, but suitable generalization to imperfectly competitive environments does not seem impossible.

<sup>12</sup> This brief statement is not accurate when the existence of a point of saturation is assumed. However, the chart is then at least oriented uniformly within each of several large segments, and the interesting economic problems presumably occur in the region where the assumption made in the text holds.

present purpose, especially since, in order to handle such problems as indivisibilities, which have been productive of so much controversy in the field of welfare economics, it is necessary to assume that some of the components of the social state are discrete variables.

As for utility functions, there is first of all the formal difficulty that, if insufficient continuity assumptions are made about the ordering, there may exist no way of assigning real numbers to the various alternatives in such a way as to satisfy the usual requirements of a utility function. In any case, we would simply be replacing the expression  $x R y$  by the expression  $U(x) \geq U(y)$ , and the structure of all proofs would be unchanged, while the elegance of the whole exposition would be marred by the introduction of the superfluous function  $U(x)$ , whose significance lies entirely in its ordinal properties. If we are concerned with ordinal properties, it seems better to represent these directly.<sup>13</sup>)

### 3. THE ORDERING OF SOCIAL STATES

In the present study the objects of choice are social states. The most precise definition of a social state would be a complete description of the amount of each type of commodity in the hands of each individual, the amount of labor to be supplied by each individual, the amount of each productive resource invested in each type of productive activity, and the amounts of various types of collective activity, such as municipal services, diplomacy and its continuation by other means, and the erection of statues to famous men. (It is assumed that each individual in the community has a definite ordering of all conceivable social states, in terms of their desirability to him. It is not assumed here that an individual's attitude toward different social states is determined exclusively by the commodity bundles which accrue to his lot under each. It is simply assumed that the individual orders all social states by whatever standards he deems relevant.) A member of Veblen's leisure class might order the states solely on the criterion of his relative income standing in each; a believer in the equality of man might order them in accordance with some measure of income equality.) Indeed, since, as mentioned above, some of the components of the social state, considered as a vector, are collective activities, purely individualistic assumptions are useless in analyzing such problems as the division of the national income between

<sup>13</sup> Similarly, in the field of production economics, it seems more natural to express the transformation restrictions by saying that the input-output vector lies in a certain point set than to introduce a transformation function and then subject the operations of the firm to the condition  $T = 0$ . In this case, the irrelevance of the functional representation is even clearer since, if  $F(t) = 0$  if and only if  $t = 0$ , then  $F(T)$  can be used as the transformation function just as well as  $T$ .

public and private expenditure. The present notation permits perfect generality in this respect. Needless to say, this generality is not without its price. More information would be available for analysis if the generality were restricted by a prior knowledge of the nature of individual orderings of social states. This problem will be touched on again.

In general, there will, then, be a difference between the ordering of social states according to the direct consumption of the individual and the ordering when the individual adds his general standards of equity (or perhaps his standards of pecuniary emulation).<sup>14</sup> We may refer to the former ordering as reflecting the *tastes* of the individual and the latter as reflecting his *values*. The distinction between the two is by no means clear-cut. An individual with esthetic feelings certainly derives pleasure from his neighbor's having a well-tended lawn. Under the system of a free market, such feelings play no direct part in social choice; yet psychologically they differ only slightly from the pleasure in one's own lawn. Intuitively, of course, we feel that not all the possible preferences which an individual might have ought to count; his preferences for matters which are "none of his business" should be irrelevant. Without challenging this view, I should like to emphasize that the decision as to which preferences are relevant and which are not is itself a value judgment and cannot be settled on an a priori basis. From a formal point of view, one cannot distinguish between an individual's dislike for having his grounds ruined by factory smoke and his extreme distaste for the existence of heathenism in Central Africa. There are probably not a few individuals in this country who would regard the former feeling as irrelevant for social policy and the latter as relevant, though the majority would probably reverse the judgment. I merely wish to emphasize here that we must look at the entire system of values, including values about values, in seeking for a truly general theory of social welfare.

(It is the ordering according to values which takes into account all the desires of the individual, including the highly important socializing desires, and which is primarily relevant for the achievement of a social maximum. The market mechanism, however, takes into account only the ordering according to tastes. This distinction is the analogue, on the side of consumption, of the divergence between social and private costs in production developed by Professor Pigou.<sup>15</sup>

<sup>14</sup> This distinction has been stressed to the author by M. Friedman, The University of Chicago.

<sup>15</sup> A. C. Pigou, *The Economics of Welfare*, London: Macmillan and Co., 1920, Part II, Chapter VI. For the analogy, see Samuelson, *op. cit.*, p. 224; Reder, *op. cit.*, pp. 64-67; G. Tintner, "A Note on Welfare Economics," *Econometrica*, Vol. 14, January, 1946, pp. 69-78.



As for notation, we will let  $R_i$  be the ordering relation for alternative social states from the standpoint of individual  $i$ . Sometimes, when several different ordering relations are being considered for the same individual, the symbols will be distinguished by adding a superscript. Corresponding to the ordering relation  $R_i$ , we have the (strict) preference relation  $P_i$  and the indifference relation  $I_i$ . If the symbol for the ordering has a prime or second attached (thus,  $R_i'$ ,  $R_i''$ ), then the corresponding symbols for preference and indifference will have the prime or second attached, respectively.

Similarly, society as a whole will be considered provisionally to have a social ordering relation for alternative social states, which will be designated by  $R$ , sometimes with a prime or second. Social preference and indifference will be denoted by  $P$  and  $I$ , respectively, primes or seconds being attached when they are attached to the relation  $R$ .

Throughout this analysis it will be assumed that individuals are rational, by which is meant that the ordering relations  $R_i$  satisfy Axioms I and II. The problem will be to construct an ordering relation for society as a whole that will also reflect rational choice-making so that  $R$  may also be assumed to satisfy Axioms I and II.)

#### 4. A DIGRESSION ON RATIONALITY AND CHOICE

The concept of rationality used throughout this study is at the heart of modern economic analysis, and it cannot be denied that it has great intuitive appeal; but closer analysis reveals difficulties. These may be illustrated by consideration of the modern developments in the theory of games and, in particular, the theory of zero-sum two-person games.<sup>16</sup>

<sup>16</sup> The theory of games involving more than two persons or games in which the sum of the payments to the various players is not independent of the methods of play is still in a dubious state despite the mathematically beautiful development in von Neumann and Morgenstern, *op. cit.*, Chapters V–XII. For example, the highly developed mechanism of compensations needed for their theory of rational behavior in such games appears to have little counterpart in the real world, as was pointed out by Professor Bain in another connection (J. S. Bain, "Output Quotas in Imperfect Cartels," *Quarterly Journal of Economics*, Vol. 62, August, 1948, pp. 617–622). On the other hand, there can be little doubt that the theory of rational play of a zero-sum two-person game has been completely solved, at least under certain restrictive assumptions as to the risk-neutrality of the players and as to the completeness of their information concerning the rules of the game. (See J. von Neumann, "Zur Theorie der Gesellschaftsspiele," *Mathematische Annalen*, Vol. 100, August, 1928, pp. 295–320; von Neumann and Morgenstern, *op. cit.*, Chapters III–IV.) Hence the theory of behavior in zero-sum two-person games affords some sort of check on the concepts of rationality derived to a large extent by analogy with the static theory of the firm under perfect competition.

As noted in Chapter II, Section 2, one of the consequences of the assumption of rationality is that the choice to be made from any set of alternatives can be determined by the choices made between pairs of alternatives. Suppose, however, that the situation is such that the chooser is never confronted with choices between pairs of alternatives; instead, the environment may always involve many alternatives. Indeed, that is precisely the situation in the theory of consumer's choice under perfect competition; the actual environment is always a whole line or plane. But, under certain plausible conditions, we can say that the choices made from the actual environments can be explained as though they were derived from choices between pairs of alternatives; and, at least conceptually, it makes sense to imagine the choices actually being made from pairs of alternatives.

Superficially, the theory of rational behavior in the zero-sum two-person game seems to fall into the same pattern. We could imagine each of the players considering all his possible strategies in turn, ordering them on the basis of the minimum profit (or maximum loss) that he could expect under each, and then choosing his best strategy by going as high up on the resulting scale as he can. But the only reason why we regard this solution as truly rational is that, if both players follow it, neither one will have any incentive to change his strategy even if he finds out the opponent's. This is the essence of the famous min-max or saddle-point theorem. The validity of this theorem, however, arises from the fact that every time we admit a set of pure strategies into the player's environment, we also admit all mixtures of them, i.e., all probability distributions over such a set of pure strategies. Hence, the environment (set of admissible strategies), if it contains more than one strategy, automatically contains an infinite number. Nor can we even conceptually imagine the choice between two strategies; for, if this limitation were real, a saddle-point would exist only in special cases, and the ordering of the strategies by minimum profit would not lead to a solution having the stability properties described above.

Thus, the model of rational choice as built up from pair-wise comparisons does not seem to suit well the case of rational behavior in the described game situation. It seems that the essential point is, and this is of general bearing, that, if conceptually we imagine a choice being made between two alternatives, we cannot exclude any probability distribution over those two choices as a possible alternative. The precise shape of a formulation of rationality which takes the last point into account or the consequences of such a reformulation on the theory of choice in general or the theory of social choice in particular cannot be foreseen; but it is at least a possibility, to which attention should be

drawn, that the paradox to be discussed below might be resolved by such a broader concept of rationality.

Many writers have felt that the assumption of rationality, in the sense of a one-dimensional ordering of all possible alternatives, is absolutely necessary for economic theorizing; for example, Professor Rothschild remarks, "Unless economic units act in conformity with some rational pattern no general theory about what would follow from certain premises would be possible."<sup>17</sup> There seems to be no logical necessity for this viewpoint; we could just as well build up our economic theory on other assumptions as to the structure of choice functions if the facts seemed to call for it.<sup>18</sup> The work of the institutionalist school may be regarded in part as such an attempt, though no systematic treatment has emerged.

The concept of choice functions not built up from orderings seems to correspond to Rothschild's "real irrationality"; however, such choice functions need not be the product of impulsive behavior but may conceivably arise from full reflection, as in the theory of games discussed above.

<sup>17</sup> K. W. Rothschild, "The Meaning of Rationality: A Note on Professor Lange's Article," *Review of Economic Studies*, Vol. 14, No. 1, 1946-47, p. 50. Rothschild also attributes this view to Professor Lange, but there seems to be a misinterpretation. Lange regards the assumption of rationality (which he identifies with ordering) as a highly convenient postulate, if true, but not necessary. (O. Lange, "The Scope and Method of Economics," *ibid.*, Vol. 13, No. 1, 1945-46, p. 30.)

<sup>18</sup> Like Lange, the present author regards economics as an attempt to discover uniformities in a certain part of reality and not as the drawing of logical consequences from a certain set of assumptions regardless of their relevance to actuality. Simplified theory-building is an absolute necessity for empirical analysis; but it is a means, not an end.

