Review of set and logic

Let f be a function from D to R. Consider the following statements. Suppose that $D_2 \subset D_1 \subseteq D$.

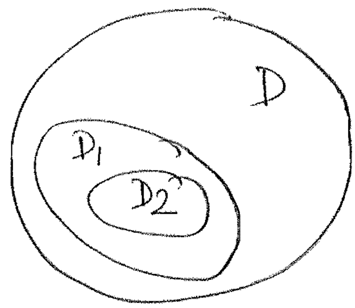
Property 1: If $x \in D_1$ then $f(x) = a$
Property 2: If $x \in D_2$ then $f(x) = a$

Claim 1: f satisfies Property 1 $\Rightarrow$ f satisfies Property 2. However there can be f which satisfies Property 2 but does not satisfy Property 1. Therefore the set of f which satisfies Property 2 is larger than that satisfies Property 1. Hence Property 2 is 'Weaker' than Property 1.

Example 1: For all citizen $i = 1, 2, \ldots, n$, let α_i, β_i denote income and education of citizen i. Let I be an index of inequality in this society. I maps $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n)$ to a number in the interval $[0, 1]$.
Property 1: If $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ then $I(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n) = 0$.
Property 2: If $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ and $\beta_1 = \beta_2 = \ldots = \beta_n$ then $I(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n) = 0$.
Note that in this example $D = \mathcal{R}^{2n}$. D_1 is the set of all vectors in D such that $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ and D_2 is the set of all vectors in D such that $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ and $\beta_1 = \beta_2 = \ldots = \beta_n$. Clearly $D_2 \subset D_1$.
An example of inequality index I, which satisfies Property 2 but not Property 1 is, $I(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n) = Var(\alpha) + Var(\beta)$.

Proof of Claim 1: Take a f which satisfies Property 1. We want to show that f satisfies Property 2. To check whether f satisfies Property 2, we must start with the antecedent, that is choose any $x \in D_2$. If we can show that $f(x) = a$ then we are done. Since $D_2 \subset D_1$, $x \in D_2 \Rightarrow x \in D_1$. We know that f satisfies Property 1 and $x \in D_1$. Hence $f(x) = a$.
Example 1 illustrates that the opposite argument, Property 2 $\Rightarrow$ Property 1, is not necessarily correct.

To follow this argument intuitively, see the following diagram. Property 1 imposes f to take value a inside D_1 (f is free to take any value, including a, outside D_1). Whereas Property 2 imposes f to take value a on a smaller set D_2. Thus restriction on f is weaker under Property 2 and hence the set of f which satisfies Property 2 is larger than that satisfying Property 1.



Similar 'If-Then' statement may arise in other contexts. To keep it simple, I shall not use proper algebraic terms in what follows. Let Q is a relation on pairs of elements in D and P is a relation on pairs in R. For examples of Q and P, see below.

Property 3: If xQ_1y then $f(x)Pf(y)$
Property 4: If xQ_2y then $f(x)Pf(y)$
Here Q_1 and Q_2 are two different relations defined on D. Suppose the following holds: for all x, y, $xQ_2y \Rightarrow xQ_1y$.

We have a result similar to claim 1.
Claim 2: f satisfies Property 3 $\Rightarrow$ f satisfies Property 4. However there can be f which satisfies Property 4 but does not satisfy Property 3. Therefore the set of f which satisfies Property 4 is larger than that satisfies Property 3. Hence Property 4 is 'Weaker' than Property 3.

I shall skip the proof of claim 2, which is similar to Claim 1. Please check it yourself. I shall illustrate Claim 2 with an example.

Example 2: For all citizen $i = 1, 2, \ldots, n$, let α_i denote income of citizen i. Let W be an index of well-being in this society. W maps $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ to a real number.

Here $D = \mathcal{R}^n$. Take two income distributions $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\alpha'_1, \alpha'_2, \ldots, \alpha'_n)$.

Property 3: If $\alpha'_i \geq \alpha_i$ for all $i = 1, \ldots, n$ and $\alpha'_k > \alpha_k$ for some k then $W(\alpha'_1, \alpha'_2, \ldots, \alpha'_n) > W(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Property 4: If $\alpha'_i > \alpha_i$ for all $i = 1, \ldots, n$ then $W(\alpha'_1, \alpha'_2, \ldots, \alpha'_n) > W(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Here relations Q_1, Q_2 and P are as follows,

$\alpha' Q_1 \alpha$: $\alpha'_i \geq \alpha_i$ for all $i = 1, \ldots, n$ and $\alpha'_k > \alpha_k$ for some k

$\alpha' Q_2 \alpha$: $\alpha'_i > \alpha_i$ for all $i = 1, \ldots, n$

P is the usual ordering on real number.

Note that $\alpha' Q_2 \alpha$ implies $\alpha' Q_1 \alpha$. Thus by Claim 2, any W that satisfies Property 3 must also satisfy Property 4. Here is a W which satisfies property 4 but violates Property 3.

$W^*(\alpha_1, \alpha_2, \ldots, \alpha_n) = \min_i \alpha_i$. Check that W^* satisfies Property 4 but violates Property 3. Can you find another such example?

The following diagram explains this argument intuitively. Property 3 imposes f to be strictly greater than $f(\alpha)$ on D_1, including the dotted boundaries (f is free to take any value, outside D_1). Whereas Property 2 imposes f to be strictly greater than $f(\alpha)$ inside D_1, not on the boundaries. f is free to take any value outside D_1 including the boundaries. Thus restriction on f is weaker under Property 4 and hence the set of f which satisfies Property 4 is larger than that satisfying Property 3.