Microeconomic Theory: Lecture 2
Choice Theory and Consumer Demand

Parikshit Ghosh

Delhi School of Economics

Summer Semester, 2014

Binary Relations

- Examples: taller than, friend of, loves, hates, etc.
- Abstract formulation: a binary relation R defined on a set of objects X may connect any two elements of the set by the statement 'xRy' and/or the statement 'yRx'.
- R may or may not have certain abstract properties, e.g.
  - Commutativity: $\forall x, y, \ \ xRy \Rightarrow yRx$. Satisfied by "classmate of" but not "son of."
  - Reflexivity: $\forall x, \ \ xRx$. Satisfied by "at least as rich as" but not "richer than."
  - Transitivity: $\forall x, y, z, \ \ xRy$ and $yRz \Rightarrow xRz$. Satisfied by "taller than" but not "friend of."
- Based on observation, we can often make general assumptions about a binary relation we are interested in studying.

The Preference Relation

- The preference relation is a particular binary relation.
- There are n goods, labeled $i = 1, 2, ..., n$.
- x_i = quantity of good i.
- A consumption bundle/vector $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$.
- Let $\succsim$ denote "at least as good as" or "weakly preferred to."
- $\mathbf{x}^1 \succsim \mathbf{x}^2$ means to the agent, the consumption bundle $\mathbf{x}^1$ is at least as good as the consumption bundle $\mathbf{x}^2$.
- $\succsim$ is a binary relation which describes the consumer's subjective preferences.

Other (Derived) Binary Relations

- The **strict preference** relation $\succ$ can be defined as:

$$\mathbf{x}^1 \succ \mathbf{x}^2 \text{ if } \mathbf{x}^1 \succsim \mathbf{x}^2 \text{ but not } \mathbf{x}^2 \succsim \mathbf{x}^1$$

- The **indifference** relation $\sim$ can be defined as:

$$\mathbf{x}^1 \sim \mathbf{x}^2 \text{ if } \mathbf{x}^1 \succsim \mathbf{x}^2 \text{ and } \mathbf{x}^2 \succsim \mathbf{x}^1$$

- Some properties of $\succsim$ (e.g. transitivity) may imply similar properties for $\succ$ and $\sim$.

The Axioms

- **Axiom 1 (Completeness):** For all $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n_+$, either $\mathbf{x}^1 \succsim \mathbf{x}^2$ or $\mathbf{x}^2 \succsim \mathbf{x}^1$ (or both).
  - The decision maker knows her mind.
  - Rules out dithering, confusion, inconsistency.

- **Axiom 2 (Transitivity):** For all $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathbb{R}^n_+$, if $\mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^3$, then $\mathbf{x}^1 \succsim \mathbf{x}^3$.
  - There are no preference loops or cycles. There is a quasi-ordering over the available alternatives.
  - Without some kind of ordering, it would be difficult to choose the best alternative.

The Axioms (contd.)

- **Axiom 3 (Continuity):** For any sequence $(\mathbf{x}^m, \mathbf{y}^m)_{m=1}^{\infty}$ such that $\mathbf{x}^m \succsim \mathbf{y}^m$ for all m, $\lim_{m\to\infty} \mathbf{x}^m = \mathbf{x}$ and $\lim_{m\to\infty} \mathbf{y}^m = \mathbf{y}$, it must be that $\mathbf{x} \succsim \mathbf{y}$.
  - Equivalent definition: For all $\mathbf{x} \in \mathbb{R}^n_+$, $\succsim(\mathbf{x})$ and $\precsim(\mathbf{x})$ are closed sets.
  - Bundles which are close in quantities are close in preference.

- **Axiom 4 (Strict Monotonicity):** For all $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n_+$, $\mathbf{x}^1 \geq \mathbf{x}^2$ implies $\mathbf{x}^1 \succsim \mathbf{x}^2$.
  - The more, the merrier.
  - Bads (e.g. pollution) can simply be defined as negative goods.

The Preference Representation Theorem

Theorem

If $\succsim$ satisfies Axioms 1-4, then there exists a continuous, increasing function $u : \mathbb{R}^n_+ \to \mathbb{R}$ which represents $\succsim$, i.e. for all $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n_+$, $\mathbf{x}^1 \succsim \mathbf{x}^2 \Leftrightarrow u(\mathbf{x}^1) \geq u(\mathbf{x}^2)$.

- The function $u(.)$ may be called an "utility function", but it is really an artifical construct that represents preferences in a mathematically tractable way.
- In cardinal choice theory, the utility function is a primitive.
- In ordinal choice theory, the preference ordering is the primitive and the utility function is a derived object.

Proof in Two Dimensions

- **Step 1:** For any $\mathbf{x}$, there is a unique symmetric bundle (z, z) such that $\mathbf{x} \sim (z, z)$.
- **Step 2:** $u(\mathbf{x}) = z$ represents $\succsim$.

- Let $Z^+ = \{z | (z, z) \succsim \mathbf{x}\}$ and $Z^= = \{z | \mathbf{x} \succsim (z, z)\}$.
- Must be of the form: $Z^+ = [\underline{z}, \infty)$ and $Z^= = [0, \overline{z}]$.
- Continuity ensures the sets are closed, monotonicity ensures there are no holes.
- To show that $\overline{z} = \underline{z}$.

Proof (contd.)

- Case 1: sets are disjoint
  - Suppose $\bar{z} < \underline{z}$.
  - Then for any $\bar{z} < z < \underline{z}$, completeness is violated.
- Case 2: sets are overlapping.
  - Suppose $\bar{z} > \underline{z}$.
  - Then for any $\underline{z} < z < \bar{z}$, $(z, z) \sim \mathbf{x}$.
  - Strict monotonicity is violated.
- Construction represents preference
  - Suppose $\mathbf{x}^1 \succsim \mathbf{x}^2$. Let $(z_1, z_1) \sim \mathbf{x}^1$ and $(z_2, z_2) \sim \mathbf{x}^2$.
  - Then $(z_1, z_1) \succ (z_2, z_2)$ (transitivity) $\Rightarrow z_1 \geq z_2$ (strict monotonicity).
  - Given the construction, $u(\mathbf{x}^1) \geq u(\mathbf{x}^2)$.

Invariance to Monotone Transformation

Theorem

If $u(.)$ represents $\succsim$, and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $v(\mathbf{x}) = f(u(\mathbf{x}))$ also represents $\succsim$.

- There is no *unique* function that represents preferences, but an entire class of functions.
- Example: suppose preferences are captured by the Cobb-Douglas utility function:

$$u(\mathbf{x}) = x_1^{\alpha} x_2^{\beta}$$

- The same preferences can also be described by:

$$v(\mathbf{x}) = \log u(\mathbf{x}) = \alpha \log x_1 + \beta \log x_2$$

Preference for Diversity

- **Axiom 5 (Convexity):** If $\mathbf{x}^1 \sim \mathbf{x}^2$, then $\lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2 \succsim \mathbf{x}^1, \mathbf{x}^2$ for all $\lambda \in [0,1]$.
- **Axiom 5A (Strict Convexity):** If $\mathbf{x}^1 \sim \mathbf{x}^2$, then $\lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2 \succ \mathbf{x}^1, \mathbf{x}^2$ for all $\lambda \in (0,1)$.

Definition

A function $f(\mathbf{x})$ is (strictly) quasiconcave if, for every $\mathbf{x}^1, \mathbf{x}^2$

$$f\left(\lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2\right) \geq (>) \min\{f(\mathbf{x}^1), f(\mathbf{x}^2)\}$$

Theorem

$u(.)$ is (strictly) quasiconcave if and only if $\succsim$ is (strictly) convex.

Indifference Curves

- The indifference curve through $\mathbf{x}^0$ is the set of all bundles just as good as $\mathbf{x}^0$

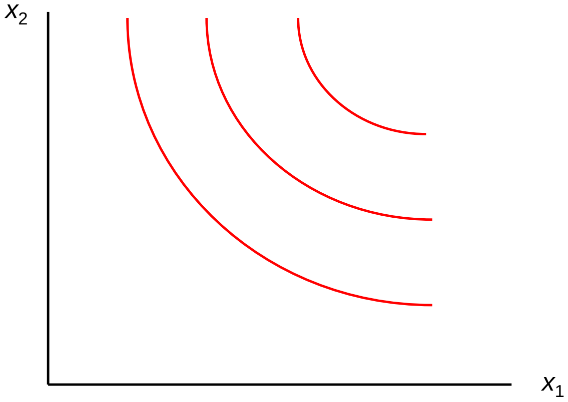
$$I(\mathbf{x}^0) = \left\{\mathbf{x} | \mathbf{x} \sim \mathbf{x}^0\right\} = \left\{\mathbf{x} | u(\mathbf{x}) = u(\mathbf{x}^0)\right\}$$

- It is also the boundary of the upper and lower contour sets, $\succsim(\mathbf{x}^0)$ and $\precsim(\mathbf{x}^0)$.
- Deriving the slope of the indifference curve (marginal rate of substitution, MRS) in two dimensions:

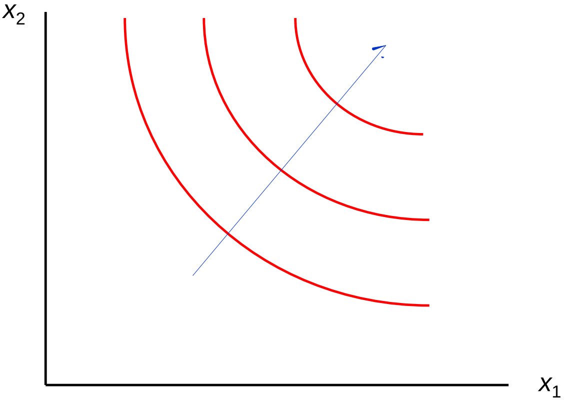
$$u(x_1, x_2) = \overline{u} \Rightarrow \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0$$

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = -\frac{u_1}{u_2} < 0$$

The Indifference Map



The Indifference Map



Properties of Indifference Curves

- Curves, not bands (strict monotonicity).
- No jumps (continuity).
- Downward sloping (strict monotonicity).
- Convex to the origin (convexity).
- Higher indifference curves represent more preferred bundles (strict monotonicity).

The Consumer's Problem

- The budget set B is the set of bundles the consumer can afford. Assuming linear prices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, income y:

$$B = \left\{ \mathbf{x} \middle| \sum_{i=1}^{n} p_i x_i \leq y \right\} = \{\mathbf{x} | \mathbf{px} \leq y\}$$

- The budget line is the boundary of the budget set.
- The consumer's problem:

$$\text{Choose} \;\; \mathbf{x}^* \in B \;\; \text{such that} \;\; \mathbf{x}^* \succsim \mathbf{x} \;\; \text{for all} \;\; \mathbf{x} \in B$$

- This can be obtained by solving:

$$\max_{\mathbf{x}} u(\mathbf{x}) \;\; \text{subject to} \;\; y - \mathbf{px} \geq 0, x_i \geq 0$$

Simplifying the Problem

- Suppose $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in S} f(\mathbf{x})$. If $\mathbf{x}^* \in S' \subset S$, then $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in S'} f(\mathbf{x})$.
- We can solve a problem by ignoring some constraints and later checking that the solution satisfies these constraints.
- If we know (by inspection) that the solution to a problem will satisfy certain constraints, we can try to solve it by adding these constraints to the problem.
- Strict monotonicity of preferences implies no money will be left unspent, i.e. $y - \mathbf{px}^* = 0$.
- Solve the simpler problem:

$$\max_{\mathbf{x}} u(\mathbf{x}) \quad \text{subject to} \quad y - \mathbf{px} = 0$$

If the solution satisfies $x_i \geq 0$, then it is the true solution.

Lagrange's Method

- Let $\mathbf{x}^*$ be the (interior) solution to:

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{subject to} \quad g_j(\mathbf{x}) = 0; j = 1, 2, \ldots, m$$

- Then there is a $\Lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ such that $(\mathbf{x}^*, \Lambda^*)$ solves:

$$\min_{\Lambda} \max_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \Lambda) \equiv f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x})$$

- We can find the solution to a constrained optimization problem (harder) by solving an unconstrained optimization problem (easier).

Application to the Consumer's Problem

- The consumer solves (assuming interior solution):

$$\max_{\mathbf{x}} u(\mathbf{x}) \quad \text{subject to} \quad y - \mathbf{px} = 0$$

- The Lagrangian is:

$$\min_{\lambda} \max_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \lambda) \equiv u(\mathbf{x}) + \lambda \left[y - \sum_{i=1}^{n} p_i x_i \right]$$

- First-order necessary conditions:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda^* p_i = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = y - \sum_{i=1}^{n} p_i x_i^* = 0$$

Simplifying and Solving

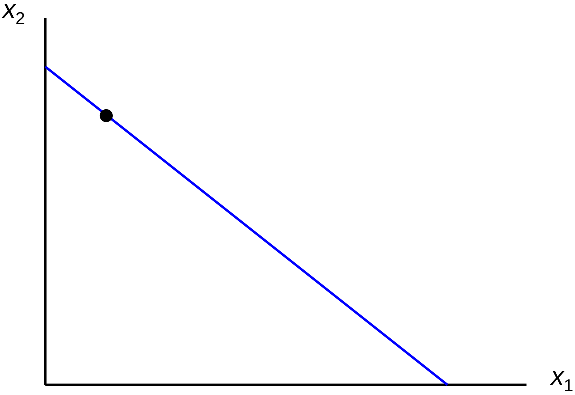
- Useful to eliminate the artificial variable λ.
- Dividing the i-th equation by the j-th:

$$\underbrace{\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}}}_{|MRS_{ij}|} = \underbrace{\frac{p_i}{p_j}}_{\text{price ratio}}$$

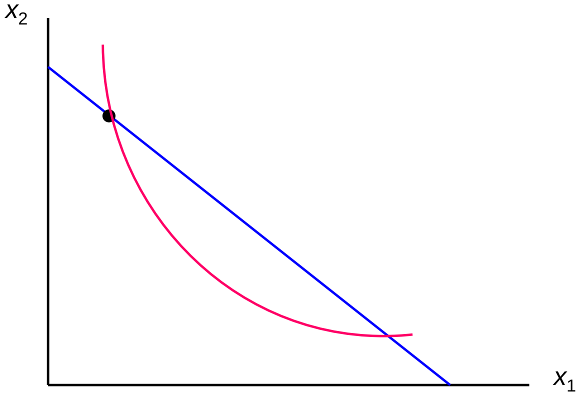
$$|MRS_{ij}| = \text{price ratio}$$

- In two dimensions, this means that at the optimum, slope of indifference curve = slope of the budget line.

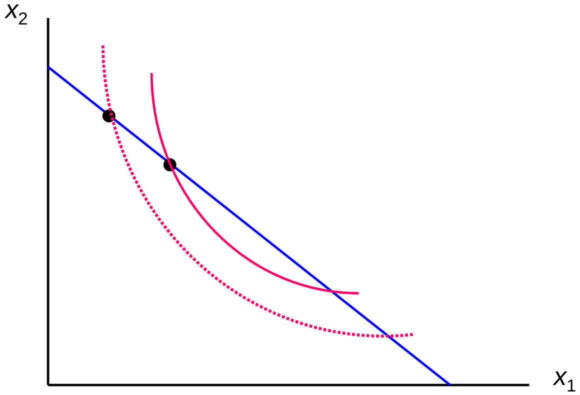
Consumer's Optimum in Pictures



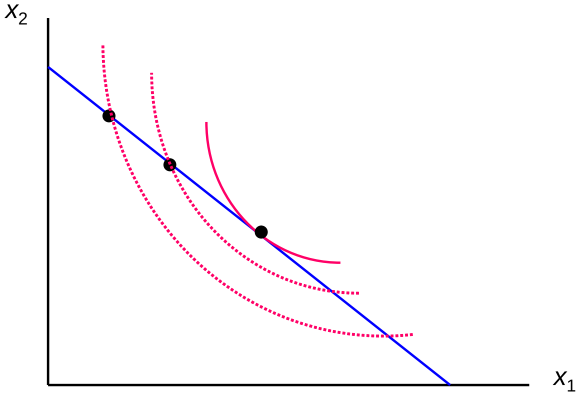
Consumer's Optimum in Pictures



Consumer's Optimum in Pictures



Consumer's Optimum in Pictures



Optimization: Read the Fine Print!

- Sometimes, the first-order conditions describe a minimum rather than a maximum.
- Need to check second-order conditions to make sure.
- It may only be a *local* maximum, not a *global* maximum.
- If there is a unique local maximum, it must be a global maximum.
- Sometimes, the true maximum is at the boundary of the feasible set (corner solution) rather than in the interior.
- The **Kuhn-Tucker conditions** generalize to both interior and corner solutions.

Second-Order Sufficient Conditions

- Consider the problem with a single equality constraint:

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{subject to } g(\mathbf{x}) = 0$$

- Suppose $\mathbf{x}^*$ satisfies the first-order necessary conditions derived by the Lagrange method.
- The bordered Hessian matrix is defined as

$$\overline{H} = \begin{bmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} \\ \vdots & & & & \vdots \\ g_n & \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn} \end{bmatrix}$$

- $\mathbf{x}^*$ is a local maximum of the constrained problem if the principal minors of $\overline{H}$ alternate in sign, starting with positive.

Uniqueness and Global Maximum

- For the consumer's problem, the bordered Hessian is

$$\overline{H} = \begin{bmatrix} 0 & p_1 & p_2 & \cdots & p_n \\ p_1 & u_{11} & u_{12} & \cdots & u_{1n} \\ p_2 & u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & & & & \vdots \\ p_n & u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix}$$

- Suppose $\mathbf{x}^* \gg \mathbf{0}$ solves the f.o.c obtained by the Lagrange method. If $u(.)$ is quasiconcave, then $\mathbf{x}^*$ is a constrained maximum.
- If $u(.)$ is strictly quasiconcave, the solution is unique.

Constrained Optimization

- The problem: $\max f(\mathbf{x}; \mathbf{a})$ subject to $\mathbf{x} \in S(\mathbf{a})$.
- $\mathbf{x}$ is a vector of endogenous variables (choices), $\mathbf{a}$ is a vector of exogenous variables (parameters).
- $f(\mathbf{x}; \mathbf{a})$ is the objective function. $S(\mathbf{a})$ is the feasible set (may be described by equalities or inequalities).
- The *choice function* gives the optimal values of the choices, as a function of the parameters:

$$\mathbf{x}^*(\mathbf{a}) = \arg \max_{\mathbf{x} \in S(\mathbf{a})} f(\mathbf{x}; \mathbf{a})$$

- The *value function* gives the optimized value of the objective function, as a function of the parameters:

$$v(\mathbf{a}) = \max_{\mathbf{x} \in S(\mathbf{a})} f(\mathbf{x}; \mathbf{a}) \equiv f(\mathbf{x}^*(\mathbf{a}); \mathbf{a})$$

The Implicit Function Theorem

- Consider a system of n continuously differentiable equations in n variables, $\mathbf{x}$, and m parameters, $\mathbf{a}$: $f^i(\mathbf{x};\mathbf{a}) = 0$.
- The Jacobian matrix J is the matrix of partial derivatives of the system of equations:

$$J = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} \ldots & \frac{\partial f^1}{\partial x_n} \\ \frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} \ldots & \frac{\partial f^2}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_2} \ldots & \frac{\partial f^n}{\partial x_n} \end{bmatrix}$$

- If $|J| \neq 0$, there exist explicit solutions described by continuously differentiable functions: $x_i^* = g^i(\mathbf{a})$, $i = 1, 2, \ldots, n$.

The Implicit Function Theorem (contd.)

- The response of the endogenous variables $\mathbf{x}^*$ to changes in some parameter a_k can be characterized without explicitly solving the system of equations.
- Using identities, we get

$$J.D\mathbf{x}^*(a_k) = D\mathbf{f}(a_k)$$

where

$$D\mathbf{x}^*(a_k)^t = \begin{pmatrix} \frac{dx_1^*}{da_k} & \frac{dx_2^*}{da_k} & \cdots & \frac{dx_n^*}{da_k} \end{pmatrix}$$

$$D\mathbf{f}(a_k)^t = \begin{pmatrix} \frac{\partial f^1}{\partial a_k} & \frac{\partial f^2}{\partial a_k} & \cdots & \frac{\partial f^n}{\partial a_k} \end{pmatrix}$$

- Applying Cramer's Rule, we get

$$\frac{dx_i^*}{da_k} = \frac{|J_k|}{|J|}$$

The Envelope Theorem

- Consider the value function:

$$v(\mathbf{a}) = \max_{\mathbf{x}} f(\mathbf{x}; \mathbf{a}) \quad \text{subject to} \quad g_j(\mathbf{x}; \mathbf{a}) = 0; j = 1, 2, \ldots, m$$

- The Lagrangian is:

$$\mathcal{L}(\mathbf{x}, \Lambda; \mathbf{a}) \equiv f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x})$$

- Suppose all functions are continuously differentiable. Then

$$\frac{\partial v(\mathbf{a})}{\partial a_k} = \frac{\partial \mathcal{L}(\mathbf{x}, \Lambda; \mathbf{a})}{\partial a_k}$$

- Intuition: changes in a parameter affects the objective function (a) directly (b) indirectly via induced changes in choices. Indirect effects can be ignored, due to f.o.c.

Illustration: Single Variable Unconstrained Optimum

- Consider the simple problem: $\max_x f(x; a)$.
- Let $v(a)$ be the value function and $x(a)$ the choice function.
- First-order condition as identity:

$$f_x(x(a); a) \equiv 0$$

- Equating derivatives of both sides (implicit function theorem):

$$f_{xx}x'(a) + f_{xa} = 0 \Rightarrow x'(a) = -\frac{f_{xa}}{f_{xx}}$$

- Since $f_{xx} < 0$ by s.o.c, sign depends on f_{xa}.
- Value function as identity: $v(a) \equiv f(x(a), a)$.
- Equating derivatives of both sides (envelope theorem):

$$v'(a) = f_x.x'(a) + f_a = f_a \quad (\text{since } f_x = 0)$$

Demand Functions

- The *Marshallian demand function* is the choice function of the consumer's problem:

$$\mathbf{x}(\mathbf{p}, y) = \arg\max_{\mathbf{x}} u(\mathbf{x}) \quad \text{subject to} \quad y - \mathbf{px} \geq 0, x_i \geq 0$$

- The *indirect utility function* is the value function of the consumer's problem:

$$v(\mathbf{p}, y) = u\left(\mathbf{x}(\mathbf{p}, y)\right)$$

- Interesting comparative statics questions:
  - How is the demand for a good (x_i) affected by changes in (i) its own price (p_i) (ii) price of another good (p_j) (iii) income?
  - What is the effect on consumer welfare (better off or worse off? by how much?) of changes in prices or incomes?

Properties of the Indirect Utility Function

- Continuous (objective function and budget set are continuous).
- Homogeneous of degree 0 (budget set remains unchanged).
- Strictly increasing in y (budget set exapands).
- Decreasing in p_i (budget set contracts).
- Quasiconvex in $(\mathbf{p}, y)$. (due to quasiconcavity of $u(.)$)
- **Roy's Identity** (assuming differentiability): Marshallian demand function can be derived from indirect utility function

$$x_i(\mathbf{p}, y) = -\frac{\frac{\partial v(\mathbf{p},y)}{\partial p_i}}{\frac{\partial v(\mathbf{p},y)}{\partial y}}$$

Proof of Roy's Identity

- The Lagrangian function (assuming interior solution):

$$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) + \lambda(y - \mathbf{px})$$

- Using the Envelope theorem:

$$\frac{\partial v(\mathbf{p}, y)}{\partial p_i} = \frac{\partial \mathcal{L}(\mathbf{p}, y)}{\partial p_i} = -\lambda^* x_i^*$$

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \frac{\partial \mathcal{L}(\mathbf{p}, y)}{\partial y} = \lambda^*$$

- Divide to get the result.

Duality Theory

- Consider the mirror image (dual) problem:

$$\min_{\mathbf{x}} \mathbf{px} \quad \text{subject to} \quad u(\mathbf{x}) \geq u, x_i \geq 0$$

- Achieve a target level of utility at the lowest cost, rather than achieve the highest level of utility for a given budget.
- The Hicksian demand function $\mathbf{x}^h(\mathbf{p}, u)$ is the choice function of this problem.
- The expenditure function $e(\mathbf{p}, u)$ is the value function.

Theorem

Suppose $f(\mathbf{x})$ and $g(\mathbf{x})$ are increasing functions. Then $f^* = \max_{\mathbf{x}} f(\mathbf{x})$ subject to $g(\mathbf{x}) \leq g^*$ if and only if $g^* = \min_{\mathbf{x}} g(\mathbf{x})$ subject to $f(\mathbf{x}) \geq f^*$.

Some Duality Based Relations

- Suppose u is the maximized value of utility at price vector $\mathbf{p}$ and income y.
- Duality says that y is the minimum amount of money needed to achieve utility u at prices $\mathbf{p}$.
- Since utility maximization and expenditure minimization are dual problems, their choice and value functions must be related.
- $x_i(\mathbf{p}, y) = x_i^h(\mathbf{p}, v(\mathbf{p}, y))$
- $x_i^h(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$
- $e(\mathbf{p}, v(\mathbf{p}, y)) = y$
- $v(\mathbf{p}, e(\mathbf{p}, u)) = u$

Properties of the Expenditure Function

- $e(\mathbf{p}, u(\mathbf{0})) = 0$.
- Continuous (objective function and feasible set are continuous).
- For all $\mathbf{p} \gg 0$, strictly increasing in u and unbounded above.
- Increasing in p_i (cost increases for *every* choice).
- Homogeneous of degree 1 in $\mathbf{p}$ (optimal choice unchanged).
- Concave in $\mathbf{p}$.
- **Shephard's Lemma** (assuming differentiability): Hicksian demand functions can be derived from the expenditure function

$$x_i^h(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}$$

Proof: Concavity and Shephard's Lemma

- Suppose $\mathbf{x}^1$ minimizes expenditure at $\mathbf{p}^1$, and $\mathbf{x}^2$ at $\mathbf{p}^2$.
- Let $\overline{\mathbf{x}}$ minimize expenditure at $\overline{\mathbf{p}} = \lambda \mathbf{p}^1 + (1-\lambda)\mathbf{p}^2$. By definition

$$\begin{aligned} \mathbf{p}^1\mathbf{x}^1 &\leq \mathbf{p}^1\overline{\mathbf{x}} \\ \mathbf{p}^2\mathbf{x}^2 &\leq \mathbf{p}^2\overline{\mathbf{x}} \end{aligned}$$

- Combining the two inequalities:

$$\lambda \mathbf{p}^1\mathbf{x}^1 + (1-\lambda)\mathbf{p}^2\mathbf{x}^2 \leq \left[\lambda \mathbf{p}^1 + (1-\lambda)\mathbf{p}^2\right]\overline{\mathbf{x}} = \overline{\mathbf{p}}.\overline{\mathbf{x}}$$

$$\text{or, } \ \lambda e(\mathbf{p}^1, u) + (1-\lambda)e(\mathbf{p}^2, u) \leq e(\lambda \mathbf{p}^1 + (1-\lambda)\mathbf{p}^2, u)$$

- Shephard's lemma obtained by applying envelope theorem.

The Slutsky Equation

Theorem

Suppose $\mathbf{p} \gg \mathbf{0}$ *and* $y > 0$, *and* $u = v(\mathbf{p}, y)$. *Then*

$$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} = \underbrace{\frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_j}}_{\substack{\textit{substitution}\\ \textit{effect}}} - \underbrace{x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}}_{\substack{\textit{income}\\ \textit{effect}}}$$

- Substitution effect: change in consumption that would arise if the consumer were compensated to preserve real income.
- Income effect: the further change in consumption which is due to drop in real income.

Proof of the Slutsky Equation

- By duality (note: identity)

$$x_i^h(\mathbf{p}, u) \equiv x_i(\mathbf{p}, e(\mathbf{p}, u))$$

- Differentiating w.r.t p_j:

$$\frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u))}{\partial p_j} + \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u))}{\partial y} \cdot \frac{\partial e(\mathbf{p}, u)}{\partial p_j}$$

- From Shephard's Lemma:

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_j} = x_j^h(\mathbf{p}, u) = x_j^h(\mathbf{p}, v(\mathbf{p}, y)) = x_j(\mathbf{p}, y)$$

- Using above:

$$\frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}$$

Testable Implications: Properties of Marshallian Demand

- Budget balancedness: $\mathbf{px}(\mathbf{p}, y) = y$ (due to strict monotonicity).
- Homogeneity of degree 0: $\mathbf{x}(\lambda\mathbf{p}, \lambda y) = \mathbf{x}(\mathbf{p}, y)$ (budget set does not change).
- The matrix H is symmetric, negative semi-definite, where

$$H = \begin{bmatrix} \frac{\partial x_1^h}{\partial p_1} & \frac{\partial x_1^h}{\partial p_2} \cdots & \frac{\partial x_1^h}{\partial p_n} \\ \frac{\partial x_2^h}{\partial p_1} & \frac{\partial x_2^h}{\partial p_2} \cdots & \frac{\partial x_2^h}{\partial p_n} \\ \vdots & & \vdots \\ \frac{\partial x_n^h}{\partial p_1} & \frac{\partial x_n^h}{\partial p_2} \cdots & \frac{\partial x_n^h}{\partial p_n} \end{bmatrix} = \left[\frac{\partial^2 e(\mathbf{p}, u)}{\partial p_i \partial p_j} \right]$$

- $\frac{\partial x_i^h}{\partial p_j}$ is observable thanks to the Slutsky equation.

The Law of Demand: A Critical Look

- Are demand curves necessarily downward sloping?
- Slutsky tells us

$$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_i} + x_i(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y} = \frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_i} = \frac{\partial^2 e(\mathbf{p}, u)}{\partial p_i^2} < 0$$

- For a normal good ($\frac{\partial x_i}{\partial y} > 0$), the law of demand holds ($\frac{\partial x_i}{\partial p_i} < 0$).
- For an inferior good ($\frac{\partial x_i}{\partial y} < 0$), it may or may not hold.
- **Giffen goods** are those which have positively sloped demand curves ($\frac{\partial x_i}{\partial p_i} > 0$).
- Must be (a) inferior (b) an important item of consumption (x_i large).

Are In-Kind Donations Inefficient?

- Many kinds of altruistic transfers are in-kind or targeted subsidies.
  - Employer matching grants to pension funds
  - Government subsidized health care
  - Tied aid by the World Bank
  - Book grants (as opposed to cash stipend) for students
  - Birthday or Diwali gifts
- The donor can make the recipient equally well off at lower cost if he gave assistance in cash rather than targeted subsidy.
- Rough idea: each Rupee of cash grant will be more valuable to the recipient since he can allocate it to suit his taste.

The Cake Eating Problem with Geometric Discounting

- A consumer has a cake of size 1 which can be consumed over dates $t = 1, 2, 3 \dots$
- The cake neither grows nor shrinks over time (exhaustible resource like petroleum).
- The consumer's utility at date t is

$$U_t = u(c_t) + \delta u(c_{t+1}) + \delta^2 u(c_{t+2}) + \dots$$

- $u(.)$ is instantaneous utility (strictly concave), $\delta \in (0, 1)$ is the discount factor.
- At date 0, the consumer's problem is to choose a sequence of consumptions $\{c_t\}_{t=0}^{\infty}$ to solve

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(c_t) \quad \text{subject to} \quad \sum_{t=0}^{\infty} c_t = 1$$

Time Consistency of the Optimal Path

- Let $\{c_t^*\}_{t=0}^{\infty}$ be the optimal consumption path at date 0.
- If the consumer gets the chance to revise her own plan at date t, will she do so (i.e. is the consumer dynamically consistent)?
- Suppose at some date t, the amount of cake left is c. At any $\widehat{t} < t$, the consumers' optimal plan for t onwards is:

$$\max_{\{c_\tau\}_{\tau=t}^{\infty}} \sum_{\tau=t}^{\infty} \delta^{\tau-\widehat{t}} u(c_\tau) \text{ subject to } \sum_{\tau=t}^{\infty} c_\tau = c$$

- The Lagrangian is

$$\mathcal{L}(\mathbf{c}, \lambda) = \sum_{\tau=t}^{\infty} \delta^{\tau-\widehat{t}} u(c_\tau) + \lambda \left[c - \sum_{\tau=t}^{\infty} c_\tau \right]$$

Time Consistency of the Optimal Path

- First-order condition:

$$\delta^{\tau-\widehat{t}}u'(c_\tau^*) = \lambda$$

- Eliminating λ:

$$\underbrace{\frac{u'(c_\tau^*)}{u'(c_{\tau+1})}}_{\text{intertemporal MRS}} = \underbrace{\delta}_{\text{discount factor}}$$

- Note that this is independent of $\widehat{t}$, the date at which the plan is being made.
- The consumer will not want to change her plans later.

Logarithmic Utility

- Suppose $u(c) = \log c$.
- From the first-order condition

$$c_{t+1}^* = \delta c_t^* \Rightarrow c_t^* = \delta^t c_0^*$$

- Using the budget constraint

$$c_0^* + \delta c_0^* + \delta^2 c_0^* + \ldots = 1 \Rightarrow c_0^* = 1 - \delta$$

$$c_t^* = \delta^t (1 - \delta)$$

- In every period, consume $1 - \delta$ fraction of the remaining cake, and save δ fraction.

Quasi-Hyperbolic Discounting and Cake Eating

- Suppose

$$U_t = u(c_t) + \beta \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u\left(c_\tau\right)$$

- The Lagrangian for the date 0 problem is:

$$\mathcal{L}(\mathbf{c}, \lambda) = u(c_t) + \beta \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u\left(c_\tau\right) + \lambda \left[1 - \sum_{\tau=t}^{\infty} c_\tau\right]$$

- First-order conditions:

$$u'\left(c_0^*\right) = \lambda^*$$

$$\beta\delta^{\tau-t} u'\left(c_\tau^*\right) = \lambda^*$$

Time Inconsistency of the Optimal Path

- Eliminating λ^*:

$$\text{MRS}_{0,1} = \frac{u'\left(c_0^*\right)}{u'\left(c_1^*\right)} = \beta\delta$$

$$\text{MRS}_{t,t+1} = \frac{u'\left(c_t^*\right)}{u'\left(c_{t+1}^*\right)} = \delta \quad \text{for all } t > 0$$

- However, when date t arrives, the consumer will want to change the plan and reallocate consumption such that

$$\text{MRS}_{t,t+1} = \beta\delta$$

- Realizing that she may change her own optimal plan later, the self aware consumer will adjust her plan at date 0 itself.
- Alternatively, the consumer may try to *commit* and restrict her own future options (e.g. Christmas savings accounts).