Microeconomic Theory: Lecture 2
Choice Theory and Consumer Demand

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Binary Relations

- Examples: taller than, friend of, loves, hates, etc.
- Abstract formulation: a binary relation $R$ defined on a set of objects $X$ may connect any two elements of the set by the statement ‘$xRy$’ and/or the statement ‘$yRx$’.
- $R$ may or may not have certain abstract properties, e.g.
  - Commutativity: $\forall x, y, \ xRy \Rightarrow yRx$. Satisfied by “classmate of” but not “son of.”
  - Reflexivity: $\forall x, \ xRx$. Satisfied by “at least as rich as” but not “richer than.”
  - Transitivity: $\forall x, y, z, \ xRy \text{ and } yRz \Rightarrow xRz$. Satisfied by “taller than” but not “friend of.”
- Based on observation, we can often make general assumptions about a binary relation we are interested in studying.
The Preference Relation

- The preference relation is a particular binary relation.
- There are \( n \) goods, labeled \( i = 1, 2, \ldots, n \).
- \( x_i \) = quantity of good \( i \).
- A consumption bundle/vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ \).
- Let \( \succsim \) denote “at least as good as” or “weakly preferred to.”
- \( \mathbf{x}^1 \succsim \mathbf{x}^2 \) means to the agent, the consumption bundle \( \mathbf{x}^1 \) is at least as good as the consumption bundle \( \mathbf{x}^2 \).
- \( \succsim \) is a binary relation which describes the consumer’s subjective preferences.
Other (Derived) Binary Relations

- The **strict preference** relation $\succ$ can be defined as:

  $$x^1 \succ x^2 \text{ if } x^1 \succeq x^2 \text{ but not } x^2 \succeq x^1$$

- The **indifference** relation $\sim$ can be defined as:

  $$x^1 \sim x^2 \text{ if } x^1 \succeq x^2 \text{ and } x^2 \succeq x^1$$

- Some properties of $\succeq$ (e.g. transitivity) may imply similar properties for $\succ$ and $\sim$. 
The Axioms

- **Axiom 1 (Completeness):** For all $x^1, x^2 \in \mathbb{R}_+^n$, either $x^1 \preceq x^2$ or $x^2 \preceq x^1$ (or both).
  - The decision maker knows her mind.
  - Rules out dithering, confusion, inconsistency.

- **Axiom 2 (Transitivity):** For all $x^1, x^2, x^3 \in \mathbb{R}_+^n$, if $x^1 \preceq x^2$ and $x^2 \preceq x^3$, then $x^1 \preceq x^3$.
  - There are no preference loops or cycles. There is a quasi-ordering over the available alternatives.
  - Without some kind of ordering, it would be difficult to choose the best alternative.
The Axioms (contd.)

- **Axiom 3 (Continuity):** For any sequence $(x^m, y^m)_{m=1}^\infty$ such that $x^m \preceq y^m$ for all $m$, $\lim_{m \to \infty} x^m = x$ and $\lim_{m \to \infty} y^m = y$, it must be that $x \preceq y$.
  - Equivalent definition: For all $x \in \mathbb{R}_+^n$, $\preceq (x)$ and $\succeq (x)$ are closed sets.
  - Bundles which are close in quantities are close in preference.

- **Axiom 4 (Strict Monotonicity):** For all $x^1, x^2 \in \mathbb{R}_+^n$, $x^1 \succeq x^2$ implies $x^1 \succeq x^2$.
  - The more, the merrier.
  - Bads (e.g. pollution) can simply be defined as negative goods.
The Preference Representation Theorem

Theorem

If \( \succeq \) satisfies Axioms 1-4, then there exists a continuous, increasing function \( u : \mathbb{R}^n_+ \rightarrow \mathbb{R} \) which represents \( \succeq \), i.e. for all \( x_1, x_2 \in \mathbb{R}^n_+ \),

\[ x_1 \succeq x_2 \iff u(x_1) \geq u(x_2). \]

- The function \( u(.) \) may be called an “utility function”, but it is really an artificial construct that represents preferences in a mathematically tractable way.
- In cardinal choice theory, the utility function is a primitive.
- In ordinal choice theory, the preference ordering is the primitive and the utility function is a derived object.
Proof in Two Dimensions

- **Step 1:** For any $x$, there is a unique symmetric bundle $(z, z)$ such that $x \sim (z, z)$.
- **Step 2:** $u(x) = z$ represents $\preceq$.

- Let $Z^+ = \{z \mid (z, z) \preceq x\}$ and $Z^- = \{z \mid x \preceq (z, z)\}$.
- Must be of the form: $Z^+ = [z, \infty)$ and $Z^- = [0, \bar{z}]$.
- Continuity ensures the sets are closed, monotonicity ensures there are no holes.
- To show that $\bar{z} = z$. 

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Proof (contd.)

- **Case 1:** sets are disjoint
  - Suppose \( \bar{z} < z \).
  - Then for any \( \bar{z} < z < z \), completeness is violated.

- **Case 2:** sets are overlapping.
  - Suppose \( \bar{z} > z \).
  - Then for any \( \bar{z} < z < \bar{z} \), \((z, z) \sim x\).
  - Strict monotonicity is violated.

- **Construction represents preference**
  - Suppose \( x^1 \succ x^2 \). Let \((z_1, z_1) \sim x^1 \) and \((z_2, z_2) \sim x^2 \).
  - Then \((z_1, z_1) > (z_2, z_2)\) (transitivity) \(\Rightarrow z_1 \geq z_2\) (strict monotonicity).
  - Given the construction, \(u(x^1) \geq u(x^2)\).
Invariance to Monotone Transformation

Theorem

If \( u(.) \) represents \( \succcurlyeq \), and \( f : \mathbb{R} \to \mathbb{R} \) is a strictly increasing function, then \( v(x) = f(u(x)) \) also represents \( \succcurlyeq \).

- There is no unique function that represents preferences, but an entire class of functions.
- Example: suppose preferences are captured by the Cobb-Douglas utility function:
  \[
  u(x) = x_1^\alpha x_2^\beta
  \]
- The same preferences can also be described by:
  \[
  v(x) = \log u(x) = \alpha \log x_1 + \beta \log x_2
  \]
Preference for Diversity

- **Axiom 5 (Convexity):** If $x^1 \sim x^2$, then
  \[ \lambda x^1 + (1 - \lambda) x^2 \succeq x^1, x^2 \text{ for all } \lambda \in [0, 1]. \]

- **Axiom 5A (Strict Convexity):** If $x^1 \sim x^2$, then
  \[ \lambda x^1 + (1 - \lambda) x^2 \succ x^1, x^2 \text{ for all } \lambda \in (0, 1). \]

**Definition**

A function $f(x)$ is (strictly) quasiconcave if, for every $x^1, x^2$

\[ f(\lambda x^1 + (1 - \lambda) x^2) \geq (>) \min \{ f(x^1), f(x^2) \} \]

**Theorem**

$u(.)$ is (strictly) quasiconcave if and only if $\succsim$ is (strictly) convex.
Indifference Curves

- The indifference curve through \( x^0 \) is the set of all bundles just as good as \( x^0 \)

\[
I(x^0) = \{ x | x \sim x^0 \} = \{ x | u(x) = u(x^0) \}
\]

- It is also the boundary of the upper and lower contour sets, \( \succsim (x^0) \) and \( \precsim (x^0) \).

- Deriving the slope of the indifference curve (marginal rate of substitution, MRS) in two dimensions:

\[
u(x_1, x_2) = \bar{u} \Rightarrow \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0
\]

\[
\frac{dx_2}{dx_1} = -\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = -\frac{u_1}{u_2} < 0
\]
The Indifference Map
The Indifference Map
Properties of Indifference Curves

- Curves, not bands (strict monotonicity).
- No jumps (continuity).
- Downward sloping (strict monotonicity).
- Convex to the origin (convexity).
- Higher indifference curves represent more preferred bundles (strict monotonicity).
The Consumer’s Problem

- The budget set $B$ is the set of bundles the consumer can afford. Assuming linear prices $p = (p_1, p_2, \ldots, p_n)$, income $y$:

$$B = \left\{ x \mid \sum_{i=1}^{n} p_i x_i \leq y \right\} = \{ x \mid px \leq y \}$$

- The budget line is the boundary of the budget set.
- The consumer’s problem:

Choose $x^* \in B$ such that $x^* \succeq x$ for all $x \in B$

- This can be obtained by solving:

$$\max_{x} u(x) \text{ subject to } y - px \geq 0, x_i \geq 0$$
Simplifying the Problem

- Suppose $\mathbf{x}^* \in \arg \max_{\mathbf{x} \in S} f(\mathbf{x})$. If $\mathbf{x}^* \in S' \subset S$, then $\mathbf{x}^* \in \arg \max_{\mathbf{x} \in S'} f(\mathbf{x})$.

- We can solve a problem by ignoring some constraints and later checking that the solution satisfies these constraints.

- If we know (by inspection) that the solution to a problem will satisfy certain constraints, we can try to solve it by adding these constraints to the problem.

- Strict monotonicity of preferences implies no money will be left unspent, i.e. $y - \mathbf{p} \mathbf{x}^* = 0$.

- Solve the simpler problem:

$$\max_{\mathbf{x}} u(\mathbf{x}) \quad \text{subject to} \quad y - \mathbf{p} \mathbf{x} = 0$$

If the solution satisfies $x_i \geq 0$, then it is the true solution.
Lagrange’s Method

Let $\mathbf{x}^*$ be the (interior) solution to:

$$\max_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } g_j(\mathbf{x}) = 0; j = 1, 2, \ldots, m$$

Then there is a $\Lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ such that $(\mathbf{x}^*, \Lambda^*)$ solves:

$$\min_{\Lambda} \max_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \Lambda) \equiv f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x})$$

We can find the solution to a constrained optimization problem (harder) by solving an unconstrained optimization problem (easier).
Application to the Consumer’s Problem

- The consumer solves (assuming interior solution):
  \[
  \max_x u(x) \quad \text{subject to} \quad y - px = 0
  \]

- The Lagrangian is:
  \[
  \min_{\lambda} \max_x L(x, \lambda) \equiv u(x) + \lambda \left[ y - \sum_{i=1}^n p_i x_i \right]
  \]

- First-order necessary conditions:
  \[
  \frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda^* p_i = 0
  \]
  \[
  \frac{\partial L}{\partial \lambda} = y - \sum_{i=1}^n p_i x_i^* = 0
  \]
Simplifying and Solving

- Useful to eliminate the artificial variable $\lambda$.
- Dividing the $i$-th equation by the $j$-th:

$$\frac{\partial u}{\partial x_i} = \frac{p_i}{p_j}$$

$$\left| MRS_{ij} \right| = \text{price ratio}$$

- In two dimensions, this means that at the optimum, slope of indifference curve = slope of the budget line.
Consumer’s Optimum in Pictures
Consumer’s Optimum in Pictures
Consumer’s Optimum in Pictures
Consumer’s Optimum in Pictures
Optimization: Read the Fine Print!

- Sometimes, the first-order conditions describe a minimum rather than a maximum.
- Need to check second-order conditions to make sure.
- It may only be a local maximum, not a global maximum.
- If there is a unique local maximum, it must be a global maximum.
- Sometimes, the true maximum is at the boundary of the feasible set (corner solution) rather than in the interior.
- The Kuhn-Tucker conditions generalize to both interior and corner solutions.
Second-Order Sufficient Conditions

- Consider the problem with a single equality constraint:

\[
\max_x f(x) \quad \text{subject to } g(x) = 0
\]

- Suppose \( x^* \) satisfies the first-order necessary conditions derived by the Lagrange method.

- The bordered Hessian matrix is defined as

\[
\overline{H} = \begin{bmatrix}
0 & g_1 & g_2 & \cdots & g_n \\
g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\
g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} \\
\vdots & & & & \vdots \\
g_n & \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn}
\end{bmatrix}
\]

- \( x^* \) is a local maximum of the constrained problem if the principal minors of \( \overline{H} \) alternate in sign, starting with positive.
Uniqueness and Global Maximum

- For the consumer’s problem, the bordered Hessian is

\[
\bar{H} = \begin{bmatrix}
0 & p_1 & p_2 & \cdots & p_n \\
p_1 & u_{11} & u_{12} & \cdots & u_{1n} \\
p_2 & u_{21} & u_{22} & \cdots & u_{2n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
p_n & u_{n1} & u_{n2} & \cdots & u_{nn}
\end{bmatrix}
\]

- Suppose \( x^* \gg 0 \) solves the f.o.c obtained by the Lagrange method. If \( u(.) \) is quasiconcave, then \( x^* \) is a constrained maximum.

- If \( u(.) \) is strictly quasiconcave, the solution is unique.
Constrained Optimization

- The problem: \( \max f(x; a) \) subject to \( x \in S(a) \).
- \( x \) is a vector of endogenous variables (choices), \( a \) is a vector of exogenous variables (parameters).
- \( f(x; a) \) is the objective function. \( S(a) \) is the feasible set (may be described by equalities or inequalities).
- The *choice function* gives the optimal values of the choices, as a function of the parameters:
  \[
  x^*(a) = \arg \max_{x \in S(a)} f(x; a)
  \]
- The *value function* gives the optimized value of the objective function, as a function of the parameters:
  \[
  v(a) = \max_{x \in S(a)} f(x; a) \equiv f(x^*(a); a)
  \]
The Implicit Function Theorem

- Consider a system of $n$ continuously differentiable equations in $n$ variables, $x$, and $m$ parameters, $a$: $f^i(x; a) = 0$.
- The Jacobian matrix $J$ is the matrix of partial derivatives of the system of equations:

$$J = \begin{bmatrix}
\frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} & \cdots & \frac{\partial f^1}{\partial x_n} \\
\frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_n}
\end{bmatrix}$$

- If $|J| \neq 0$, there exist explicit solutions described by continuously differentiable functions: $x_i^* = g^i(a)$, $i = 1, 2, \ldots, n$. 
The Implicit Function Theorem (contd.)

- The response of the endogenous variables $x^*$ to changes in some parameter $a_k$ can be characterized without explicitly solving the system of equations.
- Using identities, we get

$$J.Dx^*(a_k) = Df(a_k)$$

where

$$Dx^*(a_k)^t = \left( \frac{dx_1^*}{da_k} \quad \frac{dx_2^*}{da_k} \quad \ldots \quad \frac{dx_n^*}{da_k} \right)$$

$$Df(a_k)^t = \left( \frac{\partial f^1}{\partial a_k} \quad \frac{\partial f^2}{\partial a_k} \quad \ldots \quad \frac{\partial f^n}{\partial a_k} \right)$$

- Applying Cramer’s Rule, we get

$$\frac{dx_i^*}{da_k} = \frac{|J_k|}{|J|}$$
The Envelope Theorem

- Consider the value function:
  \[ v(a) = \max_x f(x; a) \text{ subject to } g_j(x; a) = 0; j = 1, 2, \ldots, m \]

- The Lagrangian is:
  \[ \mathcal{L}(x, \Lambda; a) \equiv f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) \]

- Suppose all functions are continuously differentiable. Then
  \[ \frac{\partial v(a)}{\partial a_k} = \frac{\partial \mathcal{L}(x, \Lambda; a)}{\partial a_k} \]

- Intuition: changes in a parameter affects the objective function (a) directly (b) indirectly via induced changes in choices. Indirect effects can be ignored, due to f.o.c.
Illustration: Single Variable Unconstrained Optimum

- Consider the simple problem: \( \max_x f(x; a) \).
- Let \( v(a) \) be the value function and \( x(a) \) the choice function.
- First-order condition as identity:
  \[
  f_x(x(a); a) = 0
  \]
- Equating derivatives of both sides (implicit function theorem):
  \[
  f_{xx}x'(a) + f_{xa} = 0 \Rightarrow x'(a) = -\frac{f_{xa}}{f_{xx}}
  \]
- Since \( f_{xx} < 0 \) by s.o.c, sign depends on \( f_{xa} \).
- Value function as identity: \( v(a) \equiv f(x(a), a) \).
- Equating derivatives of both sides (envelope theorem):
  \[
  v'(a) = f_x \cdot x'(a) + f_a = f_a \quad (\text{since } f_x = 0)
  \]
The Marshallian demand function is the choice function of the consumer’s problem:

\[ x(p, y) = \arg \max_x u(x) \quad \text{subject to} \quad y - px \geq 0, x_i \geq 0 \]

The indirect utility function is the value function of the consumer’s problem:

\[ v(p, y) = u(x(p, y)) \]

Interesting comparative statics questions:

- How is the demand for a good \( x_i \) affected by changes in (i) its own price \( p_i \) (ii) price of another good \( p_j \) (iii) income?
- What is the effect on consumer welfare (better off or worse off? by how much?) of changes in prices or incomes?
Properties of the Indirect Utility Function

- Continuous (objective function and budget set are continuous).
- Homogeneous of degree 0 (budget set remains unchanged).
- Strictly increasing in \( y \) (budget set expands).
- Decreasing in \( p_i \) (budget set contracts).
- Quasiconvex in \((p, y)\). (due to quasiconcavity of \( u(.) \))
- Roy’s Identity (assuming differentiability): Marshallian demand function can be derived from indirect utility function

\[
x_i(p, y) = -\frac{\partial v(p, y)}{\partial p_i} \cdot \frac{\partial v(p, y)}{\partial y}
\]
Proof of Roy’s Identity

- The Lagrangian function (assuming interior solution):

\[ L(x, \lambda) = u(x) + \lambda(y - px) \]

- Using the Envelope theorem:

\[ \frac{\partial v(p, y)}{\partial p_i} = \frac{\partial L(p, y)}{\partial p_i} = -\lambda^* x_i^* \]

\[ \frac{\partial v(p, y)}{\partial y} = \frac{\partial L(p, y)}{\partial y} = \lambda^* \]

- Divide to get the result.
Duality Theory

- Consider the mirror image (dual) problem:

\[
\min_x px \text{ subject to } u(x) \geq u, x_i \geq 0
\]

- Achieve a target level of utility at the lowest cost, rather than achieve the highest level of utility for a given budget.

- The Hicksian demand function \( x^h(p, u) \) is the choice function of this problem.

- The expenditure function \( e(p, u) \) is the value function.

**Theorem**

Suppose \( f(x) \) and \( g(x) \) are increasing functions. Then

\[
f^* = \max_x f(x) \text{ subject to } g(x) \leq g^* \text{ if and only if }
g^* = \min_x g(x) \text{ subject to } f(x) \geq f^*.
\]
Some Duality Based Relations

- Suppose $u$ is the maximized value of utility at price vector $\mathbf{p}$ and income $y$.
- Duality says that $y$ is the minimum amount of money needed to achieve utility $u$ at prices $\mathbf{p}$.
- Since utility maximization and expenditure minimization are dual problems, their choice and value functions must be related.

\[
x_i(\mathbf{p}, y) = x_i^h(\mathbf{p}, v(\mathbf{p}, y))
\]
\[
x_i^h(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))
\]
\[
e(\mathbf{p}, v(\mathbf{p}, y)) = y
\]
\[
v(\mathbf{p}, e(\mathbf{p}, u)) = u
\]
Properties of the Expenditure Function

- $e(p, u(0)) = 0$.
- Continuous (objective function and feasible set are continuous).
- For all $p \gg 0$, strictly increasing in $u$ and unbounded above.
- Increasing in $p_i$ (cost increases for every choice).
- Homogeneous of degree 1 in $p$ (optimal choice unchanged).
- Concave in $p$.
- **Shephard’s Lemma** (assuming differentiability): Hicksian demand functions can be derived from the expenditure function

$$x_i^h(p, u) = \frac{\partial e(p, u)}{\partial p_i}$$
Proof: Concavity and Shephard’s Lemma

- Suppose $x^1$ minimizes expenditure at $p^1$, and $x^2$ at $p^2$.
- Let $\bar{x}$ minimize expenditure at $\bar{p} = \lambda p^1 + (1 - \lambda) p^2$. By definition

$$p^1 x^1 \leq p^1 \bar{x}$$
$$p^2 x^2 \leq p^2 \bar{x}$$

- Combining the two inequalities:

$$\lambda p^1 x^1 + (1 - \lambda) p^2 x^2 \leq [\lambda p^1 + (1 - \lambda) p^2] \bar{x} = \bar{p} \bar{x}$$

or, $\lambda e(p^1, u) + (1 - \lambda) e(p^2, u) \leq e(\lambda p^1 + (1 - \lambda) p^2, u)$

- Shephard’s lemma obtained by applying envelope theorem.
The Slutsky Equation

Theorem
Suppose \( \mathbf{p} \gg \mathbf{0} \) and \( y > 0 \), and \( u = v(p, y) \). Then

\[
\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, u)}{\partial p_j} - x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}
\]

- Substitution effect: change in consumption that would arise if the consumer were compensated to preserve real income.
- Income effect: the further change in consumption which is due to drop in real income.
Proof of the Slutsky Equation

- By duality (note: identity)

\[ x_i^h(p, u) \equiv x_i(p, e(p, u)) \]

- Differentiating w.r.t. \( p_j \):

\[ \frac{\partial x_i^h(p, u)}{\partial p_j} = \frac{\partial x_i(p, e(p, u))}{\partial p_j} + \frac{\partial x_i(p, e(p, u))}{\partial y} \cdot \frac{\partial e(p, u)}{\partial p_j} \]

- From Shephard’s Lemma:

\[ \frac{\partial e(p, u)}{\partial p_j} = x_j^h(p, u) = x_j^h(p, v(p, y)) = x_j(p, y) \]

- Using above:

\[ \frac{\partial x_i^h(p, u)}{\partial p_j} = \frac{\partial x_i(p, y)}{\partial p_j} + x_j(p, y) \frac{\partial x_i(p, y)}{\partial y} \]
Testable Implications: Properties of Marshallian Demand

- **Budget balancedness**: \( px(p, y) = y \) (due to strict monotonicity).
- **Homogeneity of degree 0**: \( x(\lambda p, \lambda y) = x(p, y) \) (budget set does not change).
- **The matrix** \( H \) **is symmetric, negative semi-definite**, where

\[
H = \begin{bmatrix}
\frac{\partial x^h_1}{\partial p_1} & \frac{\partial x^h_1}{\partial p_2} & \cdots & \frac{\partial x^h_1}{\partial p_n} \\
\frac{\partial x^h_2}{\partial p_1} & \frac{\partial x^h_2}{\partial p_2} & \cdots & \frac{\partial x^h_2}{\partial p_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x^h_n}{\partial p_1} & \frac{\partial x^h_n}{\partial p_2} & \cdots & \frac{\partial x^h_n}{\partial p_n}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 e(p, u)}{\partial p_i \partial p_j}
\end{bmatrix}
\]

- \( \frac{\partial x^h_i}{\partial p_j} \) **is observable** thanks to the Slutsky equation.
The Law of Demand: A Critical Look

- Are demand curves necessarily downward sloping?
- Slutsky tells us

\[ \frac{\partial x_i(p, y)}{\partial p_i} + x_i(p, y) \frac{\partial x_i(p, y)}{\partial y} = \frac{\partial x_i(h(p, u))}{\partial p_i} = \frac{\partial^2 e(p, u)}{\partial p_i^2} < 0 \]

- For a normal good \((\frac{\partial x_i}{\partial y} > 0)\), the law of demand holds \((\frac{\partial x_i}{\partial p_i} < 0)\).
- For an inferior good \((\frac{\partial x_i}{\partial y} < 0)\), it may or may not hold.
- **Giffen goods** are those which have positively sloped demand curves \((\frac{\partial x_i}{\partial p_i} > 0)\).
- Must be (a) inferior (b) an important item of consumption \((x_i\) large).
Are In-Kind Donations Inefficient?

- Many kinds of altruistic transfers are in-kind or targeted subsidies.
  - Employer matching grants to pension funds
  - Government subsidized health care
  - Tied aid by the World Bank
  - Book grants (as opposed to cash stipend) for students
  - Birthday or Diwali gifts

- The donor can make the recipient equally well off at lower cost if he gave assistance in cash rather than targeted subsidy.

- Rough idea: each Rupee of cash grant will be more valuable to the recipient since he can allocate it to suit his taste.
The Cake Eating Problem with Geometric Discounting

- A consumer has a cake of size 1 which can be consumed over dates $t = 1, 2, 3 \ldots$
- The cake neither grows nor shrinks over time (exhaustible resource like petroleum).
- The consumer’s utility at date $t$ is
  \[ U_t = u(c_t) + \delta u(c_{t+1}) + \delta^2 u(c_{t+2}) + \ldots \]
- $u(.)$ is instantaneous utility (strictly concave), $\delta \in (0, 1)$ is the discount factor.
- At date 0, the consumer’s problem is to choose a sequence of consumptions $\{c_t\}_{t=0}^{\infty}$ to solve
  \[
  \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(c_t) \text{ subject to } \sum_{t=0}^{\infty} c_t = 1
  \]
Time Consistency of the Optimal Path

- Let \( \{c^*_t\}_{t=0}^{\infty} \) be the optimal consumption path at date 0.
- If the consumer gets the chance to revise her own plan at date \( t \), will she do so (i.e. is the consumer dynamically consistent)?
- Suppose at some date \( t \), the amount of cake left is \( c \). At any \( \hat{t} < t \), the consumers’ optimal plan for \( t \) onwards is:

\[
\max_{\{c_\tau\}_{\tau=t}^{\infty}} \sum_{\tau=t}^{\infty} \delta^{\tau-\hat{t}} u(c_\tau) \quad \text{subject to} \quad \sum_{\tau=t}^{\infty} c_\tau = c
\]

- The Lagrangian is

\[
\mathcal{L}(c, \lambda) = \sum_{\tau=t}^{\infty} \delta^{\tau-\hat{t}} u(c_\tau) + \lambda \left[ c - \sum_{\tau=t}^{\infty} c_\tau \right]
\]
Time Consistency of the Optimal Path

First-order condition:

$$\delta^{\tau-\hat{t}} u'(c^*_\tau) = \lambda$$

Eliminating $\lambda$:

$$\frac{u'(c^*_\tau)}{u'(c_{\tau+1})} = \delta$$

intertemporal MRS = discount factor

Note that this is independent of $\hat{t}$, the date at which the plan is being made.

The consumer will not want to change her plans later.
Logarithmic Utility

- Suppose $u(c) = \log c$.
- From the first-order condition

$$c^*_{t+1} = \delta c^*_t \Rightarrow c^*_t = \delta^t c^*_0$$

- Using the budget constraint

$$c^*_0 + \delta c^*_0 + \delta^2 c^*_0 + \ldots = 1 \Rightarrow c^*_0 = 1 - \delta$$

$$c^*_t = \delta^t (1 - \delta)$$

- In every period, consume $1 - \delta$ fraction of the remaining cake, and save $\delta$ fraction.
Quasi-Hyperbolic Discounting and Cake Eating

- Suppose

\[ U_t = u(c_t) + \beta \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u(c_\tau) \]

- The Lagrangian for the date 0 problem is:

\[ \mathcal{L}(c, \lambda) = u(c_t) + \beta \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u(c_\tau) + \lambda \left[ 1 - \sum_{\tau=t}^{\infty} c_\tau \right] \]

- First-order conditions:

\[ u'(c_0^*) = \lambda^* \]

\[ \beta \delta^{\tau-t} u'(c_\tau^*) = \lambda^* \]
Time Inconsistency of the Optimal Path

- Eliminating $\lambda^*$:
  \[ \text{MRS}_{0,1} = \frac{u'(c_0^*)}{u'(c_1^*)} = \beta \delta \]
  \[ \text{MRS}_{t,t+1} = \frac{u'(c_t^*)}{u'(c_{t+1}^*)} = \delta \quad \text{for all } t > 0 \]

- However, when date $t$ arrives, the consumer will want to change the plan and reallocate consumption such that
  \[ \text{MRS}_{t,t+1} = \beta \delta \]

- Realizing that she may change her own optimal plan later, the self aware consumer will adjust her plan at date 0 itself.
- Alternatively, the consumer may try to commit and restrict her own future options (e.g. Christmas savings accounts).