Ref: Advanced Microeconomic Theory by Geoffrey Jehle and Philip Reny (2nd Ed.)
Consider, an exchange economy

- There is no production - it has already taken place
- There are $N$ individuals - an individual is indexed by $i, i = 1, 2, ..., N$
- There are $M$ commodities - a commodity is indexed by $j, j = 1, 2, ..., M$
- Each individual is endowed with a bundle of commodities

Let, initial endowments be

\[
\begin{align*}
e^1 &= (e_1^1, e_2^1) = (8, 2) \\
e^2 &= (e_1^2, e_2^2) = (0, 7)
\end{align*}
\]
Definition

Allocation: An allocation is a distribution of available goods among the individuals consistent with the initial endowments.

E.g., when $e^1 = (8, 2)$ and $e^2(0, 7)$ let $x^1 = (3, 5)$, and $x^2 = (5, 3)$.

Let, $x^1 = (x^1_1, x^1_2)$, and $x^2 = (x^2_1, x^2_2)$.

Allocation $x = (x^1, x^2)$ is feasible

\[
\begin{align*}
  x^1_1 + x^2_1 & \leq e^1_1 + e^2_1 \\
  x^1_2 + x^2_2 & \leq e^1_2 + e^2_2
\end{align*}
\]

Allocation $x = (x^1, x^2)$ is non-wasteful if

\[
\begin{align*}
  x^1_1 + x^2_1 & = e^1_1 + e^2_1 \\
  x^1_2 + x^2_2 & = e^1_2 + e^2_2
\end{align*}
\]
In a $N$-person and $M$-good economy, an endowment is a vector of vectors:

$$e = (e^1, e^2, \ldots, e^N),$$

where $e^1 = (e^1_1, e^1_2, \ldots, e^1_M)$; $e^i = (e^i_1, e^i_2, \ldots, e^i_M)$, etc. So, the total endowment of goods can be written as:

- **good 1**: $e^1_1 + e^1_2 + \ldots + e^1_N = \sum_{i=1}^{N} e^i_1$
- **good j**: $e^j_1 + e^j_2 + \ldots + e^j_N = \sum_{i=1}^{N} e^i_j$
Barter: Goods for Goods IV

For the entire economy, an allocation is

\[ \mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^N), \]

where

\[ \mathbf{x}^1 = (x_1^1, x_2^1, \ldots, x_M^1) \]
\[ \mathbf{x}^i = (x_1^i, x_2^i, \ldots, x_M^i) \]

Allocation \( \mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^N) \) is non-wasteful w.r.t. good 1 if

\[ \sum_{i=1}^{N} x_1^i = \sum_{i=1}^{N} e_1^i \]
Definition

Allocation $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^N)$ is non-wasteful if

$$\sum_{i=1}^{N} x^i_j = \sum_{i=1}^{N} e^i_j$$

for all $j = 1, \ldots, M$. 

Definition

Set of (non-wasteful) allocations is the set

\[ \{x = (x^1, ..., x^N) \text{ Such that} \]

- \[ x^i_j \geq 0 \text{ for all } i \text{ and } j, \text{ and} \]
- \[ \sum_{i=1}^{N} x^i_j = \sum_{i=1}^{N} e^i_j \text{ for all } j = 1, ..., M \} \]
Assumptions I

We will make the following assumption about the choice sets and the individual preferences:

- The set of alternatives is the set of (non-wasteful) allocations
- Individuals have (self-interested) preferences defined over the set of allocations
- Each preference is/can be represented by a (self-interested) utility function
  - Each preference is complete, transitive and ????
- Each utility function is monotone, and ???
Consider a two-person, two-good economy.

**Definition**

Allocation \((\mathbf{x}^1, \mathbf{x}^2)\) is Pareto superior to the endowment, \((\mathbf{e}^1, \mathbf{e}^2)\), if

\[
    u^i(\mathbf{x}^i) \geq u^i(\mathbf{e}^i) \quad \text{holds for} \quad i = 1, 2.
\]

And

\[
    u^i(\mathbf{x}^i) > u^i(\mathbf{e}^i) \quad \text{holds for at least one} \quad i.
\]

**Remark**

If \((\mathbf{x}^1, \mathbf{x}^2)\) is Pareto superior to \((\mathbf{e}^1, \mathbf{e}^2)\), then

- \((\mathbf{e}^1, \mathbf{e}^2)\) is called Pareto inferior to \((\mathbf{x}^1, \mathbf{x}^2)\)
- \((\mathbf{e}^1, \mathbf{e}^2)\) CANNOT be Pareto Optimum
- \((\mathbf{x}^1, \mathbf{x}^2)\) may or may not be Pareto Optimum
Pareto Optimal Allocations II

**Definition**

Allocation \((x^1, x^2)\) is Pareto Optimum, if there is no (feasible) allocation \((y^1, y^2)\) such that \((y^1, y^2)\) is Pareto superior to \((x^1, x^2)\).

**Remark**

In general, there can be several Pareto Optimum allocations.
Barter in an Ideal World I

Question

*What is the best achievable outcome under Barter?*

assuming that

- individuals are free to trade/exchange
- but only if they wish to do so
- they have all the information needed for the trade.
Example

Consider the following two-person, two-goods economy:

- Endowments: $e^1 = (1, 9)$, and $e^2 = (9, 1)$
- Preferences: $u^i(x, y) = x \cdot y$. That is, $u^1(x^1_1 \cdot x^1_2) = x^1_1 \cdot x^1_2$ and $u^2(x^2_1 \cdot x^2_2) = x^2_1 \cdot x^2_2$
- Allocation: $x^1 = (3, 3)$, and $x^2 = (7, 7)$

Clearly, allocation $x = (x^1, x^2)$ is feasible. And, $u^1(x^1) \geq u^1(e^1)$, and $u^2(x^2) \geq u^2(e^2)$. In fact $u^2(x^2) > u^2(e^2)$.

That is, allocation $x = (x^1, x^2)$ is Pareto superior to $e = (e^1, e^2)$

In the above example, we say that: allocation $e = (e^1, e^2)$ is blocked by allocation $x = (x^1, x^2)$. 
Barter in an Ideal World III

Definition

For a two-person two-goods economy: Allocation \( x = (x^1, x^2) \) will block \( y = (y^1, y^2) \), if any of the following holds:

1. \( u^1(x^1) > u^1(y^1) \), and \( e^1 \geq x^1 \); or
2. \( u^2(x^2) > u^2(y^2) \) and \( e^2 \geq x^2 \); or
3. \( (x^1, x^2) \) is Pareto superior to \( (y^1, y^2) \), i.e.,

\[
\begin{align*}
    u^i(x^i) &\geq u^i(y^i) \text{ for } i = 1, 2. \text{ And} \\
    u^i(x^i) &> u^i(y^i)
\end{align*}
\]

holds for at least one \( i \).
Remark

In the above example, recall $x^1 = (3, 3)$ and $x^2 = (7, 7)$. You can verify that:

- there is no other feasible allocation $y = (y^1, y^2)$ for which the following hold (with at least one inequality)
  
  \[ u^1(y^1) \geq u^1(x^1), \quad \text{and} \quad u^2(y^2) \geq u^2(x^2). \]

- so, allocation $(x^1, x^2)$ cannot be blocked by any individual or both of them together.

Question

For the above example,

- **How many unblocked allocations are there?**
- **What is the set of possible outcomes?**
Blocking Coalition

Here is a general definition of Blocking Coalition.

Definition

Let $S \subseteq \{1, \ldots, N\}$. $S$ is called a blocking coalitions for $x = (x^1, x^2, \ldots, x^N)$ if there is some vector $y$ such that

$$
\sum_{i \in S} y^i_j = \sum_{i \in S} e^i_j \quad \text{for all } j = 1, \ldots, M
$$

$$
u^i(y^i) = u^i(y^i_1, \ldots, y^i_M) \geq u^i(x^i_1, \ldots, x^i_M) = u^i(x^i) \quad \text{for all } i \in S
$$

$$
u^i(y^i) = u^i(y^i_1, \ldots, y^i_M) > u^i(x^i_1, \ldots, x^i_M) = u^i(x^i) \quad \text{for some } i \in S
$$

Question

*What will be the outcome (equilibrium) in a barter economy, where individuals have all the information?*