CHAPTER 2

Axiomatic theory of bargaining with a fixed number of agents

2.1 Introduction

In this chapter we survey the axiomatic theory of bargaining for a fixed number of agents. Although Nash's paper has enjoyed great popularity since its publication, and the Nash solution has been used in numerous studies of actual conflict situations, the central role played by this solution was very seriously challenged in the mid-1970s by the introduction and the characterization of other solutions, notably the Kalai–Smorodinsky and Egalitarian solutions. Largely spurred by these developments, an explosion of contributions to the theory has occurred since. Here, we will describe the main aspects of the theory, with particular emphasis on results concerning the solutions that will be at the center of our own theory.

2.2 The main solutions

Although alternatives to the Nash solution were proposed soon after the publication of Nash's paper, it is fair to say that until the mid-1970s, the Nash solution was often seen by economists and game theorists as the main, if not the only, solution to the bargaining problem. This preeminence is explainable by the fact that Nash developed a natural strategic model yielding exactly the same outcomes at equilibrium, as well as by his elegant characterization. In spite of the lack of unanimous agreement on some of the specific axioms that he used (in fact, one of them, the independence axiom, was subjected to a significant amount of criticism early on), the appeal of the axiomatic methodology is such that the revival of the theory can be dated to the new characterizations developed in the mid-1970s.

In order to facilitate the transition to the rest of this book, in all of which the domain is Σ, we will present this survey on the domain Σ^n. This choice will also permit us to simplify some of the proofs.

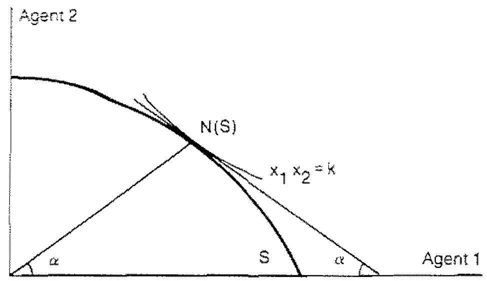

Figure 2.1. The Nash solution.

2.2.1 *The Nash solution*

Before presenting Nash's theorem in detail, we give a formal definition of the Nash solution on the domain Σ^n, and we illustrate the solution for $n=2$ (see Figure 2.1).

In what follows, e^n is the vector in $\mathbb{R}^n$ of coordinates all equal to 1.

> ***Definition.*** *The Nash solution N is defined by setting, for all $S \in \Sigma^n$, $N(S)$ equal to the maximizer in $x \in S$ of the "Nash product" $\prod x_i$.*

For $n=2$, the Nash solution can be described in two other ways. First, $N(S)$ is the undominated point x of S at which S has a line of support whose slope is equal to the negative of the slope of the line segment $[0,x]$. The point $N(S)$ is also the point of contact of the highest rectangular hyperbola touching S having the axes as asymptotes. (These geometric definitions easily extend to arbitrary n.)

Nash's theorem is based on the following axioms, abbreviated with lowercase letters. All of the axioms formulated in this chapter for solutions defined on Σ^n have counterparts for solutions defined on Σ, where population may vary. In order to keep clear the conceptual distinction between solutions defined on a fixed-population domain and solutions defined on a variable-population domain, we will abbreviate the latter with capital letters. (For example, the counterpart of *p.o*, imposed on F defined on Σ^n, is denoted *P.O.*)

Pareto-optimality (p.o): For all $S \in \Sigma^n$, for all $x \in \mathbb{R}^n$, if $x \geq F(S)$, then $x \notin S$.

A slightly weaker condition is

Weak Pareto-optimality (w.p.o): For all $S \in \Sigma^n$, for all $x \in \mathbb{R}^n$, if $x > F(S)$, then $x \notin S$.

According to *p.o*, the alternative that is selected should not be semi-strictly dominated by any feasible alternative; *w.p.o* requires only that no strict domination be possible.

Let $\Pi^n: \{1,\dots,n\} \to \{1,\dots,n\}$ be the *class of permutations of order n*. Given $\pi \in \Pi^n$, and $x \in \mathbb{R}^n$, let $\pi(x) \equiv (x_{\pi(1)},\dots,x_{\pi(n)})$. Also, given $S \subset \mathbb{R}^n$, let $\pi(S) \equiv \{x' \in \mathbb{R}^n \mid \exists x \in S \text{ with } x' = \pi(x)\}$.

Symmetry (sy): For all $S \in \Sigma^n$, if for all $\pi \in \Pi^n$, $\pi(S) = S$, then $F_i(S) = F_j(S)$ for all i, j. [Note that $\pi(S) \in \Sigma^n$.]

This says that if all agents are interchangeable in the geometric description of a problem, they should receive the same amount.

Let $\Lambda^n: \mathbb{R}^n \to \mathbb{R}^n$ be the class of *positive, independent person-by-person, and linear, transformations of order n*. Each $\lambda \in \Lambda^n$ is characterized by n positive numbers a_i such that given $x \in \mathbb{R}^n$, $\lambda(x) = (a_1 x_1,\dots,a_n x_n)$. Now, given $S \subset \mathbb{R}^n$, let $\lambda(S) \equiv \{x' \in \mathbb{R}^n \mid \exists x \in S \text{ with } x' = \lambda(x)\}$.

Scale invariance (s.inv): For all $S \in \Sigma^n$, for all $\lambda \in \Lambda^n$, $F(\lambda(S)) = \lambda(F(S))$. [Note that $\lambda(S) \in \Sigma^n$.]

In Nash's formulation, utilities are of the von Neumann–Morgenstern type, that is, they are invariant only up to arbitrary positive affine transformations; it is therefore natural to require of solutions that they be invariant under the same class of transformations. Each such transformation has an additive component and a multiplicative component. Since we have chosen $d = 0$, we do not have to require invariance under addition to the utilities of arbitrary constants. Instead, we only require invariance under multiplication of the utilities by arbitrary positive constants.

Independence of irrelevant alternatives (i.i.a): For all $S, S' \in \Sigma^n$, if $S' \subset S$ and $F(S) \in S'$, then $F(S') = F(S)$.

This says that if the solution outcome of a given problem remains feasible for a new problem obtained from it by contraction, then it should also be the solution outcome of this new problem.

Nash showed that, for $n = 2$, only one solution satisfies these four requirements. His result extends directly to arbitrary n.

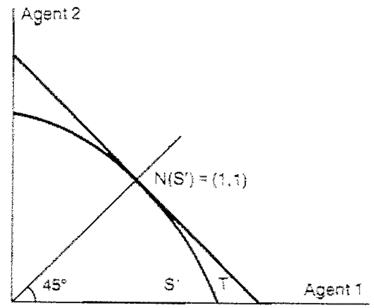

Figure 2.2. Theorem 2.1. Characterization of the Nash solution.

Theorem 2.1. A solution on Σ^n satisfies p.o, sy, s.inv, and i.i.a if and only if it is the Nash solution.

Proof. It is easy to see that N satisfies the four axioms. Conversely, to show that if a solution F on Σ^n satisfies the four axioms then $F = N$, let $S \in \Sigma^n$ be given. Note that $N(S) > 0$ (since there exists $x \in S$ with $\prod x_i > 0$ the maximum of $\prod x_i$ for $x \in S$ is also positive); therefore, the transformation $\lambda: \mathbb{R}^n \to \mathbb{R}^n$ associating with every $x \in \mathbb{R}^n$ the point $\lambda(x) \equiv (x_1/N_1(S),\dots,x_n/N_n(S))$ is a well-defined element of Λ^n. We have that $\lambda(N(S)) = (1,\dots,1) \equiv e^n$. Also, by definition of N, S is supported at $N(S)$ by the hyperplane of equation $\sum (x_i/N_i(S)) = n$. Therefore, $S' \equiv \lambda(S)$ is supported at $\lambda(N(S))$ by the hyperplane of equation $\sum x_i' = n$ (Figure 2.2). By *s.inv*, it suffices to show that $F(S') = N(S') = e^n$. To establish this, let $T \in \Sigma^n$ be defined by $T \equiv \{x' \in \mathbb{R}^n_+ \mid \sum x_i' \leqq n\}$. Note that $e^n \in PO(S) \equiv \{x \in S \mid \text{if } x' \geq x, \text{ then } x' \notin S\}$ and that T is invariant under all exchanges of agents. Therefore, by *p.o* and *sy*, $F(T) = e^n$. Also, $S' \subset T$ and $F(T) \in S'$. By *i.i.a*, $F(S') = F(T) = e^n$. Q.E.D.

This theorem constitutes the foundation of the axiomatic theory of bargaining. It shows that a *unique* point can be identified for each problem, representing an equitable compromise. In the mid-1970s, Nash's result became the object of a considerable amount of renewed attention, and the role played by each axiom in the characterization was scrutinized by several authors. Some of the axioms were shown to be of marginal importance in that their removal made admissible only very few additional solutions whereas the removal of the others made admissible unmanageably large families of solutions. A variety of other axioms were then formulated and substituted for Nash's axioms, and other appealing lists of axioms were shown to characterize other solutions. We devote the following pages to an account of these developments and, in particular, we discuss four other solutions that will be fundamental to the theory that we exposit later: the Kalai–Smorodinsky, Egalitarian, Leximin, and Utilitarian solutions.

We start with Roth's early contribution concerning the role of Pareto-optimality in Theorem 2.1. Although *p.o* is probably the most easily acceptable condition when solutions are meant to represent an arbitrator's choice, to the extent that the theory is alternatively intended to describe how agents reach compromises on their own, it is desirable that it be able to explain nonoptimal compromises, which are often observed in practice. But if optimality is violated, how is it violated? Can the theory help predict the sort of violations that will occur? This is the question addressed by Roth (1977a). The statement of the conclusion he reached requires the formulation of one more axiom.

Strong individual rationality (st.i.r): For all $S \in \Sigma^n$, $F(S) > 0$.

This says that all agents should strictly gain from the agreement. Note that on our domain, the weaker condition $F(S) \geqq 0$ is automatically satisfied since $S \subset \mathbb{R}^n_+$. (This is one of the disadvantages of our choice of domains. It obscures the significance of the individual rationality axioms. However, the requirement is natural, and the loss of generality due to this choice is limited.)

Theorem 2.2. A solution on Σ^n satisfies st.i.r, sy, s.inv, and i.i.a if and only if it is the Nash solution.

Proof. We have already pointed out that N satisfies *sy, s.inv,* and *i.i.a,* and the fact that N satisfies *st.i.r* was noted and used in the proof of Theorem 2.1. Conversely, let F be a solution on Σ^n satisfying the four axioms. The desired conclusion, that $F = N$, will follow from Theorem 2.1 and the fact that *s.inv, i.i.a,* and *st.i.r* together (*sy* is not needed) imply *p.o,* as shown next: Let $S \in \Sigma^n$ be given and suppose, by way of contradiction, that $F(S) \notin \text{PO}(S)$, that is, that there exists $x \in S$ with $x \geq F(S)$. Without loss of generality, suppose that $x_1 > F_1(S)$. Let $\lambda: \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\lambda(x') = ((F_1(S)/x_1)x'_1, x'_2, \ldots, x'_n)$. Note that $\lambda \in \Lambda^n$ since $F_1(S) > 0$, by *st.i.r,* and since $x_1 > 0$. Therefore, $S' \equiv \lambda(S) \in \Sigma^n$. We have $\lambda(x) = F(S)$. Therefore, $F(S) \in S'$. Also, $S' \subset S$. By *i.i.a,* $F(S') = F(S)$. Also, by *s.inv,* $F(S') = \lambda(F(S)) \neq F(S)$ since $F_1(S) > 0$. This contradiction between our last two conclusions establishes the claim. Q.E.D.

A straightforward corollary of Theorem 2.2 is that if *st.i.r* is not used at all, only one additional solution becomes admissible.

Definition. *The* ***Disagreement solution 0*** *is defined by setting, for all $S \in \Sigma^n$, $0(S) \equiv 0$.*

Corollary 2.1. A solution on Σ^n satisfies sy, s.inv, and i.i.a if and only if it is the Nash solution or it is the Disagreement solution.

If *sy* is dropped from the list of axioms of Theorem 2.1, a somewhat wider but still small family of additional solutions become admissible. Let Δ^{n-1} be the $(n-1)$-dimensional simplex.

Definition. *Given $\alpha \in \text{rel. int. } \Delta^{n-1}$,* ***the asymmetric Nash solution with weights*** *α, N^α, is defined by setting, for all $S \in \Sigma^n$, $N^\alpha(S) \equiv$ argmax $\prod x_i^{\alpha_i}$ for $x \in S$.*

These solutions were introduced by Harsanyi and Selten (1972).

Theorem 2.3. A solution on Σ^n satisfies st.i.r, s.inv, and i.i.a if and only if it is an asymmetric Nash solution.

Proof. It is easy to see that all N^α satisfy the three properties. The proof of the converse is very similar to that of Theorem 2.2. Given a solution F on Σ^n satisfying the three axioms, we first note, as in Theorem 2.2, that F in fact satisfies *p.o*. Let $\alpha \equiv F(\text{cch}\{\Delta^{n-1}\})$. By *p.o* and *st.i.r,* $\alpha \in \text{rel. int. } \Delta^{n-1}$. Given $S \in \Sigma^n$, let $\lambda \in \Lambda^n$ be such that $\lambda(S)$ be supported at α by Δ^{n-1}. This λ exists uniquely. Then the proof concludes as in Theorem 2.2. Q.E.D.

If *st.i.r* is not used, a few other solutions become available.

Definition. *Given $i \in \{1, \ldots, n\}$,* ***the ith Dictatorial solution D^i*** *is defined by setting, for all $S \in \Sigma^n$, $D^i(S)$ equal to the maximal point of S in the direction of the ith unit vector.*

Note that all D^i satisfy *s.inv* and *i.i.a,* but only *w.p.o* (instead of *p.o*). To recover full optimality, one may proceed as follows. First, select an ordering π of the n agents. Then given $S \in \Sigma^n$, pick $D^{\pi(1)}(S)$ if this point belongs to $\text{PO}(S)$; otherwise, among the points whose $\pi(1)$th coordinate is equal to $D^{\pi(1)}_{\pi(1)}(S)$, find the maximal point in the direction of the unit vector pertaining to agent $\pi(2)$. Pick this point if it belongs to $\text{PO}(S)$;

otherwise, repeat the operation with $\pi(3)$.... This algorithm is summarized in the following definition.

> ***Definition.*** *Given an ordering π of $\{1,\dots,n\}$, the* ***Lexicographic Dictatorial solution relative to*** *π, D^π, is defined by setting, for all $S\in\Sigma^n$, $D^\pi(S)$ to be the lexicographic maximizer over $x\in S$ of $x_{\pi(1)}, x_{\pi(2)},\dots,x_{\pi(n)}$.*

All of these solutions satisfy *p.o, s.inv,* and *i.i.a,* and there are no others if $n=2$. For them, the violations of symmetry are in a sense "maximal." The characterization of all solutions satisfying *w.p.o* (or *p.o*), *s.inv,* and *i.i.a* was accomplished by Peters, Tijs, and de Koster (1983) for $n=2$ and Peters (1983) for arbitrary n. (If $n>2$, other solutions exist defined by lexicographic maximization of certain Nash products involving subgroups of the agents.)

Omitting either *s.inv* or *i.i.a* from Theorem 2.1 makes admissible very large classes of solutions, some of which will be discussed in what follows. Roth (1977b) reformulated *i.i.a* by replacing the hypothesis that the problems have the same disagreement point (this hypothesis is part of the formulation of *i.i.a* for solutions defined on $\tilde\Sigma^n$) by the hypothesis that they have the same "ideal point" (this point enters in a fundamental way in the definition of the Kalai-Smorodinsky solution, studied next). He showed the incompatibility of this version of *i.i.a* with the other three requirements of Theorem 2.1. However, if $n=2$ and the hypothesis of equal disagreement points is replaced by the hypothesis of equal points of "minimal expectations" [the point whose ith coordinate is $D_j^j(S)$ where $j\neq i$], then the resulting axiom together with the other three axioms of Theorem 2.1 characterize a Nashlike solution defined by maximizing in the feasible set the product of utility gains not from the origin but from this point of minimal expectations. Thomson (1981a) proposes other choices of such "reference points" and similarly characterizes corresponding variants of the Nash solution.

2.2.2 *The Kalai-Smorodinsky solution*

A new impetus was given to the axiomatic theory of bargaining when Kalai and Smorodinsky (1975) provided a characterization of the following solution, illustrated in Figure 2.3.

> ***Definition.*** *The* ***Kalai-Smorodinsky solution K*** *is defined by setting, for all $S\in\Sigma^n$, $K(S)$ to be the maximal point of S on the segment connecting the origin to $a(S)$, the* ***ideal point*** *of S, defined by $a_i(S)\equiv\max\{x_i\mid x\in S\}$ for each i.*

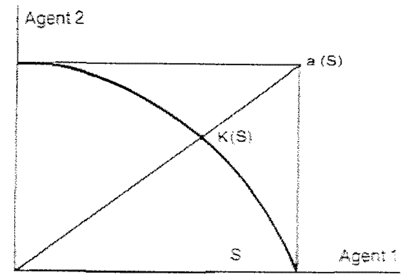

Figure 2.3. The Kalai-Smorodinsky solution.

This solution has been mainly studied for $n=2$, a case in which it satisfies a greater number of appealing properties than for arbitrary n. Consequently, we will limit our attention to that case in the next few paragraphs. An important distinguishing feature between the Nash solution and the Kalai-Smorodinsky solution is that the latter responds much more satisfactorily to expansions and contractions of the feasible set. In particular, it satisfies the following axiom.

Individual monotonicity (i.mon): For all $S,S'\in\Sigma^2$, for all i, if $a_j(S)=a_j(S')$ and $S'\supset S$, then $F_i(S')\geqq F_i(S)$.

For each utility level attainable by agent j, the maximal utility level achievable by agent i increases, whereas the range of utility levels attainable by agent j remains the same. It therefore seems natural to require that agent i not be negatively affected by the expansion. It is on this property that the characterization offered by Kalai and Smorodinsky mainly rests.

> ***Theorem 2.4.*** *A solution on Σ^2 satisfies p.o, sy, s.inv, and i.mon if and only if it is the Kalai-Smorodinsky solution.*

Proof. It is easily verified that K satisfies the four properties. Conversely, let F be a solution on Σ^2 satisfying the four properties and $S\in\Sigma^2$ be given. Let $\lambda\in\Lambda^2$ be such that $\lambda(a(S))=(1,1)$ and $S'\equiv\lambda(S)$ (see Figure 2.4). Note that $\lambda(K(S))=K(S')\equiv x$ is a point of equal coordinates. Let $T\equiv$ cch$\{(1,0),x,(0,1)\}$. Note that T is a symmetric element of Σ^2. Therefore,

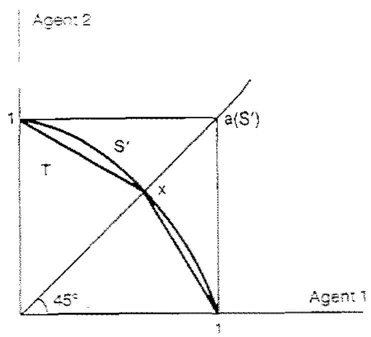

Figure 2.4. Theorem 2.4. Characterization of the Kalai-Smorodinsky solution.

by *p.o* and *sy*, $F(T)=x=K(T)$. Also, $T\subset S'$, $a_1(T)=a_1(S')$ and $a_2(T)=a_2(S')$. By *i.mon* applied twice, $F(S')\geqq F(T)=x$. Because $x\in PO(S')$, $F(S')=x$ and the desired conclusion follows by *s.inv*. Q.E.D.

By deleting *p.o* from the axioms of Theorem 2.4, a one-parameter family of solutions obtained by scaling down $K(S)$ by some number $\lambda\in[0,1]$ becomes admissible. Note that the locus of $K(S)$ as S varies in Σ^2 subject to the condition $a(S)=(1,1)$ is the segment $[(\frac{1}{2},\frac{1}{2}),(1,1)]$. More generally, given any monotone path with one endpoint on the segment $[(1,0),(0,1)]$ and the other $(1,1)$, let the solution outcome of any S normalized so that $a(S)=(1,1)$ be the intersection of the path with $PO(S)$, and that of an arbitrary S be obtained by an application of *s.inv*. Any solution constructed in this way satisfies all the axioms of Theorem 2.4 except *sy*. Peters and Tijs (1984, 1985) show that there are no others.

The removal of either *s.inv* or *i.mon* permits many additional solutions. In particular, if *s.inv* is dropped, *i.mon* can be considerably strengthened, as we will see in our discussion of the Egalitarian solution. Salonen (1985) proposed a slight reformulation of *i.mon*, which leads to a characterization of a variant of the Kalai-Smorodinsky solution.

Although the extension of the definition of the Kalai-Smorodinsky solution to the n-person case itself causes no problem, the generalization of the preceding results to the n-person case is not as straightforward as was the case of the extensions of the results concerning the Nash solution from $n=2$ to arbitrary n. First of all, for $n>2$, the n-person Kalai-Smorodinsky solution satisfies *w.p.o* only. (As noted by Roth, 1979d, if comprehensiveness of the feasible sets were not assumed, the solution could even select the origin.) This is not a serious limitation since, for most problems S, $K(S)$ in fact is (fully) Pareto optimal. In addition, Pareto-optimality can be recovered by a lexicographic operation that we will more extensively discuss in connection with the Egalitarian solution. Imai (1983), who proposed this extension, showed that the resulting solution could also be characterized by a suitable reformulation of the axioms of Theorem 2.4 together with the addition of a weak version of the independence axiom.

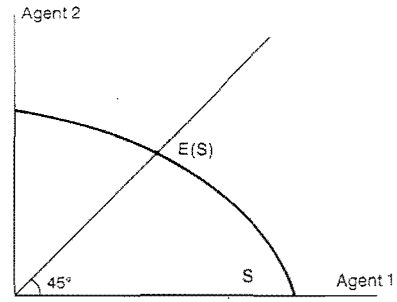

Figure 2.5. The Egalitarian solution.

The weakening of *p.o* to *w.p.o* is not the only change that has to be made in the axioms of Theorem 2.4 to extend the characterization of the Kalai-Smorodinsky solution to the case $n>2$. Indeed, several natural ways exist of generalizing the individual monotonicity condition, as discussed by Segal (1980) and Thomson (1980). One extension that will work for that purpose is: For all $S,S'\in\Sigma^n$, for all i, if $S\subset S'$ and $a_j(S)=a_j(S')$ for all $j\neq i$, then $F_i(S')\geqq F_i(S)$. This axiom is however less appealing than the two-person version since the range of utility vectors attainable by the agents different from i has not been left unaffected by the expansion.

2.2.3 *The Egalitarian solution*

We now turn to a third solution, whose main distinguishing feature from the previous two is that it involves interpersonal comparisons of utility.

> ***Definition.*** *The Egalitarian solution E is defined by setting, for all $S\in\Sigma^n$, $E(S)$ to be the maximal point of S of equal coordinates.*

The definition is illustrated in Figure 2.5.

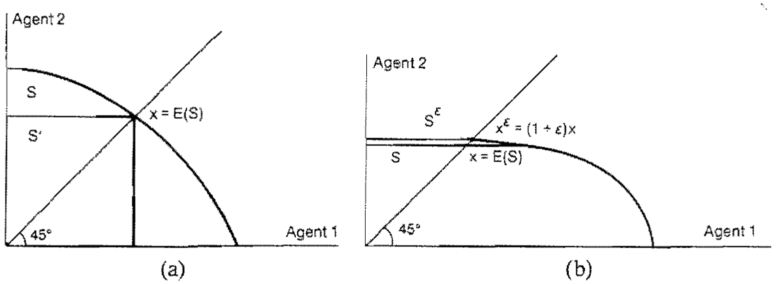

Figure 2.6. Theorem 2.5. Characterization of the Egalitarian solution. (a) $E(S) \in \mathrm{PO}(S)$; (b) $E(S) \in \mathrm{WPO}(S) \setminus \mathrm{PO}(S)$.

The most striking feature of this solution is that it satisfies the following monotonicity condition, which is very strong, since no restrictions are imposed in its hypotheses on the sort of expansions that take S into S'. In fact, this axiom can serve to provide an easy characterization of the solution.

Strong monotonicity (st.mon): For all $S, S' \in \Sigma^n$, if $S \subset S'$, then $F(S) \leqq F(S')$.

The following characterization result is a variant of a theorem due to Kalai (1977b).

Theorem 2.5. A solution on Σ^n satisfies w.p.o, sy, and st.mon if and only if it is the Egalitarian solution.

Proof. It is easily verified that E satisfies the three properties. Conversely, let F be a solution on Σ^n satisfying the three properties. Given any symmetric S, it follows from *w.p.o* and *sy* that $F(S) = E(S)$. Given any other $S \in \Sigma^n$, let $x \equiv E(S)$ and $S' \equiv \mathrm{cch}\{x\}$ (Figure 2.6a). By the previous step, $F(S') = E(S')$, and since $S' \subset S$, it follows from *st.mon* that (i) $F(S) \geqq F(S') = E(S') = E(S) = x$. We are done if $E(S) \in \mathrm{PO}(S)$. If not, for each $\epsilon > 0$, let $x^\epsilon \equiv (1+\epsilon)x$ and $S^\epsilon \in \Sigma^n$ be defined by $S^\epsilon \equiv \mathrm{cch}\{S, x^\epsilon\}$ (Figure 2.6b). Note that $x^\epsilon = E(S^\epsilon) \in \mathrm{PO}(S^\epsilon)$ so that, by the preceding argument, $F(S^\epsilon) = x^\epsilon$. Also $S^\epsilon \supset S$, so that by *st.mon*, $F(S^\epsilon) \geqq F(S)$. Since $x^\epsilon \to x$ as $\epsilon \to 0$, (ii) $F(S) \leqq x$. The desired conclusion follows from (i) and (ii).
Q.E.D.

Without *w.p.o*, the one-parameter family of *solutions of proportional character* introduced by Roth (1979a) become admissible. This family

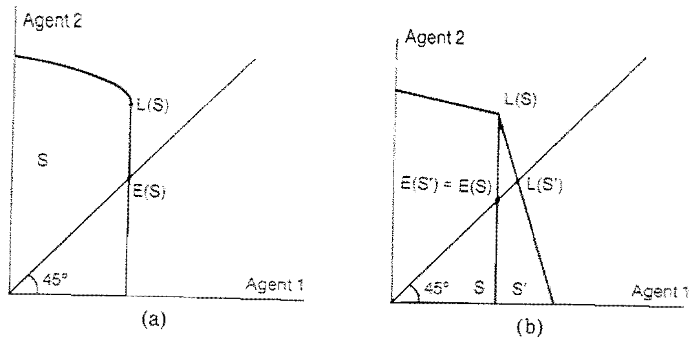

Figure 2.7. The Leximin solution L. (a) Definition of L; (b) L does not satisfy *st.mon*.

includes scaled-down versions of the Egalitarian solution obtained by choosing $\lambda \in [0,1]$ and setting the solution outcome of S equal to $\lambda E(S)$.

Conversely, one might wonder about the price of strengthening *w.p.o* to *p.o*. A lexicographic extension of the Egalitarian solution, which coincides with it on the class of strictly comprehensive problems, can be defined as follows:

Given $x \in \mathbb{R}^n$, let $\gamma(x) \in \mathbb{R}^n$ be obtained by rewriting the coordinates of x in increasing order: $\gamma_1(x) \leqq \gamma_2(x) \leqq \cdots \leqq \gamma_n(x)$. Given $x, y \in \mathbb{R}^n$, say that x is lexicographically greater than y, written as $x >_L y$, if $\gamma_1(x) > \gamma_1(y)$ or $[\gamma_1(x) = \gamma_1(y)$ and $\gamma_2(x) > \gamma_2(y)]$ or more generally if [for some $k \leqq n$, $\gamma_i(x) = \gamma_i(y)$ for all $i < k$ and $\gamma_k(x) > \gamma_k(y)$].

Definition. *The Leximin solution L is defined by setting, for all $S \in \Sigma^n$, $L(S)$ equal to the maximizer over S of $\succsim_L$.*

The definition is illustrated in Figure 2.7a for $n = 2$.

It is straightforward to verify that this solution is well defined and satisfies *p.o* and *sy*. However, it does not satisfy *st.mon* as illustrated by the example of Figure 2.7b: There $S \subset S'$ and yet $L_2(S') < L_2(S)$.

The properties of a normalized version of this solution were extensively studied by Imai (1983).

The family of Monotone Path solutions, defined next, is obtained by dropping *sy* from Theorem 2.5.

Definition. *Given a continuous, unbounded, and monotone path in $\mathbb{R}^n$ starting at the origin, the Monotone Path Solution relative* *to G, E^G is defined by setting, for all $S \in \Sigma^n$, $E^G(S)$ equal to the maximal point of S on that path.*

Any solution defined in this way satisfies *w.p.o* and *st.mon*. For the solution to be continuous, the path should be strictly increasing, except perhaps initially. These solutions are discussed in Myerson (1977) and Thomson and Myerson (1980). See also Kalai (1977b).

Theorem 2.5 shows that *st.mon* is a very strong condition indeed (in fact, Luce and Raiffa (1957) had long ago noted the incompatibility of *st.mon* and *w.p.o* on domains of nonnecessarily comprehensive problems). However, other conditions can be substituted for it in Theorem 2.5 without affecting the conclusion. For instance, Kalai (1977b) proposed the following condition.

Decomposability (dec): For all $S, S', S'' \in \Sigma^n$, if $S'' = \{x \in \mathbb{R}^n_+ \mid \exists x' \in S' \text{ s.t. } x' = x + F(S)\}$, then $F(S') = F(S) + F(S'')$.

This axiom says that the solution outcome of an expanded problem can be indifferently obtained directly or in stages by first computing the solution outcome of the initial problem and then solving the problem derived from the expanded problem by taking as disagreement point this initial solution outcome. It follows directly from Kalai that *w.p.o, sy,* and *dec* still characterize the Egalitarian solution. Other characteristics of the Egalitarian solution have been obtained by Myerson (1981) and Peters (1986).

2.2.4 *The Utilitarian solution*

We close this review with a short discussion of the Utilitarian solution. This solution has played a fundamental role in the theory of social choice in general but a marginal role in bargaining theory because it has the serious disadvantage of being independent of the disagreement point (this fact is the second one to be obscured by our choice of domain Σ^n). In spite of this limitation, we often refer to the Utilitarian solution for the purpose of comparison and because it is a limit case, permitting the most utility substitution.

The Utilitarian solution is obtained by maximizing the sum of utilities over the feasible set. Since we required solutions to associate with each problem a single point, we should specify what to do in case the sum of utilities is maximized at more than one point (having to include a tie-breaking rule in its definition is a second drawback of the solution). In the two-person case, selecting the midpoint of the segment of maximizers may be a natural choice. However, there is no equally natural choice for

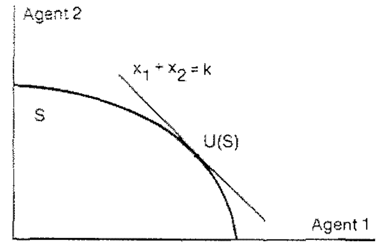

Figure 2.8. The Utilitarian solution.

more than two persons. Because of the various ways of making selections, we will sometimes use the phrase *utilitarian solutions*. (Another way to obtain a single point is of course to restrict the domain to problems with a strictly convex Pareto-optimal boundary.)

Definition. *A **Utilitarian solution** U is defined by choosing, for each $S \in \Sigma^n$, $U(S)$ among the maximizers of $\sum x_i$ for $x \in S$.*

The definition is illustrated in Figure 2.8.

Obviously, all Utilitarian solutions satisfy *p.o*. They also satisfy *sy* if appropriate selections are made. However, no Utilitarian solution satisfies *s.inv* (even on the class of problems with a strictly convex Pareto-optimal boundary). Also, no Utilitarian solution satisfies *cont* or *i.i.a* because of the impossibility of performing appropriate selections.

The Utilitarian solution has been characterized by Myerson (1981) (see also Thomson, 1981b, c).

2.3 Other solutions

Other solutions have been discussed in the literature by Raiffa (1953), Luce and Raiffa (1957), and Perles and Maschler (1981). These solutions will not play a role in our exposition of the theory of bargaining with a variable number of agents.