Microeconomic Theory: Lecture 3
Production, Costs and the Firm

Parikshit Ghosh

Delhi School of Economics

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The Firm

- Often a very large organization with thousands of workers.
- Starting assumption: objective is to maximize profits.
- Obvious exceptions: public sector organizations, non-profits, vanity projects (sports teams).
- Inside the firm: a command economy. Outside the firm: a market economy. What determines the boundary (Coase)?
- The joint stock company: separation of ownership and management/labour. Gives rise to agency problems: do managers have the incentive to maximize profits?
- Dynamic and strategic issues: there may be trade-offs between profit maximization in the short run and the long run.

The Production Function

- The firm produces one output (y) using n inputs $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.
- The input-output relationship is captured in the production function: $y = f(\mathbf{x})$, where $f(.)$ is continuous, strictly increasing and (strictly) quasiconcave, with $f(\mathbf{0}) = 0$.
- An isoquant is a set of input vectors that produce the same output:
$$Q(y) = \{\mathbf{x} \geq \mathbf{0} | f(\mathbf{x}) = y\}$$
- Monotonicity and quasiconcavity of $f(.)$ implies isoquants are convex to the origin and higher isoquants represent higher output.

The Production Function: Characteristics

- Returns to scale:
  - Constant Returns to Scale (CRS) if $f(\lambda \mathbf{x}) = \lambda f(\mathbf{x})$.
  - Decreasing Returns to Scale (DRS) if $f(\lambda \mathbf{x}) < \lambda f(\mathbf{x})$.
  - Increasing Returns to Scale (IRS) if $f(\lambda \mathbf{x}) > \lambda f(\mathbf{x})$.
- The production function is homogeneous of degree k if

$$f(\lambda \mathbf{x}) = \lambda^k f(\mathbf{x}) \quad \text{for any } \mathbf{x} \geq \mathbf{0}$$

- There is CRS, DRS, IRS if $k =, <, > 1$.

Profit Maximization

- A **perfectly competitive market** is a market where there are a large number of buyers and sellers. Each agent takes the prices as given, and assumes he will be able to buy/sell any quantity he wants at these prices.
- The firm faces some input price p and a vector of output prices, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$.
- The profit maximization problem:

$$\max_{y,\mathbf{x}} py - \mathbf{wx} \quad \text{subject to } y \leq f(\mathbf{x})$$

- Becomes an unconstrained problem after incorporating the (binding) constraint into the objective function:

$$\max_{\mathbf{x}} pf(\mathbf{x}) - \mathbf{wx}$$

Two-Step Solutions: the Cost Function

- Break up the problem into two parts.
  - Find the least costly way of producing any output level y:

$$c(\mathbf{w}, y) = \min_{\mathbf{x}} \mathbf{wx} \quad \text{subject to } f(\mathbf{x}) = y$$

  - Using this information, find the most profitable output level:

$$\max_{y} py - c(\mathbf{w}, y)$$

- The cost min problem is the dual of the consumer's problem.
- The **cost function** is the expenditure function.
- The **conditional input demand functions**, $\mathbf{x}(\mathbf{w}, y)$, are Hicksian demand functions.

Cost Functions of Homogeneous Production Functions

Theorem

Suppose $f(\mathbf{x})$ is homogeneous of degree k. Then the cost and conditional input demand functions are multiplicatively separable in y and $\mathbf{w}$, and are given by

$$\begin{aligned} c(\mathbf{w}, y) &= c(\mathbf{w}, 1).y^{\frac{1}{k}} \\ \mathbf{x}(\mathbf{w}, y) &= \mathbf{x}(\mathbf{w}, 1).y^{\frac{1}{k}} \end{aligned}$$

- The cost function is linear/convex/concave if returns to scale is constant/decreasing/increasing.
- Marginal cost is constant/increasing/decreasing if the cost function is linear/convex/concave.

Proof of the Theorem

- The cost function can be rewritten as:

$$\begin{aligned}
c(\mathbf{w}, y) &= \min_{\mathbf{x}} \mathbf{wx} \;\; \text{subject to } f(\mathbf{x}) = y \\
&= y^{\frac{1}{k}} \min_{\mathbf{x}} \mathbf{w}\left(y^{-\frac{1}{k}}\mathbf{x}\right) \;\; \text{subject to } y^{-1} f(\mathbf{x}) = 1 \\
&= y^{\frac{1}{k}} \min_{\mathbf{x}} \mathbf{w}\left(y^{-\frac{1}{k}}\mathbf{x}\right) \;\; \text{subject to } f\left(y^{-\frac{1}{k}}\mathbf{x}\right) = 1 \\
&= y^{\frac{1}{k}} \min_{\mathbf{z}} \mathbf{wz} \;\; \text{subject to } f(\mathbf{z}) = 1 \\
&= c(\mathbf{w}, 1).y^{\frac{1}{k}}
\end{aligned}$$

- Similar proof for conditional input demand functions.

Returns to Scale and Competition

- First-order condition:

$$p = \frac{\partial c(\mathbf{w}, y^*)}{\partial y}$$

- Second-order condition:

$$\frac{\partial^2 c(\mathbf{w}, y^*)}{\partial y^2} \geq 0$$

- For IRS technology, the second-order condition cannot be satisfied anywhere! y^* is either 0 or ∞. Not compatible with perfect competition. IRS typically leads to natural monopolies.
- For CRS technology, the optimum is $0, [0, \infty], \infty$ when $p <, =, > c(\mathbf{w}, 1)$. Optimum output can be indeterminate.

Returns to Scale and Competition

- The **profit function** of the firm is the value function of the profit-max problem:

$$\pi(p, \mathbf{w}) = \max_{\mathbf{x}} pf(\mathbf{x}) - \mathbf{wx}$$

- First-order condition:

$$\underbrace{p\frac{\partial f(\mathbf{x}^*)}{\partial x_i}}_{\text{Marginal revenue product}} = \underbrace{w_i}_{\text{price of input}}$$

- Second order condition: the Hessian matrix of $f(.)$ must be negative semi-definite (i.e. locally concave) at $\mathbf{x}^*$.
- The choice functions $\mathbf{x}(p, \mathbf{w})$ are the (unconditional) **input demand functions**. $y(p, \mathbf{w})$ = **supply function**.

Properties of the Profit Function

- Increasing in p (higher profit for *every* input choice).
- Decreasing in w_i (lower profit for every input choice).
- Homogeneous of degree 1 in $(p, \mathbf{w})$ (when input and output prices are scaled up, relative profits remain unchanged).
- Convex in $(p, \mathbf{w})$.
- Hotelling's Lemma (using envelope theorem):

$$\begin{aligned} \frac{\partial \pi(p, \mathbf{w})}{\partial p} &= y(p, \mathbf{w}) \\ -\frac{\partial \pi(p, \mathbf{w})}{\partial w_i} &= x_i(p, \mathbf{w}) \end{aligned}$$

Convexity of the Profit Function: Proof

- Suppose optimal input-output choices are
  - $(y^1, \mathbf{x}^1)$ at prices $(p^1, \mathbf{w}^1)$.
  - $(y^2, \mathbf{x}^2)$ at prices $(p^2, \mathbf{w}^2)$.
  - $(\overline{y}, \overline{\mathbf{x}})$ at prices $(\overline{p}, \overline{\mathbf{w}})$, where $(\overline{p}, \overline{\mathbf{w}}) = \lambda(p^1, \mathbf{w}^1) + (1-\lambda)(p^2, \mathbf{w}^2)$.
- By definition of profit maximization:

$$p^1\mathbf{x}^1 - \mathbf{w}^1\mathbf{x}^1 \geq p^1\overline{\mathbf{x}} - \mathbf{w}^1\overline{\mathbf{x}}$$

$$p^2\mathbf{x}^2 - \mathbf{w}^2\mathbf{x}^2 \geq p^2\overline{\mathbf{x}} - \mathbf{w}^2\overline{\mathbf{x}}$$

- Taking weighted averages:

$$\lambda\pi(p^1, \mathbf{w}^1) + (1-\lambda)\pi(p^2, \mathbf{w}^2) \geq \pi(\overline{p}, \overline{\mathbf{w}})$$

Implications of Convex Profit Function

- The Hessian matrix of $\pi(p, \mathbf{w})$ is symmetric, positive semi-definite:

$$H = \begin{bmatrix} \frac{\partial^2 \pi}{\partial p^2} & \frac{\partial^2 \pi}{\partial p \partial w_1} \cdots & \frac{\partial^2 \pi}{\partial p \partial w_n} \\ \frac{\partial^2 \pi}{\partial w_1 \partial p} & \frac{\partial^2 \pi}{\partial w_1^2} \cdots & \frac{\partial^2 \pi}{\partial w_1 \partial w_n} \\ \vdots & & \vdots \\ \frac{\partial^2 \pi}{\partial w_n \partial p} & \frac{\partial^2 \pi}{\partial w_n \partial w_1} \cdots & \frac{\partial^2 \pi}{\partial w_n^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial y}{\partial p} & \frac{\partial y}{\partial w_1} \cdots & \frac{\partial y}{\partial w_n} \\ -\frac{\partial x_1}{\partial p} & -\frac{\partial x_1}{\partial w_1} \cdots & -\frac{\partial x_1}{\partial w_n} \\ \vdots & & \vdots \\ -\frac{\partial x_n}{\partial p} & -\frac{\partial x_n}{\partial w_1} \cdots & -\frac{\partial x_n}{\partial w_n} \end{bmatrix}$$

- All principal minors are non-negative $\Rightarrow$ diagonal elements are non-negative.
- The supply function $y(p, \mathbf{w})$ is increasing in output price: $\frac{\partial y(p,\mathbf{w})}{\partial p} \geq 0$.
- The input demand functions, $\mathbf{x}(p, \mathbf{w})$, are decreasing in own price, $\frac{\partial x_i(p,\mathbf{w})}{\partial w_i} \leq 0$.