Microeconomic Theory I: Choice Under Uncertainty

Parikshit Ghosh

Delhi School of Economics

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Lotteries

- Set of outcomes: $\{a_1, a_2, \ldots, a_n\}$.
- A gamble/lottery is a probability distribution over outcomes: $g = (p_1 \circ a_1, p_2 \circ a_2, \ldots, p_n \circ a_n)$.
- p_i is the probabaility of outcome i.
- Sure outcomes: $(0 \circ a_1, \ldots, 1 \circ a_i, \ldots, 0 \circ a_n) = a_i$.
- Compound lotteries are probability distributions over lotteries: $(q_1 \circ g_1, q_2 \circ g_2, \ldots, q_m \circ g_m)$.
- (S_G) S is the set of all (simple) lotteries.
- $\succsim$ is a preference relation defined over S.

The von Neumann-Morgenstern Axioms

- **Axiom 1 (Completeness):** For all $g, g' \in G$, either $g \succsim g'$ or $g' \succsim g$ (or both).
- **Axiom 2 (Transitivity):** For all $g, g', g'' \in G$, if $g \succsim g'$ and $g' \succsim g''$, then $g \succsim g''$.
- **Axiom 3 (Continuity):** For any $g \in G$, there exists $\alpha \in [0, 1]$ such that

$$g \sim (\alpha \circ a_1, \ldots, (1 - \alpha) \circ a_n)$$

The von Neumann-Morgenstern Axioms

- **Axiom 4 (Monotonicity):** For any $\alpha, \beta \in [0,1]$

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n) \quad \text{iff} \quad \alpha \geq \beta$$

- **Axiom 5 (Substitution/Independence):** If $g = (p_1 \circ g_1, \ldots, p_k \circ g_k)$, $h = (p_1 \circ h_1, \ldots, p_k \circ h_k)$ and $g_i \sim h_i$ for all $i = 1, 2, \ldots, k$, then $g \sim h$.

- **Axiom 6 (Reduction to simple lotteries):** For any $g \in G$, if $g_S \in G_S$ is the simple lottery induced by g, then $g \sim g_S$.

The Expected Utility Theorem

Theorem

Suppose $\succsim$ satisfies Axioms 1 through 6. Then there exists a function $u : G \to \mathbb{R}$ such that $u(.)$

$$\text{(i) represents } \succsim \text{, i.e. } g \succsim g' \Leftrightarrow u(g) \geq u(g')$$

$$\text{(ii) has the exp utility prop, i.e. } u(g) = \sum_{i=1}^{n} p_i u(a_i)$$

- The probabilities p_i are assumed to be objective (e.g. playing roulette), not subjectively assessed (e.g. stock price).
- Savage extended the theory to subjective probabilities.
- The value of a lottery is linear in the probabilities of outcomes.

Proof: Representation

- Proof by construction: define $u(g) \in [0,1]$ such that

$$g \sim (u(g) \circ a_1, (1-u(g)) \circ a_n) \quad \text{(continuity)}$$

- Representation: $g \succsim g' \Leftrightarrow$

$$(u(g) \circ a_1, (1-u(g)) \circ a_n) \quad \succsim \quad (u(g') \circ a_1, (1-u(g')) \circ a_n)$$
$$\text{(transitivity)}$$

$$\Leftrightarrow u(g) \geq u(g') \quad \text{(monotonicity)}$$

Proof: Expected Utility Property

- Expected utility property:

$$a_i \sim (u(a_i) \circ a_1, (1 - u(a_i)) \circ a_n) \equiv q_i$$

- Then

$$\begin{aligned} g &\sim (p_1 \circ q_1, p_2 \circ q_2, \ldots, p_n \circ q_n) \quad \text{(substitution)} \\ &\sim \left(\left(\sum_{i=1}^{n} p_i u(a_i) \right) \circ a_1, \left(1 - \sum_{i=1}^{n} p_i u(a_i) \right) \circ a_n \right) \quad \text{(axiom 6)} \end{aligned}$$

- By monotonicity

$$u(g) = \sum_{i=1}^{n} p_i u(a_i)$$

Invariance to Positive Affine Transformations

Theorem

Suppose the VNM function $u(.)$ represents $\succsim$ over G. Then the VNM function $v(.)$ represents $\succsim$ if and only if there exist real numbers α and $\beta > 0$ such that

$$v(g) = \alpha + \beta u(g) \quad \textit{for all } g \in G$$

- As in choice under certainty, there is no unique function that represents preferences.
- Representation is more restrictive: only positive linear transformations preserve preference.

Proof of 'Only If' Part

- Sufficiency is trivial. Proving necessity.
- Let

$$a_i \sim (\alpha_i \circ a_1, (1-\alpha_i) \circ a_n) \quad \text{(continuity)}$$

- Since both $u(.)$ and $v(.)$ represent $\succsim$ and are VNM (expected utility) functions

$$\begin{aligned} u(a_i) &= \alpha_i u(a_1) + (1-\alpha_i)\, u(a_n) \\ v(a_i) &= \alpha_i v(a_1) + (1-\alpha_i)\, v(a_n) \end{aligned}$$

- Solving for α_i:

$$\alpha_i = \frac{u(a_i) - u(a_n)}{u(a_1) - u(a_n)} = \frac{v(a_i) - v(a_n)}{v(a_1) - v(a_n)}$$

Proof (contd.)

- Solving for $v(a_i)$:

$$v(a_i) = \underbrace{\frac{u(a_1)v(a_n) - u(a_n)v(a_1)}{u(a_1) - u(a_n)}}_{\alpha} + \underbrace{\left[\frac{v(a_1) - v(a_n)}{u(a_1) - u(a_n)}\right]}_{\beta} u(a_i)$$

- There are two degrees of freedom while choosing the utility function.

The Allais Paradox

- **Decision problem 1:** which do you prefer?

Lottery A: 1 crore (1)

Lottery B: 5crore (.1), 1 crore (.89), 0 (.01)

- **Decision problem 2:** which do you prefer?

Lottery C: 1 crore (.11), 0 (.89)

Lottery D:5 crore (.1), 0 (.9)

- In surveys, most people say:

$$A \succ B, D \succ C$$

What is Wrong?

- Suppose $u(.)$ represents these preferences.
- $A \succ B$ implies

$$u(1) > .1u(5) + .89u(1) + .01u(0)$$

$$\text{or} \quad .1u(5) - .11u(1) + .01u(0) < 0$$

- $D \succ C$ implies

$$.1u(5) + .9u(0) > .11u(1) + .89u(0)$$

$$\text{or} \quad .1u(5) - .11u(1) + .01u(0) > 0$$

- These preferences cannot be represented by a VNM function since it leads to a contradiction.

The Ellsberg Paradox

- An urn contains 300 balls, out of which 100 are known to be red, and the remaining 200 are known to be either blue or green.
- **Decision problem 1:** which do you prefer?

  Lottery A: Rs. 100 if Red

  Lottery B: Rs. 100 if Blue

- **Decision problem 2:** which do you prefer?

  Lottery C Rs. 100 if Not Red

  Lottery D: Rs. 100 if Not Blue

- In surveys, most people say:

$$A \succ B, C \succ D$$

What is Wrong?

- Suppose $u(.)$ represents these preferences, and suppose the decision maker conjectures $\Pr[\text{blue}] = p$.
- $A \succ B$ implies
$$p < \frac{1}{3}$$
- $C \succ D$ implies
$$\frac{2}{3} > 1 - p \Rightarrow p > \frac{1}{3}$$
- These preferences cannot be represented by any expected utility function (ambiguity aversion).

Non-Consequentialism: Machina's Mom

- A mother has two children but only one (indivisible) toy.
- Outcomes: b (boy gets it), g (girl gets it).
- Preference: $b \sim g$, $(0.5 \circ b, 0.5 \circ g) \succ b, g$.
- Violates monotonicity axiom.
- Why does Machina's mom strictly prefer tossing a coin?
- To guarantee equal opportunity, since she cannot ensure equal outcome.

Bayes' Rule

- Suppose 1% of the population is infected with swine flu virus.
- Suppose there is a test of 90% accuracy (10% chance of false positive or false negative).
- A patient tests positive. What is the probability he is actually infected?
- Bayes' Rule says $\Pr(\text{infected}|\text{positive})$

$$= \frac{\Pr(\text{inf})\Pr(\text{positive}|\text{inf})}{\Pr(\text{inf})\Pr(\text{positive}|\text{inf}) + \Pr(\text{uninf})\Pr(\text{positive}|\text{uninf})}$$

$$= \frac{(.01)(.9)}{(.01)(.9) + (.99)(.1)} = \frac{1}{12}$$

- The small prior nullifies the effect of the large test accuracy.

Framing Effect

- Kahnemann and Tversky (1981): suppose 600 people will be subjected to a medical treatment against some deadly disease.
- **Decision problem 1:** which do you prefer?

$$\text{Treatment A: 200 people will be saved}$$

$$\text{Treatment B: everyone saved (prob } \frac{1}{3}\text{) or no one saved (prob } \frac{2}{3}\text{)}$$

- **Decision problem 2:** which do you prefer?

$$\text{Treatment C: 400 people will die}$$

$$\text{Treatment D: everyone dies (prob } \frac{2}{3}\text{) or no one dies (prob } \frac{1}{3}\text{)}$$

- In surveys, most people say:

$$A \succ B \ (72\%), D \succ C \ (78\%)$$

Monetary Payoffs

- Let $a_i = w_i$ (some amount of wealth).
- Expected value of a lottery: $\mathbf{E}(g) = \sum_{i=1}^{n} p_i w_i$.
- Expected utility of a lottery: $u(g) = \sum_{i=1}^{n} p_i u(w_i)$.
- Definition: $u(.)$ exhibits
  - risk neutrality if $u(g) = u(\mathbf{E}(g))$ for all $g \in G$.
  - risk aversion if $u(g) < u(\mathbf{E}(g))$ for all $g \in G$.
  - risk loving if $u(g) > u(\mathbf{E}(g))$ for all $g \in G$.
- Certainty equivalent: $C(g)$ is such that $u(g) = u(C(g))$.
- Risk premium $R(g) = \mathbf{E}(g) - C(g)$.
- Risk neutrality/aversion/loving $\Rightarrow R(g) =, >, < 0$.

Optimum Purchase of Insurance

- An agent with wealth w faces a loss L with probability p.
- She has a concave (risk averse) utility function $u(w)$.
- She can insure her wealth at a premium of ρ per rupee insured.
- The agent's problem is to insure an amount $x \leq w$ to solve:

$$\max_{x} pu(w - L - \rho x + x) + (1 - p)u(w - \rho x)$$

- First order condition:

$$p(1 - \rho)u'(w - L - \rho x + x) = (1 - p)\rho u(w - \rho x)$$

- $x < (=) L$ if $\rho > (=) p$.
- Zero profit condition for insurance companies:

$$(1 - p)\rho - p(1 - \rho) = 0 \Rightarrow p = \rho$$

Degree of Risk Aversion

- The Arrow-Pratt measure of absolute risk aversion:

$$r(w) = -\frac{u''(w)}{u'(w)}$$

- Interpretation: a more risk averse agent will accept a strictly smaller set of lotteries.
- Consider lotteries of the form $(p \circ x_1, (1-p) \circ x_2)$. Let $x_2(x_1)$ be the boundary of the acceptable set.
- By definition:

$$pu(w + x_1) + (1-p)u(w + x_2(x_1)) \equiv u(w)$$

- Differentiating with respect to x_2 at $(0, 0)$:

$$pu'(w) + (1-p)u'(w)x_2'(0) = 0 \Rightarrow x_2'(0) = -\frac{p}{1-p}$$

Degree of Risk Aversion

- The more curved the boundary at $(0, 0)$, the smaller is the acceptance set.
- Differentiating a second time at $(0, 0)$:

$$pu''(w) + (1-p)u''(w)\left[x_2'(0)\right]^2 + (1-p)u'(w)x_2''(0) = 0$$

- Since $x_2'(0) = -\frac{p}{1-p}$

$$x_2''(0) = \frac{p}{(1-p)^2}\left[-\frac{u''(w)}{u'(w)}\right]$$

- Agents with larger $r(w)$ have smaller acceptance sets.