Microeconomic Theory I: Choice Under Uncertainty

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September 8, 2014
Lotteries

- Set of outcomes: \( \{a_1, a_2, \ldots, a_n\} \).
- A gamble/lottery is a probability distribution over outcomes: 
  \[ g = (p_1 \circ a_1, p_2 \circ a_2, \ldots, p_n \circ a_n) \].
- \( p_i \) is the probability of outcome \( i \).
- Sure outcomes: \( (0 \circ a_1, \ldots, 1 \circ a_i, \ldots, 0 \circ a_n) = a_i \).
- Compound lotteries are probability distributions over lotteries: 
  \[ (q_1 \circ g_1, q_2 \circ g_2, \ldots, q_m \circ g_m) \].
- \( (S_G) \) \( S \) is the set of all (simple) lotteries.
- \( \succ \) is a preference relation defined over \( S \).
The von Neumann-Morgenstern Axioms

- **Axiom 1 (Completeness):** For all $g, g' \in G$, either $g \succeq g'$ or $g' \succeq g$ (or both).

- **Axiom 2 (Transitivity):** For all $g, g', g'' \in G$, if $g \succeq g'$ and $g' \succeq g''$, then $g \succeq g''$.

- **Axiom 3 (Continuity):** For any $g \in G$, there exists $\alpha \in [0, 1]$ such that

$$g \sim (\alpha \circ a_1, \ldots, (1 - \alpha) \circ a_n)$$
The von Neumann-Morgenstern Axioms

- **Axiom 4 (Monotonicity):** For any $\alpha, \beta \in [0, 1]$

  $$(\alpha \circ a_1, (1 - \alpha) \circ a_n) \succeq (\beta \circ a_1, (1 - \beta) \circ a_n) \iff \alpha \geq \beta$$

- **Axiom 5 (Substitution/Independence):** If $g = (p_1 \circ g_1, \ldots, p_k \circ g_k)$, $h = (p_1 \circ h_1, \ldots, p_k \circ h_k)$ and $g_i \sim h_i$ for all $i = 1, 2, \ldots, k$, then $g \sim h$.

- **Axiom 6 (Reduction to simple lotteries):** For any $g \in G$, if $g_S \in G_S$ is the simple lottery induced by $g$, then $g \sim g_S$. 
The Expected Utility Theorem

**Theorem**

Suppose $\preceq$ satisfies Axioms 1 through 6. Then there exists a function $u : G \rightarrow \mathbb{R}$ such that $u(.)$

\[(i)\text{ represents } \preceq, \text{ i.e. } g \preceq g' \iff u(g) \geq u(g')\]

\[(ii)\text{ has the exp utility prop, i.e. } u(g) = \sum_{i=1}^{n} p_i u(a_i)\]

- The probabilities $p_i$ are assumed to be objective (e.g. playing roulette), not subjectively assessed (e.g. stock price).
- Savage extended the theory to subjective probabilities.
- The value of a lottery is linear in the probabilities of outcomes.
Proof: Representation

- Proof by construction: define $u(g) \in [0, 1]$ such that

$$g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n) \text{ (continuity)}$$

- Representation: $g \succsim g' \iff$

$$\left( u(g) \circ a_1, (1 - u(g)) \circ a_n \right) \succsim \left( u(g') \circ a_1, (1 - u(g')) \circ a_n \right) \text{ (transitivity)}$$

$$\iff u(g) \geq u(g') \text{ (monotonicity)}$$
Proof: Expected Utility Property

- Expected utility property:

\[ a_i \sim (u(a_i) \circ a_1, (1 - u(a_i)) \circ a_n) \equiv q_i \]

- Then

\[
g \sim (p_1 \circ q_1, p_2 \circ q_2, \ldots, p_n \circ q_n) \quad \text{(substitution)}
\]

\[
\sim \left( \left( \sum_{i=1}^{n} p_i u(a_i) \right) \circ a_1, \left( 1 - \sum_{i=1}^{n} p_i u(a_i) \right) \circ a_n \right) \quad \text{(axiom 6)}
\]

- By monotonicity

\[
u(g) = \sum_{i=1}^{n} p_i u(a_i)
\]
Invariance to Positive Affine Transformations

Theorem

Suppose the VNM function $u(.)$ represents \( \sim \) over $G$. Then the VNM function $v(.)$ represents \( \sim \) if and only if there exist real numbers $\alpha$ and $\beta > 0$ such that

$$v(g) = \alpha + \beta u(g) \quad \text{for all} \ g \in G$$

- As in choice under certainty, there is no unique function that represents preferences.
- Representation is more restrictive: only positive linear transformations preserve preference.
Proof of ‘Only If’ Part

- Sufficiency is trivial. Proving necessity.
- Let
  \[ a_i \sim (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n) \] (continuity)
- Since both \( u(.) \) and \( v(.) \) represent \( \succeq \) and are VNM (expected utility) functions
  \[
  u(a_i) = \alpha_i u(a_1) + (1 - \alpha_i) u(a_n) \\
  v(a_i) = \alpha_i v(a_1) + (1 - \alpha_i) v(a_n)
  \]
- Solving for \( \alpha_i \):
  \[
  \alpha_i = \frac{u(a_i) - u(a_n)}{u(a_1) - u(a_n)} = \frac{v(a_i) - v(a_n)}{v(a_1) - v(a_n)}
  \]
Proof (contd.)

- Solving for $v(a_i)$:

$$v(a_i) = \frac{u(a_1)v(a_n) - u(a_n)v(a_1)}{u(a_1) - u(a_n)} \alpha + \left[ \frac{v(a_1) - v(a_n)}{u(a_1) - u(a_n)} \right] u(a_i) \beta$$

- There are two degrees of freedom while choosing the utility function.
The Allais Paradox

- **Decision problem 1**: which do you prefer?
  - Lottery A: 1 crore (1)
  - Lottery B: 5 crore (.1), 1 crore (.89), 0 (.01)

- **Decision problem 2**: which do you prefer?
  - Lottery C: 1 crore (.11), 0 (.89)
  - Lottery D: 5 crore (.1), 0 (.9)

- In surveys, most people say:
  \[ A \succ B, \; D \succ C \]
What is Wrong?

- Suppose \( u(\cdot) \) represents these preferences.
- \( A \succ B \) implies
  \[
  u(1) > .1u(5) + .89u(1) + .01u(0)
  \]
  or \( .1u(5) - .11u(1) + .01u(0) < 0 \)
- \( D \succ C \) implies
  \[
  .1u(5) + .9u(0) > .11u(1) + .89u(0)
  \]
  or \( .1u(5) - .11u(1) + .01u(0) > 0 \)
- These preferences cannot be represented by a VNM function since it leads to a contradiction.
An urn contains 300 balls, out of which 100 are known to be red, and the remaining 200 are known to be either blue or green.

**Decision problem 1:** which do you prefer?

Lottery A: Rs. 100 if Red  
Lottery B: Rs. 100 if Blue

**Decision problem 2:** which do you prefer?

Lottery C Rs. 100 if Not Red  
Lottery D: Rs. 100 if Not Blue

In surveys, most people say:

\[ A \succ B, C \succ D \]
What is Wrong?

- Suppose \( u(.) \) represents these preferences, and suppose the decision maker conjectures \( \text{Pr}[\text{blue}] = p \).
- \( A \succ B \) implies
  \[
  p < \frac{1}{3}
  \]
- \( C \succ D \) implies
  \[
  \frac{2}{3} > 1 - p \Rightarrow p > \frac{1}{3}
  \]
- These preferences cannot be represented by any expected utility function (ambiguity aversion).
Non-Consequentialism: Machina’s Mom

- A mother has two children but only one (indivisible) toy.
- Outcomes: \(b\) (boy gets it), \(g\) (girl gets it).
- Preference: \(b \sim g\), \((0.5 \circ b, 0.5 \circ g) \succ b, g\).
- Violates monotonicity axiom.
- Why does Machina’s mom strictly prefer tossing a coin?
- To guarantee equal opportunity, since she cannot ensure equal outcome.
Bayes’ Rule

- Suppose 1% of the population is infected with swine flu virus.
- Suppose there is a test of 90% accuracy (10% chance of false positive or false negative).
- A patient tests positive. What is the probability he is actually infected?
- Bayes’ Rule says $P(\text{infected} | \text{positive})$

\[
= \frac{P(\text{inf}) P(\text{positive} | \text{inf})}{P(\text{inf}) P(\text{positive} | \text{inf}) + P(\text{uninf}) P(\text{positive} | \text{uninf})} \\
= \frac{(.01)(.9)}{(.01)(.9) + (.99)(.1)} = \frac{1}{12}
\]

- The small prior nullifies the effect of the large test accuracy.
Framing Effect

- Kahnemmann and Tversky (1981): suppose 600 people will be subjected to a medical treatment against some deadly disease.

- **Decision problem 1:** which do you prefer?

  Treatment A: 200 people will be saved
  
  Treatment B: everyone saved (prob $\frac{1}{3}$) or no one saved (prob $\frac{2}{3}$)

- **Decision problem 2:** which do you prefer?

  Treatment C: 400 people will die
  
  Treatment D: everyone dies (prob $\frac{2}{3}$) or no one dies (prob $\frac{1}{3}$)

- In surveys, most people say:

  $A \succ B$ (72%), $D \succ C$ (78%)
Monetary Payoffs

- Let $a_i = w_i$ (some amount of wealth).
- Expected value of a lottery: $E(g) = \sum_{i=1}^{n} p_i w_i$.
- Expected utility of a lottery: $u(g) = \sum_{i=1}^{n} p_i u(w_i)$.
- Definition: $u(.)$ exhibits
  - risk neutrality if $u(g) = u(E(g))$ for all $g \in G$.
  - risk aversion if $u(g) < u(E(g))$ for all $g \in G$.
  - risk loving if $u(g) > u(E(g))$ for all $g \in G$.
- Certainty equivalent: $C(g)$ is such that $u(g) = u(C(g))$.
- Risk premium $R(g) = E(g) - C(g)$.
- Risk neutrality/aversion/loving $\Rightarrow R(g) =, >, < 0$. 
Optimum Purchase of Insurance

- An agent with wealth $w$ faces a loss $L$ with probability $p$.
- She has a concave (risk averse) utility function $u(w)$.
- She can insure her wealth at a premium of $\rho$ per rupee insured.
- The agent’s problem is to insure an amount $x \leq w$ to solve:

$$\max_x pu(w - L - \rho x + x) + (1 - p)u(w - \rho x)$$

- First order condition:

$$p(1 - \rho)u'(w - L - \rho x + x) = (1 - p)\rho u(w - \rho x)$$

- $x < (\leq)L$ if $\rho > (\leq)p$.
- Zero profit condition for insurance companies:

$$(1 - p)\rho - p(1 - \rho) = 0 \Rightarrow p = \rho$$
Degree of Risk Aversion

- The Arrow-Pratt measure of absolute risk aversion:

\[ r(w) = -\frac{u''(w)}{u'(w)} \]

- Interpretation: a more risk averse agent will accept a strictly smaller set of lotteries.

- Consider lotteries of the form \((p \circ x_1, (1 - p) \circ x_2)\). Let \(x_2(x_1)\) be the boundary of the acceptable set.

- By definition:

\[ pu(w + x_1) + (1 - p)u(w + x_2(x_1)) \equiv u(w) \]

- Differentiating with respect to \(x_2\) at \((0, 0)\):

\[ pu'(w) + (1 - p)u'(w)x_2'(0) = 0 \Rightarrow x_2'(0) = -\frac{p}{1 - p} \]
Degree of Risk Aversion

- The more curved the boundary at \((0, 0)\), the smaller is the acceptance set.
- Differentiating a second time at \((0, 0)\):

\[
pu''(w) + (1 - p)u''(w) \left[x'_2(0)\right]^2 + (1 - p)u'(w)x''_2(0) = 0
\]

- Since \(x'_2(0) = -\frac{p}{1-p}\)

\[
x''_2(0) = \frac{p}{(1 - p)^2} \left[-\frac{u''(w)}{u'(w)}\right]
\]

- Agents with larger \(r(w)\) have smaller acceptance sets.