Delhi School of Economics
Course 001: Microeconomic Theory
Solutions to Problem Set 2.

1. Properties of $\succsim$ extend to $\succ$ and $\sim$.
   (a) Assume $x_1 \succ x_2 \succ x_3$. This implies: $x_1 \succsim x_2 \succsim x_3 \Rightarrow x_1 \succsim x_3$.
   Now suppose $x_3 \succsim x_1$. Combined with $x_1 \succsim x_2$ and transitivity of $\succsim$, we get $x_3 \succsim x_2$, which contradicts the fact that $x_2 \succ x_3$. Hence it is not true that $x_3 \succsim x_1$.
   Combining, we get $x_1 \succ x_3$.
   (b) $x_1 \sim x_2 \sim x_3 \Rightarrow x_1 \succsim x_2 \succsim x_3 \Rightarrow x_1 \succsim x_3$.
   $x_3 \sim x_2 \sim x_1 \Rightarrow x_3 \succsim x_2 \succsim x_1 \Rightarrow x_3 \succsim x_1$.
   Combining, we get $x_1 \sim x_3$.

2. $x_1 \sim x_2 \succ x_3 \Rightarrow x_1 \succsim x_2 \succsim x_3 \Rightarrow x_1 \succsim x_3$.
   (a) Suppose $x_3 \succsim x_1$. Combined with $x_1 \succsim x_2$ and transitivity of $\succsim$, we get $x_3 \succsim x_2$, which contradicts the fact that $x_2 \succ x_3$. Hence it is not true that $x_3 \succsim x_1$.
   Combining, we get $x_1 \succ x_3$.

3. The continuity axiom is violated. Consider the following sequence of bundles: $(x_1^n, x_2^n) = (a, 0)$ and $(y_1^n, y_2^n) = (a - \frac{1}{n}, b)$, where $a, b > 0$. For the preferences described, $(x_1^n, x_2^n) \succ (y_1^n, y_2^n)$ for any n. However, $(x_1^n, x_2^n) \to (a, 0)$ and $(y_1^n, y_2^n) \to (a, b)$. Then $(a, b) \succ (a, 0)$, i.e., preference is reversed in the limit, which is a violation of continuity. These preferences cannot be represented by a *continuous* utility function.

4. This is the Stone-geary utility function.
   (a) After power and log transformations:
   $$\gamma = \frac{\beta_1}{\beta_1 + \beta_2}$$
   (b) Solution will be interior. From the condition MRS = price ratio, we get
   $$\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{x_2 - \alpha_2}{x_1 - \alpha_1}\right) = \frac{p_1}{p_2}$$
   The other equation is that of the budget line:
   $$p_1 x_1 + p_2 x_2 = y$$
   Solving these two equations gives us the Marshallian demand functions:
   $$\begin{aligned} x_1 &= \alpha_1 + \frac{y - p_1\alpha_1 - p_2\alpha_2}{p_1} \\ x_2 &= \alpha_2 + \frac{y - p_1\alpha_1 - p_2\alpha_2}{p_2} \end{aligned}$$

5. I will skip some of the easier details.
   (a) For Cobb-Douglas preferences, note that $MU_i \to \infty$ as $x_i \to 0$, hence the solution will always be interior for any strictly positive price vector. Also, since the utility function is strictly quasi-concave, SOC will always be satisfied. The FOC, using the standard tangency condition (MRS = price ratio) gives
   $$\frac{\alpha_i x_j}{\alpha_j x_i} = \frac{p_i}{p_j}$$

This can be rewritten as

$$\frac{p_i x_i}{\alpha_i} = k \quad \text{for all } i \tag{1}$$

where k is some constant. The value of k can be solved by using the above in the budget constraint:

$$\sum_{i=1}^{n} \alpha_i k = y \Rightarrow k = y \quad \text{since } \sum_{i=1}^{n} \alpha_i = 1$$

Using this in (1), we get the Marshallian demand functions:

$$x_i(\mathbf{p}, y) = \frac{\alpha_i y}{p_i}$$

Plugging back these demands in the utility function and simplifying, we get the indirect utility function:

$$v(\mathbf{p}, y) = \left(\prod_{i=1}^{n} \alpha_i^{\alpha_i} p_i^{-\alpha_i} \right) y$$

For the dual problem (expenditure minimization), (1) must still be satisfied but the value of k will be different. To find this value, insert (1) into the constraint of the problem to get

$$\left(\prod_{i=1}^{n} \alpha_i^{-\alpha_i} p_i^{\alpha_i} \right) u = k$$

Upon replacing back in (1), we get the Hicksian demand functions:

$$x_i^h(\mathbf{p}, u) = \frac{\left(\prod_{i=1}^{n} \alpha_i^{-\alpha_i} p_i^{\alpha_i} \right) \alpha_i u}{p_i}$$

Plugging back these expressions into the expression for spending,i.e., in $\sum_i^n p_i x_i^h$, we get the expenditure function

$$e(\mathbf{p}, u) = \left(\prod_{i=1}^{n} \alpha_i^{-\alpha_i} p_i^{\alpha_i} \right) u$$

(b) The utility function is of a piecewise linear form:

$$\begin{aligned} u(x_1, x_2) &= ax_1 + x_2 \quad \text{for } x_2 \leq x_1 \\ &= x_1 + ax_2 \ \text{for } x_2 > x_1 \end{aligned}$$

The slopes above and below the 45-degree line are $\frac{1}{a}$ and a respectively. If $a > 1$, the indifference curves are concave to the origin. In this case, optimal solutions will always be at a corner. Leaving the details to you, I will focus on the case of convex preferences ($a < 1$). Here, the optimal choice is either at a corner or at the kink, depending on the slope of the budget line. Specifically, Marshallian demands are given by

$$\begin{aligned} (x_1, x_2) &= \left(0, \frac{y}{p_2}\right) \quad \text{if } \frac{p_1}{p_2} > \frac{1}{a} \\ &= \left(\frac{y}{p_1 + p_2}, \frac{y}{p_1 + p_2}\right) \quad \text{if } a < \frac{p_1}{p_2} < \frac{1}{a} \\ &= \left(\frac{y}{p_1}, 0\right) \quad \text{if } \frac{p_1}{p_2} < a \end{aligned}$$

In the case where the price ratio is either a or $\frac{1}{a}$, the budget line coincides with one of the arms of the indifference curve and any choice on that arm is optimal. The indirect utility function is

$$\begin{aligned} v(\mathbf{p}, y) &= \frac{ay}{p_2} \quad \text{if } \frac{p_1}{p_2} > \frac{1}{a} \\ &= \frac{(a+1)y}{p_1 + p_2} \quad \text{if } a < \frac{p_1}{p_2} < \frac{1}{a} \\ &= \frac{ay}{p_1} \quad \text{if } \frac{p_1}{p_2} < a \end{aligned}$$

The Hicksian demand functions are

$$\begin{aligned}(x_1^h, x_2^h) &= \left(0, \frac{u}{a}\right) \text{ if } \frac{p_1}{p_2} > \frac{1}{a} \\ &= \left(\frac{u}{a+1}, \frac{u}{a+1}\right) \text{ if } a < \frac{p_1}{p_2} < \frac{1}{a} \\ &= \left(\frac{u}{a}, 0\right) \text{ if } \frac{p_1}{p_2} > \frac{1}{a}\end{aligned}$$

The expenditure functions is

$$\begin{aligned}e(\mathbf{p}, u) &= \frac{p_2 u}{a} \text{ if } \frac{p_1}{p_2} > \frac{1}{a} \\ &= \frac{(p_1+p_2)u}{a+1} \text{ if } a < \frac{p_1}{p_2} < \frac{1}{a} \\ &= \frac{p_1 u}{a} \text{ if } \frac{p_1}{p_2} > \frac{1}{a}\end{aligned}$$

(c) The utility functions are concave to the origin, hence the point of tangency represents a minimum rather than a maximum. Obviously there will be a corner solution. Marshallian Demand functions are:

$$\begin{aligned}(x_1, x_2) &= \left(\frac{y}{p_1}, 0\right) \text{ if } p_1 < p_2 \\ &= \left(0, \frac{y}{p_2}\right) \text{ if } p_1 > p_2\end{aligned}$$

When $p_1 = p_2$, either corner is optimal.
Indirect utility function:

$$\begin{aligned}v(\mathbf{p}, y) &= \frac{y^2}{p_1^2} \text{ if } p_1 \leq p_2 \\ &= \frac{y^2}{p_2^2} \text{ if } p_1 > p_2\end{aligned}$$

Hicksian demand functions:

$$\begin{aligned}(x_1^h, x_2^h) &= (\sqrt{u}, 0) \text{ if } p_1 < p_2 \\ &= (0, \sqrt{u}) \text{ if } p_1 > p_2\end{aligned}$$

Expenditure function:

$$\begin{aligned}e(\mathbf{p}, u) &= p_1\sqrt{u} \text{ if } p_1 \leq p_2 \\ &= p_2\sqrt{u} \text{ if } p_1 > p_2\end{aligned}$$

(d) This is the case of perfect substitutes which yields corner solutions for almost all price vectors (draw the picture). Marshallian demands are:

$$\begin{aligned}(x_1, x_2) &= \left(0, \frac{y}{p_2}\right) \text{ if } \frac{p_1}{p_2} > \frac{a}{b} \\ &= \left(\frac{y}{p_1}, 0\right) \text{ if } \frac{p_1}{p_2} < \frac{a}{b}\end{aligned}$$

The indirect utility function is

$$\begin{aligned}v(\mathbf{p}, y) &= \frac{by}{p_2} \text{ if } \frac{p_1}{p_2} \geq \frac{a}{b} \\ &= \frac{ay}{p_1} \text{ if } \frac{p_1}{p_2} < \frac{a}{b}\end{aligned}$$

The Hicksian demands are:

$$
\begin{aligned}
(x_1^h, x_2^h) &= \left(0, \frac{u}{b}\right) \text{ if } \frac{p_1}{p_2} > \frac{a}{b} \\
&= \left(\frac{u}{a}, 0\right) \text{ if } \frac{p_1}{p_2} < \frac{a}{b}
\end{aligned}
$$

The expenditure function is

$$
\begin{aligned}
e(\mathbf{p}, u) &= \frac{p_2 u}{b} \text{ if } \frac{p_1}{p_2} \geq \frac{a}{b} \\
&= \frac{p_1 u}{a} \text{ if } \frac{p_1}{p_2} < \frac{a}{b}
\end{aligned}
$$

(e) First, focus on cases where there is an interior.solution. Applying the principle MRS = price ratio, we get

$$\frac{1+x_2}{1+x_1} = \frac{p_1}{p_2}$$

Combining with the equation of the budget line, we get the Marshallian demands

$$(x_1, x_2) = \left(\frac{y - p_1 + p_2}{p_1}, \frac{y - p_2 + p_1}{p_2}\right)$$

Sticking with the interior solution case, the other functions are easy to derive (you can exploit the symmetry to simplify calculations):

$$
\begin{aligned}
(x_1^h, x_2^h) &= \left(\sqrt{\frac{p_2(1+u)}{p_1}}, \sqrt{\frac{p_1(1+u)}{p_2}}\right) \\
v(\mathbf{p}, y) &= \frac{y^2 + (p_1 + p_2)y}{p_1 p_2} \\
e(\mathbf{p}, u) &= 2\sqrt{p_1 p_2 (1+u)}
\end{aligned}
$$

Some caveat has to be added to the solution, however. Sometimes, there will be corner solutions. Note from the Marshallian demand expressions above, whenever $y < p_1 - p_2$, we have $x_1 < 0$. This is inadmissible since negative quantities are not allowed. In this case, the optimal choice will involve $x_1 = 0$ and all the income spent on good 2, i.e., $x_2 = \frac{y}{p_2}$. The opposite corner solution arises whenever $y < p_2 - p_1$. A compact and accurate description of the Marshallian demands will then be

$$
\begin{aligned}
x_1(\mathbf{p}, y) &= \max\left\{0, \min\left\{\frac{y - p_1 + p_2}{p_1}, \frac{y}{p_1}\right\}\right\} \\
x_2(\mathbf{p}, y) &= \max\left\{0, \min\left\{\frac{y - p_2 + p_1}{p_2}, \frac{y}{p_2}\right\}\right\}
\end{aligned}
$$

The other functions have to be similarly adjusted to take account of corner solutions. The details are ommitted.

6. First, we argue that with additively separable utility, all goods must be normal goods. Suppose x_i falls when income y goes up. For any pair (i, j), the FOC is

$$\frac{u_i'(x_i)}{u_j'(x_j)} = \frac{p_i}{p_j}$$

If x_i decreases, by concavity, x_j must also decrease to keep the MRS constant (remember prices haven't changed, only income). Hence, if any one good is inferior, all goods are inferior. But this violates the monotonicity axiom, so we have a contradiction. Hence none of the goods can be inferior. The argument is completed by noting only inferior goods can be Giffen goods.

7. The agent solves (taking log transformation of the utility function to simplify calculations):

$$\max_{L,F} \frac{1}{3} \ln L + \frac{2}{3} \ln F$$

subject to the budget constraint

$$m + w(T - L) = pF$$

Substituting this into the objective function, the problem reduces to

$$\max_{L} \frac{1}{3} \ln L + \frac{2}{3} \ln [m + w(T - L)] - \frac{2}{3} \ln p$$

and the FOC for interior solution is

$$\frac{1}{L^*} = \frac{2w}{m + w(T - L^*)}$$

The solution for optimal choice of leisure is

$$L^* = \min \left\{ \frac{m + wT}{3w}, T \right\}$$

since total leisure cannot exceed the time endowment T. Therefor labour supply $(T - L^*)$ is given by

$$\max \left\{ \frac{2wT - m}{3w}, 0 \right\}$$

For strictly positive labour supply, we need

$$w > \frac{m}{2T}$$

The agent needs a high enough wage and insufficient non-wage income to be willing to work. Food demand is obtained by replacing $(T - L^*)$ in the budget constraint:

$$F^* = \frac{1}{p} \max \left\{ \frac{2}{3}(wT + m), m \right\}$$

8. The agent's problem is (again taking log transformation):

$$\max_{x_1, x_2} \alpha \ln x_1 + (1 - \alpha) \ln x_2$$

subject to the budget constraint

$$p_1 x_1 + p_2 x_2 = 5p_1 + 3p_2$$

Usual tangency condition (MRS = price ratio) gives

$$\frac{\alpha x_2}{(1 - \alpha) x_1} = \frac{p_1}{p_2}$$

(a) Using this in the euation for the budget line, we get the demands:

$$x_1 = \frac{\alpha (5p_1 + 3p_2)}{p_1}$$

$$x_2 = \frac{(1 - \alpha)(5p_1 + 3p_2)}{p_2}$$

(b) The agent is a net buyer of good 1 if $x_1 > 5$, i.e.,

$$\frac{p_1}{p_2} < \frac{3\alpha}{5(1 - \alpha)}$$

9. This exercise focuses on the cash-vs-kind debate. The main point is that the best means of delivering aid depends on whether the donor judges the recipient's welfare by the recipient's preferences (welfarism) or the donor's (paternalism).

   (a) The utility function can be transformed into:

   $$u(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$$

   which gives rise to demand functions

   $$\begin{aligned} x(p_x, p_y, I) &= \frac{I}{3p_x} \\ y(p_x, p_y, I) &= \frac{2I}{3p_y} \end{aligned}$$

   Indirect utility is obtained by substituting demands into the utility function:

   $$v(p_x, p_y, I) = \frac{2^{\frac{2}{3}} I}{3\,(p_x)^{\frac{1}{3}}\,(p_y)^{\frac{2}{3}}}$$

   (b) From the demand functions: $x = 2$, $y = 2$.

   (c) Let the subsidy be s per school. Banana Republic's demand functions become:

   $$\begin{aligned} x(p_x, p_y, I) &= \frac{I}{3\,(p_x - s)} \\ y(p_x, p_y, I) &= \frac{2I}{3p_y} \end{aligned}$$

   Here $s = 10$. Using this, we can calculate: $x = 4$, $y = 2$. The value of the subsidy is $xs = 40$. The utility attained is $16^{\frac{1}{3}}$.

   (d) Calculate the expenditure function for these preferences:

   $$e(p_x, p_y, u) \equiv \min\,(p_x x + p_y y) \quad \text{sub to } x^{\frac{1}{3}} y^{\frac{2}{3}} = u$$

   The FOC are

   $$\frac{1}{2} \cdot \frac{y}{x} = \frac{p_x}{p_y} \quad \text{and} \quad x^{\frac{1}{3}} y^{\frac{2}{3}} = u$$

   Solving, we get the Hicksian demand functions

   $$\begin{aligned} x(p_x, p_y, u) &= \left(\frac{p_y}{2p_x}\right)^{\frac{2}{3}} u \\ y(p_x, p_y, u) &= \left(\frac{2p_x}{p_y}\right)^{\frac{1}{3}} u \end{aligned}$$

   Using these expressions gives us the expenditure function

   $$e(p_x, p_y, u) = \left(2^{\frac{1}{3}} + 2^{-\frac{2}{3}}\right)(p_x)^{\frac{1}{3}}\,(p_y)^{\frac{2}{3}}\,.u$$

   Insert the prices and target utility ($u = 16^{\frac{1}{3}}$) to calculate the minimum expenditure. The difference between this amount and the income $I = 120$ is the cash grant that will be necessary. Show that this is less than the 40 Bleeding Heart was spending in subsidies.

   (e) For any cash award c, the demand function for schools would become

   $$x(p_x, p_y, I) = \frac{I + c}{3p_x}$$

   To increase the demand for schools from 2 to 4, we must have $c = 120$, which is much more than the 40 spent by the agency under the subsidy scheme.

10. This question tries to understand the effect of tax rates and laws on charitable donations.

   (a) It is easy to show that at the optimum:

$$\begin{aligned} x_1 &= \alpha(1-t)y \\ x_2 &= (1-\alpha)(1-t)y \end{aligned}$$

   A tax cuts increases charitable donations because it has a pure income effect.

   (b) The problem can be written as

$$\max \alpha \ln x_1 + (1-\alpha) \ln x_2 \text{ subject to } x_1 + x_2 = (1-t)y + tx_2$$

   The budget constraint can be rewritten as

$$\frac{x_1}{1-t} + x_2 = y$$

   where $\frac{1}{1-t}$ can be interpreted as the price of own consumption in the sense of opportunity cost (the amount of charitable donations forgone for every rupee spent on own needs). This gives rise to optimal choices:

$$\begin{aligned} x_1 &= \alpha(1-t)y \\ x_2 &= (1-\alpha)y \end{aligned}$$

   (c) A tax cut makes own consumption cheaper and hence the substitution effect tends to increase selfish spending and reduce donations. However, the tax cut also leaves more disposable income in the agent's pocket and so the income effect tends to increase donations. For Cobb-Douglas utility, the two effects exactly cancel each other out.

   (d) Yes. It is the property that under Cobb-Douglas preferences, total *spending* on each good is a constant. In general, income and substitution effects may not cancel out exactly and the effect of tax cuts (when donations are tax deductible) would be theoretically ambiguous.