Problem set, Social Choice Theory , 2014

1. Let A be the set of alternatives and N be the set of citizens. Strict preference of citizen $i \in N$ over A is denoted by P_i.
(a) Majority rule: $a \in A$ is socially as good as $b \in A \Leftrightarrow |\{i \in N \mid aP_ib\}| \geq |\{i \in N \mid bP_ia\}|$
(b) Approval voting rule: Fix an integer, $1 \leq k \leq |A|$. Take $a \in A$ and $i \in N$, $s(a,i) = 1$ if i ranks a among top k alternatives, otherwise $s(a,i) = 0$. $a \in A$ socially as good as $b \in A \Leftrightarrow \sum_{i \in N} s(a,i) \geq \sum_{i \in N} s(b,i)$
(c) Reverse dictatorship: Fix $k \in N$. $a \in A$ is socially preferred to $b \in A \Leftrightarrow bP_ka$
Check whether the above social rankings satisfy transitivity, IIA and Praeto.

2. $A = \{x, y, z\}$ and $N = \{1, 2\}$. Strict preference of citizen $i \in N$ over A is denoted by P_i. A social aggregation rule f is defined as follows, $f(P_1^*, P_2^*) = P_1^*$, where $P_1^* : xP_1^*yP_1^*z$, $P_2^* : yP_2^*xP_2^*z$ and $f(P_1, P_2) = P_2$ for all other P_1, P_2. Does this aggregation rule satisfy Pareto and IIA?

3. Consider the Arrow domain.
(i) Show that if there are only two alternatives, then there are non-dictatorial rules which satisfy IIA and Pareto.
(ii) Consider the following voting problem. There are three candidates $\{L, C, R\}$ and n voters. Voters are of the following types: left wing ($L \succ C \succ R$), right wing ($R \succ C \succ L$) and centrists (either $C \succ L \succ R$ or $C \succ R \succ L$). Find a social ranking in this setting which satisfies IIA and Pareto. Does it violate Arrow's impossibility theorem?

4. A social preference R is 'Acyclic' if the preference has at least one maximal element in every subset of A, that is for all $A' \subseteq A$, $\{x \in A' \mid xRy, \ \forall y \in A'\} \neq \emptyset$.
(a) Show that Acyclicity is weaker condition than transitivity.
(b) Consider the following situation: $A = \{x, y, z\}$, $N = \{1, 2\}$ and strict preference of citizen $i \in N$ over A is denoted by P_i. An aggregation rule called 'Veto rule' is defined as follows. Citizen 1 is the dictator with one qualification that citizen 2 can veto the possibility that x be socially preferred to y. In other words, social preference coincide with citizen 1's preference except the situation when xP_1y but yP_2x, in which case xIy, that is x is socially indifferent to y. Show that the veto rule satisfies Acyclicity, IIA and Pareto but fails transitivity. Does this violate Arrow's impossibility theorem.

5. Check whether the following social welfare rankings satisfy 'Invariance under cardinal unit comparable utilities', 'Invariance under ordinal comparable utilities', 'Anonimity' and 'Hammond equity'. Suppose there are only two agents in a society. Let $x_{(k)}$ denote the $k-th$ highest utility under policy x, where $1 \leq k \leq 2$.
Ranking1: x is as good as $y \Leftrightarrow x_{(k)} \leq y_{(k)}$.
Ranking2: x is as good as $y \Leftrightarrow \sum_k \omega_k x_{(k)} \geq \sum_k \omega_k y_{(k)}$, $\omega_k > 0$ for $k = 1, 2$.

6. A family of social welfare rankings called 'generalized Gini rankings' is defined as follows. Let $x_{(k)}$ denote the $k-th$ highest utility under policy x, where $1 \leq k \leq n$.
x is as good as $y \Leftrightarrow \frac{1}{n^\delta} \sum_{k=1}^{n} [k^\delta - (k-1)^\delta] x_{(k)} \geq \frac{1}{n^\delta} \sum_{k=1}^{n} [k^\delta - (k-1)^\delta] y_{(k)}$
(i) Show that 'generalized Gini rankings' satisfy 'Invariance under cardinal full comparable utilities' and 'Anonymity'.
(ii) Show that $\delta = 1$ is the same as the utilitarian rule.
(iii) Show that 'generalized Gini rankings' converge to 'Rawlsian ranking' as $\delta \to \infty$.

7. Let X be the (finite) set of alternative policies and N be the set of citizens. Citizen $i \in N$ has a utility function $U_i : X \longrightarrow \mathcal{R}$ (real number); that is $U_i(x)$ denotes i's utility under policy x.
a) In this context define and justify 'Independence of Irrelevant Alternatives' (IIA) and 'Pareto Indifference' (PI).
b) Provide an example to illustrate some limitations of the Pareto Indifference principle.
c) Define 'Welfarism' ('Strong Neutrality'). Show that IIA and PI implies Welfarism under full domain.
d) Do the following social choice rankings satisfy Welfarism - explain?
(i) x is socially as good as $y \Leftrightarrow [\theta \max_{i \in N} U_i(x) + (1-\theta) \min_{i \in N} U_i(x)] \geq [\theta \max_{i \in N} U_i(y) + (1-\theta) \min_{i \in N} U_i(y)]$
(ii) x is socially as good as $y \Leftrightarrow [\sum_{i \in N} (U_i(x) - d_i)^2]^{\frac{1}{2}} \geq [\sum_{i \in N} (U_i(y) - d_i)^2]^{\frac{1}{2}}$ where $d_i = \frac{1}{|X|} \sum_{x \in X} U_i(x)$ for all i, and $|X|$ denotes the cardinality of X.

8. Consider a profile of individual utility function $(u_1, u_2, \ldots, u_n)$. Consider another profile $(u'_1, u'_2, \ldots, u'_n)$, where $u'_i = u_j$ and $u'_j = u_i$, $u'_k = u_k$ for all $k \neq i, j$. A social welfare ranking satisfies Anonimity* if social raking remains the same under profiles $(u_1, u_2, \ldots, u_n)$ and $(u'_1, u'_2, \ldots, u'_n)$. Show that under 'Welfarism', Anonimity is equivalent to Anonimity*.

9. Either prove or provide counterexample:

(a) Invariance under ordinal non-comparability is strictly stronger than Invariance under cardinal unit comparability.
(b) Invariance under ordinal comparability is strictly stronger than Invariance under cardinal unit comparability.
(c) IIA is strictly weaker than Welfarism (strong-neutrality).

10. Let $\widehat{R}$ be a ranking of policies which satisfies Welfarism, that is policies are ranked just by comparing utility vectors. Moreover, assume that $u_i(x) > 0$ for all $i \in N$ and for all $x \in A$. Let us construct a ranking R^* from $\widehat{R}$ as follows,
$(e^{u_1(x)}, \ldots, e^{u_n(x)})R^*(e^{u_1(y)}, \ldots, e^{u_n(y)}) \Leftrightarrow (u_1(x), \ldots, u_n(x))\widehat{R}(u_1(x), \ldots, u_n(y))$
(a) Show that if $\widehat{R}$ satisfies Weak Pareto, Continuity and Anonimity then R^* also satisfies these axioms.
(b) Let us define two invariance properties,
Invariance under non-comparable unit shift:
$(u_1(x), \ldots, u_n(x))R(u_1(y), \ldots, u_n(y)) \Leftrightarrow (a_1 + u_1(x), \ldots, a_n + u_n(x))R(a_1 + u_1(y), \ldots, a_n + u_n(y))$
Invariance under non-comparable scale shift:
$(u_1(x), \ldots, u_n(x))R(u_1(y), \ldots, u_n(y)) \Leftrightarrow (b_1 u_1(x), \ldots, b_n u_n(x))R(b_1 u_1(y), \ldots, b_n u_n(y))$
Show that: $\widehat{R}$ satisfies Invariance under non-comparable unit shift $\Leftrightarrow$ R^* satisfies Invariance under non-comparable scale shift.
(c) Show that there is only one aggregation rule which satisfies Invariance under non-comparable scale shift, Weak Pareto, Continuity and Anonimity. Identify this rule.

11. Consider an axiomatic bargaining problem. Define Efficiency, Symmetry, Invariance Under Scale Transformation and Independence of Irrelevant Alternatives. Drop one of the above axioms at a time and show that there are outcomes (other than the Nash outcome) which satisfy the rest.

12. Consider an economy with two goods and two agents. Agents 1 has an endowment of 0.6 units of good X and 0.6 units of good Y. Whereas agent 2 has an endowment of 0.4 units of both the goods. Utility functions are as follows, $u_1(x, y) = x + 2y$ and $u_2(x, y) = 2x + y$.
$a)$ Model this situation as a bargaining problem, that is, find the utility possibility set and the disagreement point.
$b)$ Find the Nash bargaining outcome and the Kalai-Smorodinsky bargaining outcome.

13. Consider the following class of bargaining problems, $\Sigma = \{(S, \mathbf{0}) \mid S \subset \mathcal{R}^2_+\}$ where S is compact, convex and comprehensive bargaining set and $\mathbf{0}$

is the disagreement payoff. A bargaining outcome is a function $F(S)$, where $F_k(S)$ denotes the allocation of agent k. Consider the following properties,
Strong Monotonicity: F satisfies Strong Monotonicity if $S \subset T$ implies $F_k(S) \leq F_k(T)$ for all k.
Weak monotonicity: Given a bargaining problem S, let $m_k(S)$ denote the maximum utility of agent k in S. F satisfies Weak Monotonicity if $S \subset T$ and $m_k(S) = m_k(T)$ for all k implies $F_k(S) \leq F_k(T)$ for all k.
a) Can you justify Strong Monotonicity?
b) Do you find Weak Monotonicity appealing?
c) Show that if F satisfies Strong monotonicity then it should also satisfy Weak Monotonicity.

14. Suppose that a society consists of two groups U and D, where U is landed elites and D landless labourers. Suppose that a typical education policy is a redistribution from U to D (tax U and subsidise D). Given a redistribution $0 \leq y \leq 5$, wage of D is uniform random variable in the range $[y, 4+y]$ and wage of U is uniform random variable in the range $[5-y, 10-y]$.
a) Explain equal opportunity policy - clearly specify the underlying philosophical position.
b) Find the equal opportunity policy for the above society.

15. Suppose that a society consists of two groups, Red and Blue, of equal population. The society has a per capita education budget of $\frac{1}{2}$. A typical education policy is a division of this budget between Reds and Blues, where x_R and x_B denote per capita education expenditures on Reds and Blues respectively. For each group t, given any policy, earning (w_t) is a strictly increasing function of years of schooling. Given a policy (x_R, x_B), w_R is uniform random variable in the range $[x_R, 2]$ and w_B is uniform random variable in the range $[0.5 + x_B, 2]$.
(a) Find and interpret the equal opportunity policy.
(b) Is there any efficiency loss under the equal opportunity policy?