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Equity, Envy, and Efficiency*

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Consider the problem of dividing a fixed amount of goods among a fixed number of agents. If, in a given allocation, agent i prefers the bundle of agent j to his own, we will say *i envies j*. If there are no envious agents at allocation x, we will say x is *equitable*. If x is both pareto efficient and equitable, we will say x is *fair*.

Section 1 motivates and examines these definitions, and compares this approach to some other theories of normative economics. Section 2 examines the relationship between envy and efficiency and establishes some quite general results concerning the existence of fair allocations.

Section 3 considers the problem of fair allocation of goods and leisure when production is possible. It is found that fair allocations will not in general exist in this case, even under very regular conditions. Accordingly, the concept is generalized in two ways which will exist under weak conditions, and these new concepts are characterized in terms of income and wealth distribution.

Finally, Section 4 considers an extension of the concept of equity where we allow comparisions to be made between coalitions of agents. In this case it is shown that the *only* allocations that are coalition-fair in a large economy are competitive equilibria with equal incomes.

1. THE CONCEPT OF FAIRNESS

What is a fair way to divide society's product? The importance of this question can hardly be denied, but the amount of economic analysis relevant to it is rather small. In this paper I attempt to apply some of the standard tools of theoretical economics to the analysis of certain formal definitions of fairness.

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I begin by considering the case of pure fair division: There is some fixed amount of resources to be divided among n agents. I will define an allocation as *equitable* if no agent prefers some other agent's bundle to his own.[1] If an allocation is both equitable and pareto efficient, I will say it is *fair*.[2] Finally, if some agent does prefer another agents' bundle to his own at a given allocation, I will say that the first agent *envies* the second.

These definitions formalize a recurrent theme in ethical thought: namely, considerations of *symmetry* in distributive justice. The equity comparison allows each agent to put himself in the place of each other agent and then forces him to evaluate the other agent's position on the same terms that he judges his own. Thus it allows an interpersonal comparison of a sort, but it restricts the way in which this comparison can be made; in particular, there can be no "double standard" for evaluating others' positions as compared to one's own position.

Of course this definition can only be a minimal requirement for fairness; after all, the only facts taken into account are the preferences of the agents and the physical amount of goods to be divided. In many cases other facts may be relevant to the fair division problem; examples of such other considerations might be the strengths of the agents preferences, the moral worth of the agents, or the history of how each of the agents contributed to the formation of the original bundle.

But the simplicty and minimal informational requirements of this definition should count as a strength of this approach rather than a weakness. As I expand the problem of fair allocations to include the possibility of production, coalition formation, and so on, the criteria for what should count as a "fair" allocation may certainly change. But we must walk before we can run, and it will pay us to examine the implications of this simple definition in some detail.

Before I proceed to that task, I want to spend a small amount of time comparing this approach to the "standard" approach of specifying a social welfare function of the form $W(u_i(x))$ and choosing a division that maximizes it. The "fairness" of the allocation resulting from this approach depends critically on the particular welfare function used. Furthermore, it is well known that finding a "reasonable" social welfare function may be a very difficult problem; I am referring, of course, to the various impossibility results of social decision theory. (For a good survey of these results see [10].)

Social decision theory views the specification of the social welfare function as a problem in aggregating individual preferences. Its chief

[1] The definition of equity is due to Foley [6].

[2] The definition of fairness is due to Schmeidler and Yaari [12]. Schmeidler and Vind have also considered the related notion of "fair net trades." [13].

results are of the form "There are no reasonable way to aggregate individual preferences."

Why do we get such a pessimistic conclusion from this approach? It seems to me that there are two problems:

(i) Social decision theory asks for too much out of the aggregating process.

(ii) Social decision theory does not put enough into the aggregating process.

Social decision theory asks for too much out of the process in that it asks for an entire *ordering* of the various social states (allocations in this case). The original question asked only for a "good" allocation; there was no requirement to rank all allocations. The fairness criterion in fact limits itself to answering the original question. It is limited in that it gives no indication of the merits of two nonfair allocations, but by restricting itself in this way it allows for a reasonable solution to the original problem.

Social decision theory puts too little into the social decision problem in that we generally allow individual preferences to be defined over the entire set of social states. In the particular problem of distributive justice, this means that individual preferences are defined over entire allocations. I think that this degree of generally contains too little structure to produce any satisfactory *positive* results. The fairness approach, on the other hand, restricts preferences to be defined only on individual bundles and thus allows for a symmetric comparison of the agents' relative positions.

Besides the generally negative results concerning the specifications of such functions, the specific welfare functions that have actually been used are all of the Bergsonian variety; that is where the utility functions were defined only on the individual bundles: $W(u_i(x_i))$. The allocations that maximize such functions have the desirable property of being pareto efficient; however, the restriction to the Bergsonian form eliminates the information available for the "envy" comparison. Welfare functions consistent with the idea of fairness would have a form where utility evaluations of other agents bundles were allowed; that is, the welfare function would have the form of $W(u_i(x_j))$. A specific example would be: $W(x) = \alpha \sum u_i(x_i) - \beta \sum (u_i(x_j) - u_i(x_i))\, \delta_{ij}$, where δ_{ij} is unity if the "envy" term is positive and zero otherwise. The parameters α and β can be interpreted as weighing the relative importance of the "efficiency" and the "equity" considerations.

A. *Comparison to Rawls*

John Rawls has considered in some detail the meaning of the concept of justice. Since many economists are familiar with his work, it may be useful to compare the fairness idea to Rawls' approach.

Rawls argues that the principles of justice in a society should be principles that would be agreed upon by free and rational persons in an "original position" of "ignorance" as to their actual positions in the society in question. The original position is to be regarded only as a hypothetical state. Formally it adds nothing to an analysis of what a just state is; it only gives us a restriction on what type of *reasons* can be given for choosing one particular principle of justice over another. The restriction is of course that the only reasons allowed are reasons that could be appealed to in the original position. Thus Rawls spends considerable effort in delineating exactly what information is available to the agents in the original position.

This, it seems to me, is a well-directed inquiry. For what should count as reasons in a moral discussion is an important and interesting question; although the idea of the original position simply leaves us with the same problem, its particular picturesque description allows for a certain insight. By appealing to the initial anonymity of the agents, Rawls appeals to the same symmetry instinct to which fairness appeals.

When Rawls eventually tries to answer the question of what principles would be chosen in the original position, he arrives at two principles, which I abbreviate as the "equal liberties principle" and the "difference principle." Much of the book is spent in clarifying these principles and analyzing their consequences. To justify the choice of these particular principles of justice, Rawls states that "it is useful as a heuristic device to think of the two principles as the maximin solution to the problem of social justice." [9, p. 152].

Many economists have jumped on this statement as implying that Rawls favors a maximin social welfare function. The arguments against such a maximin welfare function are rather strong, primarily resting on the fact that people are usually not all that pessimistic in their choice behavior [1]. But note that Rawls appeals to the maximin argument only as a *heuristic* principle. His fundamental assertion is that the two principles of justice mentioned before would be chosen; the maximum behavior is only an attempted explanation of *why* they would be chosen.

The question that concerns me here is not how the theory of fairness compares to a maximin social welfare function, but rather whether the theory of fairness could be the outcome of the original position as described by Rawls. It seems to me that it could, and in fact I believe Rawls himself argues to this effect.

Rawls discusses the concept of "envy" in Sections 80 and 81. It is important to take note of his terminology; Rawls thinks of envy as "the propensity to view with hostility the greater good of others even though their being more fortunate than we are does not detract from our advantages" [9, p. 532]. In the particular case of distributive justice I am considering, this definition seems to describe a case where preferences are defined over entire allocations, and increasing the bundle of one agent results in decreasing the utility of the other agents. Hence it is clear that the theory of fairness rules out what Rawls calls envious behavior since preferences are required to be defined on individual bundles.

On the other hand, Rawls does allow that *resentment* is legitimate moral category. For Rawls claims,

"If we resent our having less than others, it must be because we think that their being better off is the result of unjust institutions. Those who express resentment must be prepared to show why certain institutions are unjust..." [9, p. 533].

I believe that envy, as I have defined it, is very similar to Rawls' concept of resentment, for the existence of envy is clear-cut evidence that agents are being treated asymmetrically. In the above quotes, Rawls implies that a just society would be free from resentment.

Hence it would seem that a just allocation of goods in Rawls' sense must satisfy the criterion of fairness as I have defined it.

2. Fair Division

In this section I will present some theorems concerning the problem of fair division previously introduced and discuss some of the relationships between the concepts of equity, envy, and efficiency. We will first restate the previous definitions in somewhat more formal terms.

An allocation x is *weakly efficient* (x *is in* PW) iff there is no feasible allocation y such that $y_i \succ_i x_i$ for all agents i. An allocation x is *strongly efficient* (x *is in* PS) iff there is no feasible allocation y such that $y_i \succsim_i x_i$ for all agents i and there is some agent j such that $y_j \succ_j x_j$. An allocation x is *equitable* iff $x_i \succsim_i x_j$ for all agents i and j. If $x_i \prec_i x_j$, we will say that i *envies* j at the allocation x. If an allocation x is both equitable and strongly efficient, we will say x is *fair*. If the allocation x is equitable but only weakly efficient, we will say x is *weakly fair*.

A fundamental relationship between envy and efficiency is given in the following theorem.

THEOREM 2.1. *If x is a strongly efficient allocation, then there is some agent that envies no one and there is some agent that no one envies.*

Thus there is a "top" and a "bottom" to the set of agents in a strongly efficient allocation. It is possible to extend this partial order to the whole set of agents by disregarding the nonenvious agents and their bundles and considering the resulting allocation; this allocation is still strongly efficient, and thus there are nonenvious agents. (These are the agents who envied only the original nonenvious agents.) We can consider these agents to be the "second best off," and then continue to extend the ordering. Unfortunately, simple examples show that the ordering which comes from disregarding the *unenvied* agents, those at the bottom of the pile, will not in general be consistent with the ordering just described. Nevertheless, it is of some interest to note that we can get a natural measure of how well off each agent is in any strongly efficient allocation.

Moving on to the concept of equity, we recall that a classical notion of equity in the context of a market economy is that of an equal-income competitive equilibrium, which is also, of course, an efficient allocation. It is therefore reassuring to notice that equal-income competitive allocations are indeed fair by our definition.

THEOREM 2.2. *Suppose that preferences are monotonic. Then if (x, p) is a competitive equilibrium with $p \cdot x_i = p \cdot x_j$ for all i and j, then x is fair.*[3]

Interestingly enough, a competitive equilibrium from an *equitable* allocation is not necessarily fair, and not all fair allocations have equal incomes. Furthermore, there will in general be points in the equal division core which are not fair.

A primary concern about the usefulness of the concept of fair allocations is the question of whether they exist in general circumstances. The above theorem gives us an immediate result on this existence question.

THEOREM 2.3. *If preferences are convex and monotonic, then fair allocations exist.*[4]

The primary restriction of the above theorem is that of convexity of preferences. As fair allocations can easily exist in the absence of this condition, the above result is somewhat unsatisfactory.

It turns out that a more general condition for the existence of fair allocations is that the topological structure of the set of efficient allocations be especially simple; that is, that it consist of one piece with no

[3] The first part of Theorem 2.2 is of course Koopmans' first optimality theorem. Since the definitions are slightly different, I have repeated the proof, inserting the necessary changes. The assumptions can be relaxed somewhat.

[4] The idea of Theorem 2.3 is due to Schmeidler and Yaari [12].

The situation is radically different when production is possible. For in this case the agents may contribute differently to the social product, and thus there is an inherent asymmetry to the problem. Indeed, the deepest problems of distributive justice are concerned precisely with this question: How do we divide the social product when agents can contribute differently to the formation of that product? We will try to investigate this question by extending the approach of fair division discussed in the last section.

We will assume that there is a fixed initial bundle of consumption goods w in R_+^k; furthermore, each agent can hold up to one unit of his own leisure. Thus the final bundles held by the ith agent are of the form $(x_i, 1 - q_i)$, where x_i is the ith agent's bundle of commodities, q_i is his amount of labor time, and thus $1 - q_i$ is his amount of leisure time. We will incorporate the technological production possibilities into the analysis by considering the set of all feasible allocations X, a subset of $R^{n(1+k)}$. The definition of strongly efficient allocations is similar to the previous definition: an allocation is equitable iff $(x_i, 1 - q_i) \succsim_i (x_j, 1 - q_j)$ for all agents i and j, and an allocation is fair iff it is both equitable and strongly efficient. Thus, if we have an efficient allocation where each agent (weakly) prefers his consumption–leisure bundle to any other agent's, that allocation is fair.

The problem with this approach is simply this: Fair allocations, as defined above, will not in general exist even in very regular cases. The problem becomes apparent when we examine the proof of Theorem 2.6: for this theorem we need the results of both Theorem 2.4 (that the efficient set is homeomorphic to a simplex) and Theorem 2.1 (that at an efficient allocation there is some agent that no one envies.) There is no problem with Theorem 2.6; the efficient set will still be homeomorphic to a simplex if we assume that (i) zero consumption and zero leisure is the worst possible bundle, and (ii) the set of feasible allocations is *regular*—i.e., it is compact and convex, and if $(x, 1 - q)$ is in X, every allocation that is smaller than $(x, 1 - q)$ is in X.

The problem comes in Theorem 2.1. Surprising as it may seem, it is possible to have strongly efficient allocations where two agents each envy the other. Consider the following two-person two-good example:

$$u_1(x_1, q_1) = \log x_1 + \log (35 - q_1),$$

$$u_2(x_2, q_2) = \log x_2 + \log (25 - q_2),$$

$$x_1 + x_2 = (1/5)\, q_1 + q_2 .$$

Consider the allocation $((6, 5), (10, 15))$. It is easy to check that the marginal rates of substitution equal the marginal rates of transformation so that this allocation is efficient. However,

$$u_1(x_1, q_1) = 6 \times 30 = 180,$$
$$u_1(x_2, q_2) = 10 \times 20 = 200,$$
$$u_2(x_2, q_2) = 10 \times 10 = 100,$$
$$u_2(x_1, q_1) = 6 \times 20 = 120,$$

so that each agent envies the other. Since the crucial relationship between envy and efficiency does not go through to the production case, the proof of Theorem 2.6 does not work. In fact the following economy[8] has no fair allocations at all, even though it exhibits constant returns to scale and homogeneous utility functions:

$$u_1(x_1, q_1) = (11/10)\, x_1 + (1 - q_1),$$
$$u_2(x_2, q_2) = 2x_2 + (1 - q_2),$$
$$x_1 + x_2 = (1/10)\, q_2 + q_1,$$
$$0 \leqslant q_1 \leqslant 1, \qquad 0 \leqslant q_2 \leqslant 1.$$

The intuitive reason for this is clear: Efficiency will always require that agent 1 do all the work and agent 2 compensates him for it by allowing him larger consumption. But in such a situation agent 2 will envy agent 1 because he consumes more of the goods and agent 1 will envy agent 2 because he consumes more leisure.

The fundamental problem here is that agent 2 really "envies" the ability of agent 1 as revealed in any efficient allocation. Since this ability cannot be traded, we cannot hope to get a fair allocation. Similarly, one person might envy another person's talent or good looks. However, there is an important difference between talent and productive ability; I may not be able to produce as fine a painting as Picasso could, but I could produce as many if I just worked more (and lived long enough). In economic activites with a well-defined product an agent with less ability may be able to produce as much as an agent with more ability simply by working longer and harder. It is this type of substitution that will allow us to define another notion of "fairness" in the productive case.

It is also important to notice that this nonexistence is not due solely to the fact that there are different types of labor or different abilities. The effect of different tastes is crucial, for one can show that Theorem 2.8 goes through unchanged so that, if all agents have the same preferences, a fair allocation exists even though agents' abilities may differ.

Apparently, to get a satisfactory notion of fairness in the production

[8] This example is due to Pazner and Schmeidler [7].

context, we will have to change our definition of "equity." We will consider two possible generalizations of the notion of fair allocations, each concentrating on different aspects of the concept of equity.

We will consider production technologies where it makes sense to associate with each agent the amount of goods that agent produces at a given allocation. Thus we can consider bundles of the form $(x_i, 1 - q_i, z_i)$, where x and q are as before and z_i is the amount of all commodities produced by the ith agent, so that z_i is an element of R^k. In what follows we will make the Independence Assumption: that z_i is independent of permutations of the other z_j's. Thus the production of the ith agent may depend on *what* others produce, but not on *who* produces it.

Under this assumption, it makes sense to ask how much j would have to work to produce what i produces at some particular allocation $(x, 1 - q, z)$; this will just be the amount of j's labor necessary to produce the output z_i assuming $z - z_i$ is held constant, and we will denote this amount of labor by $Q_j(z_i)$.

We shall now define an allocation as equitable* iff $(x_i, 1 - q_i) \succsim_i (x_j, 1 - Q_i(z_j))$ for all agents i and j. Of course, if it is impossible for agent i to produce what j produces, $Q_i(z_j)$ will be undefined, and we will regard the equity* condition as being vacuously satisfied for these two agents.

Admittedly this definition is not entirely ethically satisfactory. Perverse cases arise when one agent is the sole producer of some good, since in that case no complaint against him could be justified. It is especially bad if this agent is the sole producer of some good that gives utility only to him! However, in cases where there is a reasonable amount of substitution possibilities between agents' labor, the definition has a certain appeal. It only allows you to complain about another agent's consumption if you are willing to match his contribution to the social product. Otherwise your complaint cannot be considered legitimate. Thus I may "envy" a doctor who only works one day a week doing brain surgery and yet has substantial consumption; but unless I am willing to put in enough labor time to match his production of services—for example, 6 years of medical school required—my complaint against him cannot count as legitimate in the sense of equity*.

This definition does happily provide us with an existence theorem for fair* allocations; for the analog of Theorem 2.1 goes through.

THEOREM 3.1. *If the Independence Assumption is satisfied, and if $(x, 1 - q)$ is a strongly efficient allocation, then there is some agent that envies* no one and some agent that no one envies*.*

And so the existence theorem works.

THEOREM 3.2. *Let X be regular, let preferences be monotonic, and suppose that there are no two allocations in* PW, *which all agents regard as indifferent. Then fair* allocations exist.*

(Of course the analog of Theorem 2.6 concerning the existence of fair sets goes through also.)

Recall that in the case of fair division, a competitive equilibrium from an equal division was fair. An analogous result holds for fair* allocations. Suppose that we have a particularly simple kind of technology where there we can associate with each agent an "ability," a_i, so that if the ith agent works for q_i hours he contributes $a_i q_i$ "labor power." Production then depends only on the amount of "labor power," not on the amount of time worked by agents. In this special case, an allocation is equitable* iff $(x_i, 1 - q_i) \gtrsim_i (x_j, 1 - (a_j/a_i)\, q_j)$ for all i and j. Then we have the following theorem.

THEOREM 3.3. *Suppose that preferences are convex and monotonic; then if we choose an initial endowment where each agent gets w/n of the consumption goods and one unit of his own leisure, the resulting competitive equilibrium is fair*.*

So the "natural" equilibrium, with equal division and no compensation for abilities, has the property of equity*; if any agent preferred some other agent's bundle to his own, he would not be willing to produce what that other agent produces.

The intuition here is clear: If two agents produce the same output, efficiency requires that they be paid the same total amount, even though their wages may differ. Hence, if I prefer to produce what another agent produces and our initial endowments of goods are the same, I should be able to also afford his consumption bundle. This argument also shows how we could extend the theorem to more complex technologies; we only need require that agents evaluate consumption–output bundles rather than consumption–leisure bundles. Then the result should go through for any technology where an individual's output is defined.

We will now discuss the second concept of "fair allocations" that I mentioned earlier. If we have the classical conditions of convexity, monotonicity, and so on, every efficient allocation is a competitive equilibrium for some initial indowment. Thus with each efficient allocation $(x, 1 - q)$ we can associate a competitive price vector (p, r), where p is the price vector of the consumption goods and r is the vector of wage rates for the various kinds of labor. We can then associate with each agent an implicit income $y_i = (p, r) \cdot (x_i, 1 - q_i)$, where each agent's leisure is evaluated at his particular wage rate. We will then say that an allocation is income-fair iff $y_i = y_j$ for all agents i and j.

It is easy to prove that income-fair allocations will always exist; we simply divide the total consumption–leisure bundle up evenly by giving each agent an equal share of all consumption goods and an equal share of each other agent's leisure time and then trade to a competitive equilibrium. Since a given agent presumably only cares about his own leisure, pareto efficiency implies that no agent will hold any other agent's leisure time at the competitive equilibrium. Stated formally:

THEOREM 3.4. *Suppose that preferences are convex and monotonic; then if we choose an initial endowment where each agent gets w/n of the consumption goods and l/n of each agent's leisure, the resulting competitive equilibrium is income fair.*[9]

Theorems 3.3 and 3.4 demonstrate the fundamental ambiguity of equity in the production case: Should we view labor time on an *individual* basis and give each agent the same amount of his individual leisure, or should we view labor time on a *social* basis and give each agent the same amount of "labor power"? In the first case we have equal wealth but no correction for ability, and in the second case we have equal incomes and total correction for ability.

4. Coalition Fairness

The concept of equity allows comparisons between agents to be made only on an individualistic basis; each agent compares his own bundle to the bundle of each of the other agents. A stronger notion of equity might be one in which comparisons were allowed between *groups* of agents. For example, each group of agents could compare its aggregate bundle to the aggregate bundle of any other group of the same size. A concept of this type will be called *coalition fairness*, or, more briefly, c-fairness. Before we can state the formal definitions, we will need to set up some machinery.

We consider a collection of agents, C, which may be finite or infinite, and the set of coalitions of agents in C, $\mathscr{C}$, which we will assume to be a sigma-algebra of C. We have a measure on C, $\lambda : \mathscr{C} \to R_+$, which measures the size of a coalition. If C is finite, λ will just be the normalized counting measure, while if C is a continuum we will assume that λ is an atomless measure, normalized so that $\lambda(C) = 1$.

[9] Pazner and Schmeidler prove a similar result in [8].

Suppose that we have some fixed bundle of goods, w in R_+^k, to be divided among the agents of C. An allocation α will be a measure, $\alpha : C \to R^k$, $\alpha(C) = w$, that assigns to each coalition its aggregate bundle. [In the finite case it is sometimes convenient to use the standard definition of allocation as an nk vector $x = (x_1, ..., x_n)$.] Coalitions are assumed to have a preference ordering over possible allocations; we write "coalition A prefers α to β" as "$\alpha \succ_A \beta$." Preferences are interpreted as "$\alpha \succ_A \beta$" means "(almost) all agents in A prefer α to β" (see Debreu [4]).

Finally, we denote by $R_e(\alpha)$ the e range of α : $R_e(\alpha) = \{\alpha(A) : \lambda(A) = e\}$, and by $P_e(\alpha)$ the e-preferred bundles, $P_e(\alpha) = \{\beta(A) : \beta \succ_A \alpha$ and $\lambda(A) = e\}$. $R_e(\alpha)$ consists of all aggregate bundles held by coalitions of size e in the allocation α, and $P_e(\alpha)$ consists of all aggregate bundles that can be distributed among the agents of a coalition of size e to form a preferred (partial) allocation.

We can now succinctly state the definition of c-fairness.

DEFINITION. An allocation α is *c-fair* iff $P_e(\alpha) \cap R_d(\alpha) = \varnothing$ for $0 \leqslant d \leqslant 1$, $0 < e \leqslant 1$, $d \leqslant e$.

In other words, an allocation α is c-fair iff no coalition of size e prefers any aggregate bundle of any coalition of the same size or smaller.[10]

Notice that this definition requires that a c-fair allocation be (weakly) pareto efficient, since for $e = \lambda(C) = 1$ we require that $P_1(\alpha) \cap R_1(\alpha) = \varnothing$, so that there is no way to rearrange the allocation to make every agent better off.

The first question is, of course, when do c-fair allocations exist? Since the concept of c-fairness includes the concept of fairness, every c-fair allocation must certainly be fair but not vice verse. However, we do have the following.

THEOREM 4.1. *If α is a competitive equilibrium with initial endowment $i(A) = \lambda(A)\, w$ for all A in $\mathscr{C}$, then α is c-fair.*[11]

In other words, equal-income competitive equilibria are c-fair.

Are there any other c-fair allocations? In general, the answer is "yes"; it is easy to construct examples in the two-person two-good Edgeworth box case. However, in an important sense, equal-income competitive equilibria are the only c-fair allocations for a fair division problem with many agents.

There are two approaches to formalizing and demonstrating this proposition: one is by considering a replicated economy in the manner of

[10] The definition of c-fairness is due to Vind [15].

[11] Theorem 4.1 was stated and proved by Vind in [15]. A quite general theorem on the existence of a competitive equilibrium with a continuum of agents may be found in [3].

Debreu and Scarf [5]; the other is by considering an economy with an atomless continuum of agents in the manner of Vind [14, 15]. We begin with the replicated economy.

Suppose that we have only a finite number of types of agents $i = 1,..., m$. Here, by saying that two agents are of the same type, I only mean that they have the same preferences. We will assume that these preferences are strongly convex, continuous, and insatiable, as do Debreu and Scarf [5]. Under these assumptions, it is clear that c-fair allocations must have the equal-treatment property; that is, if x is c-fair, then agents of the same type must get the same bundle. (This is proved formally in Lemma 4 in the appendix to section 4).

Now consider a given allocation $x = (x_1 ,..., x_m)$ which is c-fair, and let the economy replicate; that is, consider an economy with r agents of each type and r times the original bundle w to be divided among them. (Admittedly this is not quite the fair division problem, since the bundle to be divided keeps increasing as the economy replicates; however, since the number of agents keeps growing also, the problem is essentially the same from the viewpoint of an individual agent.)

Since c-fair allocations have the equal treatment property, we only need to consider their projection into the m-type space, so clearly we will get no more of them in the replicated economy; the question is, will we get fewer? The answer is "yes"; in fact, we have the following theorem.

THEOREM 4.2. *If $(x_1 ,..., x_m)$ is c-fair for all replications r, then it is a competitive equilibrium with equal incomes—that is, with initial endowment $\omega_i = w/m$ for $i = 1,..., m$.*

In the two-person two-good Edgeworth box case there is a simple diagrammatic argument which is presented as Example 4.1 (see the appendix to Section 4).

If c-fair allocations are equal-income competitive allocations in the limit, we would expect that to be the case when we start out with a continuum of agents. As usual, we can also dispense with the assumption of convexity of preferences in the continuum case.

First we note the following.

THEOREM 4.3. *If $(C, \mathscr{C}, \lambda)$ is an atomless economy and x is a c-fair allocation, then x is atomless.*[12]

Thus we only need to consider atomless allocations as candidates for c-fair allocations. We now have the following theorem.

[12] Theorem 4.3 was stated by Vind in [15].

THEOREM 4.4. *If $(C, \mathscr{C}, \lambda)$ is an atomless economy, then α is c-fair implies that α is a competitive equilibrium with $i(A) = \lambda(A)\, w$ for all A in $\mathscr{C}$.*[13]

The implication of Theorems 4.2 and 4.4 seems to be that, if we wish to divide things fairly among a large number of agents so that the allocation is stable with respect to envy among coalitions, then our only choice is an allocation that is a competitive equilibrium with equal incomes.

It has been suggested to me that a more general and more symmetric definition of coalition fairness might be one where each coalition compares its "average" bundle to the "average" bundle of each other coalition.[14] In this way, each coalition can consider the aggregate bundles of all other coalitions, not just coalitions of the same size or smaller. We can formalize this notion in the following definition:

DEFINITION. An allocation α is c′-fair iff $P_e(\alpha) \cap (e/d)\, R_d(\alpha) = \varnothing$ for all $0 < d \leqslant 1$ and $0 < e \leqslant 1$.

Thus each coalition examines the aggregate bundle of each other coalition, weighting the aggregate bundle by their relative sizes; if any such weighted bundle is preferred by the examining coalition, the allocation cannot be c′-fair.

It is clear that the notion of c′-fairness implies the notion of c-fairness, and one would suspect that it is a strictly stronger notion; that is, that there are c-fair allocations that are not c′-fair. However, that is not the case.

THEOREM 4.5. *Let $(C, \mathscr{C}, \lambda)$ be an atomless economy; then an allocation α is c′-fair if and only if it is c-fair.*

APPENDIX TO SECTION 2

The set of feasible allocations will be denoted by

$$X = \left\{x \text{ in } R^{nk}: \sum x_i \leqslant w\right\}.$$

We assume $w \gg 0$. We will assume that each agent i has preferences $\succsim_i$ defined on the commodity space R_+^k and that these preferences are

[13] Theorem 4.4 is a generalization of a theorem by Vind in [15], which had required the additional hypothesis that the dimension of $\bigcup R_e(\alpha)$ be k. Since this hypothesis is economically meaningless, the present version is a substantial improvement.

[14] Andreu Mas-Collel and Michael Scriven made this suggestion.

complete, transitive, and closed. Thus these preferences can be represented by continuous utility functions $u_i : R_+^{k} \to R$.

Preferences are said to be convex iff for $x' \neq x$, $x' \gtrsim_i x$ implies

$$ax + (1 - a)\,x' \gtrsim_i x \text{ for } 0 \leqslant a \leqslant 1.$$

Preferences are strictly convex iff, in the above case, $ax + (1 - a)\,x' \succ_i x$ for $0 < a < 1$. Preferences are monotonic iff $x \leqslant x'$ and $x \neq x'$ implies $x \prec_i x'$.

An allocation x is a competitive equilibrium with prices p and initial endowment ω iff $x' \succ_i x_i$ implies that $p \cdot x' > p \cdot x_i$ and $p \cdot x_i \leqslant p \cdot \omega_i$ for $i = 1,..., n$.

$$S^{n-1} = \left\{x \text{ in } R_+^{n} : \sum x_i = 1\right\} \text{ — the unit simplex.}$$

$$\operatorname{int} S^{n-1} = \{x \text{ in } S^{n-1} : x_i > 0 \text{ for all } i = 1,..., n\}.$$

$$S_j^{n-1} = \{x \text{ in } S^{n-1} : x_j = 0\} \text{ — the } j\text{th side of } S^{n-1}.$$

THEOREM 2.1. *If x is a strongly efficient allocation, then there is some agent that envies no one and there is some agent that no one envies.*

Proof. Suppose to the contrary that each agent envies some other agent. Then, since there are only a finite number of agents, there is some cycle $(i_1, ..., i_m)$ such that i_1 envies i_2 envies $\cdots$ envies i_m envies i_1. Then the allocation x' where each agent in the cycle receives the bundle of the agent he envies and agents outside the cycle remain the same is feasible and dominates the original allocation x. This contradicts the fact that x is strongly efficient.

The proof of the second assertion is similar. ∎

THEOREM 2.2. *Suppose that preferences are monotonic. Then if (x, p) is a competitive equilibrium with $p \cdot x_i = p \cdot x_j$ for all i and j, x is fair.*

Proof. First we will show that x is strongly efficient. Assume not; then there is some allocation y such that $y_i \gtrsim_i x_i$ for $i = 1,..., n$ and, for some j, $y_j \succ_j x_j$. We can choose y so that it itself is strongly efficient.

For each j that strictly prefers y_j to x_j we have $p \cdot y_j > p \cdot x_j$. Consider some agent i that is indifferent between x_i and y_i, if any such agents exist. If $p \cdot y_i < p \cdot x_i$, the agent could afford to buy a slightly more expensive bundle, and by monotonicity he could find a bundle strictly better than x_i, contradicting the fact that x_i is a competitive equilibrium. Thus

$$p \cdot y_i \geqslant p \cdot x_i,$$

so that

$$\sum p \cdot y_i > \sum p \cdot x_i .$$

Since preferences are monotonic,

$$\sum y_i = \sum x_i = \sum \omega_i ,$$

but this gives us

$$p \cdot \sum \omega_i > p \cdot \sum \omega_i .$$

This is a contradiction, so it must be that x is strongly efficient.

To show that x is also equitable, we suppose that agent i envies agent j: $x_i <_i x_j$. Then by definition of competitive equilibrium, $p \cdot x_i < p \cdot x_j = p \cdot x_i$, which is a contradiction. ∎

THEOREM 2.3. *If preferences are convex and monotonic, then fair allocations exist.*

Proof. Let the initial allocation ω be defined by $\omega_i = w/n$. Under the assumptions of the theorem, standard existence proofs show that a competitive equilibrium (x, p) will exist, be in PS, and $p \cdot x_i = p \cdot x_j = p \cdot (w/n)$. By Theorem 2.2 this will be fair. ∎

THEOREM 2.4. *Suppose that every agent prefers any nonzero bundle to the zero bundle; then* $u(\mathrm{PW}_+)$ *is homeomorphic to the interior of an* $(n - 1)$*-dimensional simplex. Furthermore, if there are no two allocations in* PW_+ that *all agents regard as indifferent, then* PW_+ *is itself homeomorphic to the interior of an* $(n - 1)$*-dimensional simplex.*

Proof. We will, without loss of generality, normalize the utility functions so that $u_i(0) = 0$ and $u_i(w/n) = 1$. The proof procedes in a number of steps.

STEP 1. If x is feasible, $x_i \neq 0$ for any i, and x is not in PW, then there exists a feasible allocation z and a real number $t > 1$ such that $u_i(z_i) = tu_i(x_i)$ for all $i = 1,..., n$.

Proof. If x is not in PW, then there exists some feasible y such that $u_i(y_i)/u_i(x_i) > 1$ for $i = 1,..., n$. Let $t = \min_i u_i(y_i)/u_i(x_i)$. The functions $f_i : [0, 1] \to R$ defined by $f(e) = u_i(ey)/u_i(x)$ are continuous, $f_i(1) \geqslant t$, $f_i(0) = 0 < t$. Therefore, by the intermediate value theorem, there is a set of e_i' such that $f_i(e_i') = t$; the allocation defined by $z_i = e_i' y_i$ is fesiable and satisfies the above requirements.

STEP 2. Let $p : u(\mathrm{PW}_+) \to \operatorname{int} S^{n-1}$ be defined by $p(u) = u/\sum u_i$. The function p is certainly continuous on $u(\mathrm{PW}_+)$, since the denominator cannot vanish. Furthermore, I claim it is one to one on $u(\mathrm{PW}_+)$.

Proof. Assume not; then there exists u, v in $u(\mathrm{PW}_+)$ such that

$$u/\sum u_i = v/\sum v_i = z \text{ in int } S^{n-1}.$$

Therefore the u and v are scalar multiples of each other with no zero coordinates, so that one is strictly greater than the other, which contradicts the fact that both are in $u(\mathrm{PW}_+)$.

STEP 3. The set $T(y) = \{t \text{ in } R\colon ty \text{ is in } u(X)\}$ contains a nonzero element for each y in S^{n-1}.

Proof. The set of functions $g_i : [0, 1] \to R$ defined by

$$g_i(e_i) = u_i(e_i(w/n))$$

are continuous, $g_i(1) = 1$, $g_i(0) = 0$. Applying the intermediate value theorem again, we find that for y in S^{n-1} there exists e_i' in $[0, 1]$ such that $u_i(e_i'(w/n)) = y_i$ and the allocation $e_i'(w/n)$ for $i = 1,\dots, n$ is certainly feasible. Therefore 1 is in $T(y)$.

STEP 4. The function $p : u(\mathrm{PW}_+) \to \operatorname{int} S^{n-1}$ is onto.

Proof. Given y in int S^{n-1}, the set $T(y)$ defined above is compact and nonempty. Then there exists $0 \neq t' =$ the maximum t such that t is in $T(y)$. Suppose that $t'y$ is not in $u(\mathrm{PW}_+)$; then by Step 1 there is some t'' such that $t''t'y$ is in $u(X)$. But then $t''t' > t'$, which contradicts the maximality of t'.

STEP 5. p^{-1} is continuous on int S^{n-1}.

Proof. Let K be an arbitrary closed set in $u(\mathrm{PW}_+)$. Then K is compact since it is a subset of a compact set—namely, $u(\mathrm{PW})$. For p^{-1} to be continuous, we need $(p^{-1})^{-1}(K)$ to be closed. Since p is one to one and onto, this set is just $p(K)$, which is closed by compactness of K.

STEP 6. p is a homeomorphism between $u(\mathrm{PW}_+)$ and int S^{n-1}.

Proof. It is one to one, continuous, and onto. Furthermore, if there are no two allocations x and y in PW_+ such that $u_i(x_i) = u_i(y_i)$ for $i = 1,\dots, n$ the map u restricted to PW_+ will be one to one. It is clearly continuous and onto, so composing it with p will give us a homeomorphism between PW_+ and int S^{n-1}. ∎

COROLLARY. *If preferences are strongly monotonic and strictly convex, then* PS = PW *and both are homeomorphic to an* $(n-1)$*-dimensional simplex.*

Proof. That PW contains PS is obvious. We will show that PS contains PW. Suppose the allocation x is not in PS; then there is some allocation y such that $u_i(y_i) \geqslant u_i(x_i)$ for all i, and $u_j(y_j) > u_j(x_j)$ for some j. By continuity we can remove a positive fraction $1-\theta$ of all commodities from y_j and still have $u_j(\theta y_j) > u_j(x_j)$. Then define an allocation z by $z_j = \theta_{y_j}$, $z_i = y_i + (1-\theta)\, y_j/(n-1)$ for $i = 1,..., n$, $i \neq j$. The allocation z has the property that $u_i(z_i) > u_i(x_i)$ for all $i = 1,..., n$, so that x is not in PW.

We now need to show that there are no two allocations in PW which all agents regard as indifferent. Suppose that x and y are two such allocations; then $\frac{1}{2}x + \frac{1}{2}y$ is feasible, at least as good for all agents, and strictly preferred to both x and y by agents for whom $x_i \neq y_i$, which contradicts the efficiency of x and y.

The fact that the homeomorphism p is one to one and onto on the boundary of PW can be verified from the fact that PS = PW and the steps of the theorem. ∎

THEOREM 2.5. *If preferences are monotonic and there are no two allocations in* PW *which all agents regard as indifferent, then fair allocations exist.*

Proof. By the remarks in Step 6 of Theorem 2.4, we see that u is a homeomorphism between PW_+ and $u(\text{PW}_+)$. In Theorem 2.6 we see that the intersection of the $u(M_j)$'s is nonempty, and thus the intersection of the M_j's is nonempty. Any allocation in this intersection is fair. ∎

LEMMA (*Knaster, Kuratowski, and Mazurkiewicz*). *Let* $M_1,..., M_n$ *be a family of closed subsets of* S^{n-1} *with the property that the* jth *face of* S^{n-1} *is contained in* M_j, *and that* S^{n-1} *is contained in the union of the* M_j. *Then the intersection of the* M_j *is nonempty.*

Proof. See Scarf [10, p. 68]. ∎

THEOREM 2.6. *If preferences are monotonic, then fair sets exist.*

Proof. Define the set of allocations where no agent envies agent j: $M_j = \{x \text{ in PS}: u_i(x_i) \geqslant u_i(x_j) \text{ for all } i = 1,..., n\}$. Then the union of these sets for $j = 1,..., n$ covers PS by Theorem 2.1, and by the Corollary to Theorem 2.4 PW = PS. Since the functions u_i are continuous, M_j is closed, and, since any bundle is preferred to the zero bundle, M_j contains all allocations in PS where $x_j = 0$.

Now take the image of these sets M_j under the map pu : PS $\to S^{n-1}$ defined by $\text{pu}(x) = (u_1(x_1) \cdots u_n(x_n))/\sum u_i(x_i))$. The sets pu$(M_j)$, $j = 1,..., n$, satisfy the hypotheses of the lemma of Knaster, Kuratowski, and Mazurkiewicz, so that their intersection is nonempty.

Let z be in this intersection; then $z \gg 0$ since, if $z_i = 0$ for some i, there would be some j such that $z_j > 0$ and therefore $p^{-1}(z)$ would not be in $u(M_j)$. Since the points where $z \gg 0$ form the interior of S^{n-1}, p is a homeomorphism on such points, so that the intersection of the $u(M_j)$ is nonempty for $j = 1,..., n$. Let v be in this intersection and consider the set $u^{-1}(v)$, which I claim is a fair set, F.

This is true because

(i) $u_i(x_i) = u_i(y_i)$ for all x, y in F, $i = 1,..., n$.

(ii) Suppose that $u_i(x_j) > u_i(x_i)$, so that x is not in M_j for some x in $F = u^{-1}(v)$. But v is in $u(M_j)$, so there must be some other allocation y in $u^{-1}(v)$ that is in M_j, which means $u_i(y_i) \geq u_i(y_j)$ for $i = 1,..., n$ and $u_i(y_i) = u_i(x_i)$ for $i = 1,..., n$. ∎

THEOREM 2.7. *If the preferences of agent i are identical with those of j and both are strictly convex, then, if x is a fair allocation, $x_i = x_j$.*

Proof. Assume that $x_i \neq x_j$, and consider the allocation $z = (1/2)\, x_i + (1/2)\, x_j$. Since x is fair and i and j have the same preferences, $x_i \sim_i x_j$ and $x_j \sim_j x_i$, so that $z \succ_i x_i$ and $z \succ_j x_j$. Since giving z to both i and j is feasible, this contradicts strong efficiency. ∎

THEOREM 2.8. *If any bundle is preferred to the zero bundle and all agents have identical preferences, then weakly fair allocations exist.*

Proof. If an allocation is to be fair in these circumstances, it must give equal utility to each agent and also be weakly efficient. Let pu be the map defined in Theorem 2.4, Step 2; then $(\text{pu})^{-1}(1/n,..., 1/n)$ is a set of weakly efficient allocations with equal utilities. ∎

EXAMPLE 2.1. Monotonicity does not imply the existence of fair allocations. (However, the allocations x and y form a fair set.)

To see that there are no fair allocations in Fig. 1, imagine a point such as x_0 or y_0 that moves along one of the components of PS. As x_0 moves

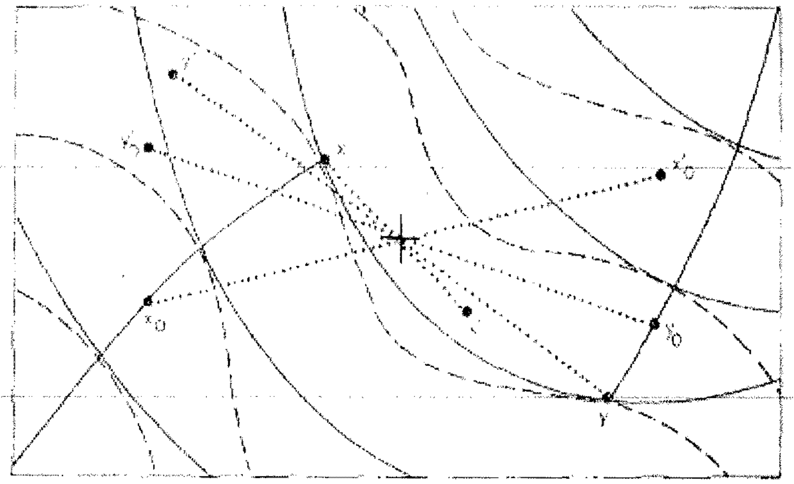

FIGURE 1.

along the left component, its swap x_0' always lies on a higher indifference curve than does x_0, showing that x_0 cannot be fair. The point y_0 behaves similarly.

APPENDIX TO SECTION 3

THEOREM 3.1. *If the Independence Assumption is satisfied and if $(x, 1-q)$ is a strongly efficient allocation, then there is some agent that envies* no one, and some agent that no one envies*.*

Proof. The proof is similar to the proof of Theorem 2.1. Suppose that each agent envies* some other agent. Since there are only a finite number of agents, there must be some cycle. Performing the "swap" among the agents in the cycle is feasible because of the definition of envy* and the Independence Assumption; the resulting allocation dominates the original one, contradicting efficiency. ∎

THEOREM 3.2. *Let X be regular, let preferences be monotonic, and suppose that there are no two allocations in* PW *which all agents regard as indifferent. Then fair* allocations exist.*

Proof. The proof is similar to those of Theorems 2.5 and 2.6. The first hypotheses allow the proof of Theorem 2.4 to work in this case, which makes the efficient set homeomorphic to a simplex. Since Theorem 4.1 provides the analog to Theorem 2.1, the application of the Knaster–Kuratowski–Mazurkiewicz Lemma can procede as before. ∎

THEOREM 4.3. *Suppose that preferences are convex and monotonic; then, if we choose an initial endowment where each agent gets w/n of the consumption goods and one unit of his own leisure, the resulting competitive equilibrium is fair*.*

Proof. The assumptions imply the existence of a competitive equilibrium with prices p and wages r. We can normalize the wage of labor with ability of unity to have $r_i = 1$; since in the competitive equilibrium all wages will be proportional to ability in this case, the normalization will make $r_i = a_i$ for $i = 1,..., n$. By the definition of competitive equilibrium, we have

$$(p, a_i) \cdot (x_i, 1 - q_i) = p \cdot w/n + a_i,$$

or, by rewriting,

$$p \cdot x_i - a_i q_i = p \cdot w/n \qquad \text{for} \quad i = 1,..., n.$$

It is clear that $(x, 1 - q)$ is efficient; assume then that some agent i envies* some agent j. Then

$$(x_i, 1 - q_i) <_i (x_j, 1 - (a_j/a_i) q_j),$$

which implies

$$(p, a_i) \cdot (x_i, 1 - q_i) < (p, a_i) \cdot (x_j, 1 - (a_j/a_i) q_j).$$

Expanding and substituting, we get

$$p \cdot w/n = p \cdot x_i - a_i q_i < p \cdot x_j - a_j q_j = p \cdot w/n$$

which gives us the contradiction. ∎

EXAMPLE 3.1. The following economy has no fair allocations:

$$u_1(x_1, q_1) = (11/10)\, x_1 + (1 - q_1),$$
$$u_2(x_2, q_2) = 2x_2 + (1 - q_2),$$
$$x_1 + x_2 = q_1 + (1/10)\, q_2 \qquad \text{for} \quad 0 \leqslant q_1 \leqslant 1,\ 0 \leqslant q_2 \leqslant 1$$

Proof.

(a) In any efficient allocation we must have

(i) $q_1 = 1$, since if q_1 were strictly less than 1 we would have $u_1(x_1 + (1 - q_1), 0)$ strictly greater than $u_1(x_1, q_1)$, and that bundle would be feasible for agent 1;

(ii) $q_2 = 0$, since if q_2 were strictly greater than 0 we would have $u_2(x_2 - e/10, q_2 - e)$ strictly greater than $u_2(x_2, q_2)$, and that bundle would be feasible for agent 2, for small enough e.

(b) If $x_2 = 0$ we would have

$$u_2(x_1, q_1) = 2 > 1 = u_2(x_2, q_2),$$

which is certainly not fair.

(c) We are left which the case where the allocation is of the form $(x_1, 1), (x_2, 0))$ with $x_1 + x_2 = 1$. For an allocation of this form to be fair, we must have

$$u_1(x_1, q_1) \geqslant u_1(x_2, q_2),$$
$$(11/10)\, x_1 \geqslant (11/10)\, x_2 + 1,$$

and

$$u_2(x_2, q_2) \geqslant u_2(x_1, q_1),$$
$$2x_2 + 1 \geqslant 2x_1 .$$

But these two inequalities, along with the equality $x_1 + x_2 = 1$, imply that $3/4 \geqslant x_1 \geqslant 21/22$, which is a contradiction. ∎

Appendix to Section 4

Definitions.

An allocation α is in the (equal division) core iff there is no allocation β and coalition B such that $\beta \succ_B \alpha$ and $\beta(B) = \lambda(B)\, w$ for $\lambda(B) > 0$.

An allocation α is a competitive equilibrium from $i(A) = \lambda(A)\, w$ iff there exists a price vector p in R^k such that $p \cdot \alpha(A) = p \cdot i(A)$ for all A in $\mathscr{C}$ and $p \cdot x > p \cdot i(A)$ for all x in $P_e(\alpha)$, A in $\mathscr{C}$ such that $\lambda(A) = e > 0$.

A coalition A is an atom for a measure μ iff $\mu(A) > 0$ and $B \subset A$ implies that $\mu(B) = \mu(A)$ or $\mu(B) = 0$.

Assumptions on Preferences

(a) In the replication case we make the assumptions of Debreu and that Scarf [5]: namely,

(i) *Insatiability.* Given a commodity bundle x, we assume there is a commodity bundle x' such that $x' \succ_i x$.

(ii) *Strong convexity.* Let x' and x be arbitrary commodity bundles, $x' \neq x$, $x' \succ_i x$, and let $0 < a < 1$. We assume that

$$ax + (1 - a)\, x' \succ_i x.$$

(iii) *Continuity.* We assume that $\{x: x \succsim_i x'\}$ and $\{x: x' \succsim_i x\}$ are closed.