Delhi School of Economics
Course 001: Microeconomic Theory
Solutions to Problem Set 3

1. (a) The firm solves

$$\min_{L,K} wL + rK \quad \text{subject to } L^\alpha K^\beta = y$$

The Lagrangian is

$$wL + rK + \lambda\left[y - L^\alpha K^\beta\right]$$

This yields the FOCS

$$\begin{aligned} w &= \lambda\alpha L^{\alpha-1}K^\beta \\ r &= \lambda\beta L^\alpha K^{\beta-1} \\ y &= L^\alpha K^\beta \end{aligned}$$

From the first two equations, by dividing, you get

$$\frac{\alpha K}{\beta L} = \frac{w}{r}$$

Using this in the production function, we get the conditional input demands

$$\begin{aligned} L &= \left(\frac{\alpha r}{\beta w}\right)^{\frac{\beta}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} \\ K &= \left(\frac{\beta w}{\alpha r}\right)^{\frac{\alpha}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} \end{aligned}$$

Therefore the cost function is

$$\begin{aligned} c(y) &= wL + rk \\ &= Ay^{\frac{1}{\alpha+\beta}} \end{aligned}$$

where

$$A = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] w^{\frac{\alpha}{\alpha+\beta}} r^{\frac{\beta}{\alpha+\beta}}$$

The supply function is given by $p = c'(y)$, i.e.,

$$\frac{A}{(\alpha+\beta)} y^{\frac{1-\alpha-\beta}{\alpha+\beta}} = p$$

which can be inverted to write

$$y(p) = \left[\frac{p(\alpha+\beta)}{A}\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

(b) Here, for any production target y, there is a corner solution, i.e., the firm uses either only L or only K. More precisely,

$$\begin{aligned} (L, K) &= \left(\frac{q}{a}, 0\right) \text{ if } \frac{a}{b} > \frac{w}{r} \\ &= \left(0, \frac{q}{b}\right) \text{ if } \frac{a}{b} < \frac{w}{r} \end{aligned}$$

Therefore the cost function is

$$\begin{aligned} c(q) &= \frac{wq}{a} \text{ if } \frac{a}{b} > \frac{w}{r} \\ &= \frac{rq}{b} \text{ if } \frac{a}{b} < \frac{w}{r} \end{aligned}$$

The firm's average and marginal costs are constant and given by $\min\left\{\frac{w}{a}, \frac{r}{b}\right\}$. Therefore the supply function is given by

$$\begin{aligned} q(p) &= 0 \text{ if } p < \min\left\{\frac{w}{a}, \frac{r}{b}\right\} \\ &\in [0,\infty) \text{ if } p = \min\left\{\frac{w}{a}, \frac{r}{b}\right\} \\ &= \infty \text{ if } p > \min\left\{\frac{w}{a}, \frac{r}{b}\right\} \end{aligned}$$

(c) The firm will always choose L and K such that

$$\frac{L}{a} = \frac{K}{b} = y$$

Therefore its cost function is

$$c(y) = wL + rK = (wa + rb)y$$

Since marginal and average costs are constant at $wa + rb$, we have the horizontal supply function

$$\begin{aligned} q(p) &= 0 \text{ if } p < wa + rb \\ &\in [0,\infty) \text{ if } p = wa + rb \\ &= \infty \text{ if } p > wa + rb \end{aligned}$$

2. See example 3.1, 3.3 and 3.5 in Jehle and Reny.

3. (a) The profit maximization problem is

$$\max_{l,k} p l^{\frac{1}{4}} k^{\frac{1}{2}} - wl - rk$$

The FOC are

$$\begin{aligned} \frac{1}{4} l^{-\frac{3}{4}} k^{\frac{1}{2}} &= \frac{w}{p} \\ \frac{1}{2} l^{\frac{1}{4}} k^{-\frac{1}{2}} &= \frac{r}{p} \end{aligned}$$

Dividing we get

$$\frac{2k}{l} = \frac{w}{r}$$

Using this above we get the unconditional input demand functions:

$$\begin{aligned} l &= \frac{p^4}{1024 w^2 r^2} \\ k &= \frac{p^4}{2048 w r^3} \end{aligned}$$

Using these in the production function, we get the firm's supply function:

$$y = \frac{p^3}{256 wr}$$

(b) The firm solves

$$\max_{y_1, y_2} p(y_1 + y_2) - a y_1^2 - b y_2^2$$

From the FOC, we get

$$y_1 = \frac{p}{2a} \text{ and } y_2 = \frac{p}{2b}$$

Therefore the supply is

$$y = \frac{p(a+b)}{2ab}$$

4. (a) Both are homogeneous of degree $\frac{1}{2}$, and hence represent decreasing returns to scale.

   (b) First, derive the cost function for each individual factory. In factory 1, $l_1 = k_1 = q_1^2$. Therefore

$$c_1(q_1) = (w+r)q_1^2$$

   In factory 2, $l_2 = k_2 = \frac{q_2^2}{4}$. Therefore

$$c_2(q_2) = \frac{1}{4}(w+r)q_2^2$$

   Optimum quantities are given by $p = c_1'(q_1)$ and $p = c_2'(q_2)$. This yields

$$\begin{aligned} q_1 &= \frac{p}{2(w+r)} \\ q_2 &= \frac{2p}{(w+r)} \end{aligned}$$

   Hence the total supply is

$$q = \frac{5p}{2(w+r)}$$

   Input demand functions are

$$\begin{aligned} l &= l_1 + l_2 = q_1^2 + \frac{1}{4}q_2^2 = \frac{5p^2}{4(w+r)^2} \\ k &= l = \frac{5p^2}{4(w+r)^2} \end{aligned}$$

   (c) Let $k_1 = k$ and $k_2 = 1-k$. Then optimal labour choice is also $l_1 = k$ and $l_2 = 1-k$. Note that capital costs are sunk, only labour costs matter. The firm solves

$$\max_k p\left[\sqrt{k} + 2\sqrt{1-k}\right] - w(k+1-k)$$

   This is maximized at $k = \frac{1}{5}$. Therefore, the firm's supply is

$$q = \sqrt{\frac{1}{5}} + 2\sqrt{\frac{4}{5}} = \sqrt{5}$$

   and the total demand for labour is 1 unit, the same as available capital. The one caveat is maximized profits should be non-negative, which is true if

$$\sqrt{5}p \geq w$$

   Otherwise, the firm should shut down the factories.

5. (a) For each firm supply is given by $p = c'(q)$ and hence $q = \frac{p}{4}$. Also note that the firm's average cost is minimized where average and marginal costs are equal, i.e.,

$$\frac{c(q)}{q} = c'(q) \Rightarrow \frac{8}{q} + 2q = 4q \Rightarrow q = 2$$

   Hence the min average cost is 8. The supply curve is then

$$\begin{aligned} q &= \frac{p}{4} \text{ if } p \geq 8 \\ &= 0 \text{ otherwise} \end{aligned}$$

   Given there are 4 firms, market supply is given by

$$\begin{aligned} Q &= p \text{ if } p \geq 8 \\ &= 0 \text{ otherwise} \end{aligned}$$

Equating demand and supply:

$$p^* = 20 - p^* \Rightarrow p = 10 \text{ and hence } Q^* = 10$$

Since this is greater than 8, firms will indeed supply the requisite amount since they are making positive profits.

(b) In the long run, profits are 0, hence $p = AC_{\min} = 8$. Then, from the demand function, $Q = 12$. Each firm produces 2 units (that is the quantity where average cost is lowest). Therefore there are 6 firms in the long run equilibrium.

(c) This increases $AC_{\min}$ from 8 to 10, hence long run $p = 10$, $Q = 10$. The number of firms in the industry is 5.