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Source: *The Quarterly Journal of Economics*, Vol. 92, No. 4 (Nov., 1978), pp. 671-687
Published by: Oxford University Press
Stable URL: http://www.jstor.org/stable/1883182

EGALITARIAN EQUIVALENT ALLOCATIONS: A NEW CONCEPT OF ECONOMIC EQUITY*

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FOREWORD

The conceptual difficulties involved in the quest for a normative criterion for social choice are well-known. Arrow's celebrated impossibility theorem has taught us to be more modest in our search for such a criterion than earlier pioneers of the new welfare economics had been hoping for. In the context of assessing the relative social desirability of alternative economic allocations, the concept of Pareto efficiency still stands out as the central cornerstone of normative economics. However, since it is recognized that some Pareto allocations may be rather inequitable from some intuitive distributional viewpoint, one would like to supplement the Pareto condition with some notion of economic justice. If possible, attempts should be made to design notions of distributive justice that, like the Pareto criterion itself, are ordinal in nature and do not involve questionable interpersonal utility comparisons. This paper presents such an attempt.

I. INTRODUCTION

While a systematic review of the vast literature on normative economics is beyond the scope of this paper, some brief remarks regarding the present state of the art will help put the problem of economic equity in proper perspective.

When one rereads such classics of the new welfare economics as Bergson (1938), Samuelson (1947, 1950, 1956), and Graaff (1957), one cannot help but feel how disturbing for normative economics Arrow's (1963) general impossibility theorem really is. If Bergsonian social welfare functions had been able to satisfy Arrow's minimal conditions

* Pazner's research was done while he visited Northwestern University. Schmeidler's research was done partly in Tel-Aviv and partly while he visited the University of Illinois. The research in Tel-Aviv was supported by the Foerder Institute for Economic Research.

The Quarterly Journal of Economics, November 1978 0033-5533/78/0092-0671$01.00

on such functions, a very powerful analytical concept would indeed have become available. Significant insights to normative economics have nevertheless been provided by the social welfare function approach. But the explicit underlying (ordinal) interpersonal welfare comparisons (effectively ruled out by Arrow's result) imply that a robust equity criterion, normatively compelling for any possible economy, cannot be derived from this particular approach.

The question that suggests itself therefore is whether any reasonable equity criterion, the normative significance of which is equally valid in any particular society (economy), can be advanced.

A few years ago Foley (1967) advanced the concept of fair (or envy-free) allocations as a reasonable equity criterion. An allocation is said to be fair if nobody prefers anybody else's bundle over his own. The concept of fairness is appealing from an equity viewpoint in that it treats economic agents symmetrically, is ordinal in nature, and is free of interpersonal comparisons of utility. However, as shown by Pazner and Schmeidler (1974), standard Arrow-Debreu production economies may display the disturbing feature that among all Pareto-efficient allocations none can be found that is fair. In light of the general acceptance of the Pareto criterion, it would be desirable to have a concept of equity that never conflicts with Pareto efficiency (under the standard assumptions on the economic environment). Since the fairness criterion does not possess this property and since there is also the question of whether an equity concept based on envy can be morally acceptable in the first place (see Rawls, 1971), the issue of defining an adequate criterion is still open.

The present paper introduces a concept of economic equity that, as fairness, possesses an appealing symmetry property, is ordinal in nature, and is free of interpersonal welfare comparisons. Specifically, an allocation is said to be egalitarian-equivalent if there exists a fixed commodity bundle (the same for each agent) that is considered by each agent to be indifferent to the bundle that he actually gets in the allocation under consideration. In other words, an egalitarian-equivalent allocation has the special property that its underlying welfare distribution could have been generated by an egalitarian economy. It is shown that Pareto-efficient and egalitarian-equivalent allocations always exist under (even weaker than) the standard conditions on the economic environment. When supplemented by the egalitarian-equivalence criterion, the set of Pareto-efficient allocations is thus restricted to those allocations having the property that there exists an egalitarian economy (i.e., an economy in which everybody gets an identical bundle) in which every agent enjoys precisely the same welfare level as that experienced by him at the Pareto allocation under consideration. Another equivalent way of characterizing the set of Pareto allocations that are admissible according to the egalitarian-equivalence criterion is to visualize those for which all the underlying indifference surfaces have at least one point of intersection (when drawn in the same commodity space with respect to the same origin).

Two remarks on our approach are called for at this point. First, note that the normative significance of an egalitarian-equivalent allocation derives from its being agent-wise indifferent to the egalitarian distribution of commodities in some hypothetical economy. This may bring to mind the hypothetical compensation tests of Kaldor (1939), Hicks (1939), Scitovsky (1941), and Samuelson (1950) in some of which a state of the economy was said to be better than another if in the first state everybody could be made better off than in the second by suitable lump-sum transfers (without requiring actual compensation to be effected). But note that in our case the welfare distribution is *not* hypothetical (only the reference egalitarian economy conducive to the same welfare distribution is). Since our interest here is precisely in the distributional aspects of any allocation, the distribution of welfare is all-important; in the compensation tests, efficiency was the issue so that the actual distribution of welfare was essentially ignored. Potential welfare was the issue there; actual welfare is the issue here.

The second remark we wish to make deals with the horizontal and vertical equity aspects of any egalitarian-equivalent allocation (see Musgrave, 1959, for the concepts of horizontal and vertical equity). Any two individuals having identical preferences will enjoy precisely the same welfare level in any egalitarian-equivalent allocation (since the egalitarian reference economy assigns to everybody an identical bundle that is preference-wise indifferent for each individual to his actual bundle). Hence horizontal equity is satisfied; that is, equal treatment of equals is assured. For any two individuals having different preferences, the inequality in their actual bundles in any egalitarian-equivalent allocation is limited by the requirement that their underlying indifference curves (surfaces) contain at least one common bundle (since they must meet precisely at the egalitarian allocation of the hypothetical reference economy). Hence a particular notion of vertical equity is implicit in the egalitarian-equivalence criterion; an egalitarian economy exists that would assign to each individual the same welfare level as in the actual allocation.

The main results in the present paper are as follows. The new concept of equity is shown to be consistent with Pareto efficiency in both pure exchange and production economies (Sections II and V). In the case of two-person economies, it is shown that any fair allocation is also egalitarian-equivalent (Section III). In the case of n-person economies this relationship no longer holds, but it is seen that the larger becomes the number of agents, the smaller becomes the set of Pareto-efficient and egalitarian-equivalent allocations in relation to the set of Pareto allocations (Section II). It is also interesting to note that some very natural maximin interpretations (justifications?) can be given to the set of Pareto-efficient and egalitarian-equivalent allocations (Section IV).

The precise plan of the paper is as follows. In Section II the concept of egalitarian-equivalent allocation is further explained, and its consistency with the Pareto criterion established. Section III relates the new concept to that of fairness and presents an arbitration scheme for allocations based on the new concept. Section IV discusses some maximin interpretations of the results and relates the present approach to Rawls's theory of justice. While Sections II through IV deal mainly with pure exchange economies, it is made explicit in Section V that the results carry over to production economies.

The Mathematical Appendices at the end of the paper contain the relevant definitions and statements for each section. A section containing the proofs concludes the appendices. The text and the appendices have been designed to make it possible for them to be read independently. Both the text and the appendices are self-contained, but the reader of the appendices should consult the text for interpretive purposes.

II. The Concept of Pareto-Efficient-Egalitarian-Equivalent-Allocations (PEEEA)

Consider a standard pure exchange economy in which externalities are absent and where each consumer has preferences over the nonnegative orthant of the (Euclidean) commodity space that can be represented by a continuous and monotone-increasing utility function. Given any vector of aggregate endowments, modern welfare economics has singled out the set of Pareto-efficient allocations (the contract curve) as the relevant set of allocations from the viewpoint of normative social choice.

The problem of course is that the contract curve (more generally, the Pareto set) contains some rather unappetizing allocations (for instance all those at which one individual gets everything, or more generally those in which one or some individuals do not get anything). The subject matter of the theory of economic equity, as we see it, is to supplement the Pareto criterion by an ethically appealing distributional criterion, the role of which should essentially be to restrict the set of admissible Pareto-efficient allocations. As the theoretical justification for any equity criterion depends heavily on its being consistent with the existing conceptual framework of modern economic theory, our objective is to present an equity criterion that does not presume interpersonal welfare comparisons and that does not stand in conflict with the ordinal nature of Pareto-efficiency.

For the sake of expositional simplicity, consider the case where there are two consumers and two commodities (but note that every step in the argument carries over to any number of agents and commodities; see the Mathemtical Appendix to this section). Suppose that each consumer is given precisely half the total endowments. This egalitarian distribution will in general not be Pareto-efficient. Consider the ray in commodity space that goes from the origin through the vector of aggregate endowments. The egalitarian distribution is represented by each man being given the same bundle along this ray. If the egalitarian distribution is not Pareto-efficient, then (by monotonicity and continuity of preferences) moving each man slightly up along the ray yields distributions of utilities that are still feasible, since the starting utility distribution is in the interior of the utility possibility set. In particular, if we simultaneously move each man up along the commodity ray in precisely the same manner, we eventually shall hit a utility distribution that lies on the utility possibility frontier. This means that there exists a Pareto-efficient allocation that is equivalent from the viewpoint of each consumer to the hypothetical (nonfeasible) distribution along the ray that would give to each consumer the same bundle (which, by being strictly greater than the egalitarian distribution of the aggregate endowments, is itself not feasible). This Pareto-efficient allocation is thus equivalent to the egalitarian distribution in the hypothetical (larger than the original) economy.

It is now clear that we can repeat this experiment along any positive direction in the commodity space and by so doing generate the set of Pareto-efficient allocations, each of them having the following property: there is an egalitarian allocation in some hypothetical economy (in which the preferences of the agents are identical to those of the economy under consideration and the aggregate endowments of some or all commodities are larger than those of the original economy), so that the welfare levels in the hypothetical economy are agent-wise equal to those of the allocation under consideration.

The resulting set of allocations is what we call the set of Pareto-efficient and egalitarian-equivalent allocations (PEEEA in the sequel). It is a restriction of the Pareto set of the economy to those allocations having the specified equity property that their underlying utility levels distribution could have been generated by some egalitarian economy.

That this method indeed restricts the set of admissible Pareto-efficient allocations is clear, since, for instance, under strictly monotone preferences the "end-points" of the utility possibility frontier (in which some consumers get nothing) are excluded. How substantial the restriction is in fact depends of course on the precise form of the preference orderings. However, it is obvious that, under strict convexity of preferences when the number of agents is much greater than that of commodities, the restriction is significant—a desirable result.

From the viewpoint of the ethical appeal of the concept, observe that it is equivalent to restricting the set of admissible Pareto-efficient allocations to those for which the underlying indifference surfaces all meet at least once at a bundle common to all. Since each man is indifferent between having his actual bundle and a given bundle, that is also indifferent from the viewpoint of every other consumer, the concept treats agents symmetrically in this sense.

III. PEEEA AS A FAIR ARBITRATION SCHEME FOR ALLOCATIONS

In this section we relate the concept of Pareto-efficient-egalitarian-equivalent allocations to that of fair and Pareto-efficient allocations. We do so by confining most of the discussion to two-person economies. The special interest of this case stems from its applicability to the well-known two-person bargaining problem. We note in passing that the search for plausible arbitration schemes is very closely related to the quest for an acceptable normative criterion for the division of economic resources; in either case we are looking for solution concepts that will be deemed "fair" when all possible social environments (preferences, resources, etc.) are considered at the outset. The bargaining problem is usually formulated in terms of cardinal utilities and admits of a number of interesting solutions (e.g., Harsanyi-Zeuthen, 1956; Nash, 1950; and the recent work of Kalai-Smorodinsky, 1975). The case of ordinal preferences on which so much of economic theory is based, presents special difficulties and does not yet lead to comparable solution concepts. The notion of a fair allocation can be thought of as presenting a plausible way of restricting the set of admissible solutions in the ordinal case. Before discussing the relevance of the PEEEA concept in the present context, it thus seems appropriate to review briefly some results of the fairness literature.

Recall the definition of a fair allocation. An allocation is said to be fair (equitable) if no individual prefers the bundle of any other individual in the allocation over his own. In other words, a fair allocation is an allocation in which there is no envy; no individual would like to switch bundles with any other. A Pareto-efficient allocation that is fair is called a fair and Pareto-efficient allocation. Fair and Pareto-efficient allocations have been the subject of some recent literature (Foley, 1967; Pazner and Schmeidler, 1974; and Varian, 1974). The main result in the present context is that while fair and Pareto-efficient allocations always exist in standard pure exchange economies, they may fail to exist in standard production economies (Pazner and Schmeidler, 1974). This failure is not restricted to the two-person case; we shall come back to this matter later on. Hence the need for a different normative criterion which one would like, as in the case of fairness, to be ordinal in nature and free of interpersonal welfare comparisons.

As argued below, a major appeal of the concept of PEEEA in the two-person case lies in its including all fair and efficient allocations whenever they exist and importantly in its consistency in the general n-person case (in the sense that Pareto-efficient-egalitarian-equivalent allocations always exist) even when fair and efficient allocations fail to exist.

For the sake of expositional simplicity we shall conduct the discussion in this section in terms of two commodities. As shown in the Mathematical Appendix to this section, the results carry over to any number of commodities.

Note first that if the indifference curves of the two individuals (each drawn with respect to the same origin and passing through the corresponding agent's bundle) do not intersect, then the allocation under consideration cannot be fair, for, in such a case the indifference curve of one of the agents must lie below that of the other, implying that the first agent envies the second. Since the PEEEA restriction of the Pareto set is precisely via the exclusion of all Pareto-efficient allocations for which the indifference curves do not intersect, we conclude that this method does not rule out any fair and Pareto-efficient allocation.

Second, observe that while the intersection of the underlying indifference curves is necessary for any allocation to be fair, it is by no means sufficient. It is easy to see that if each agent's bundle lies on the same side of the intersection point (when the latter is unique), it must be the case that one agent envies the other.

We conclude thus that the set of Pareto-efficient-egalitarian-equivalent allocations (PEEEA) includes the set of Pareto-efficient and fair allocations, and that the latter is usually a proper subset of the former.

Coming back to the bargaining problem, consider the following proposed solution. Suppose that the preferences in a two-person exchange economy are convex. The suggested solution is the PEEEA, which is equivalent to the egalitarian reference bundle lying on the ray through the aggregate endowment. The choice of this ray is motivated by the fact that the particular PEEEA induced by it is fair. To see this, note that the average bundle for this allocation must lie at or below the egalitarian reference bundle, both being on the same ray. This fact coupled with the convexity (toward the origin) of the indifference curves precludes the possibility of envy.

It is easy to see that for any other ray, one can find convex preferences such that the resulting PEEEA is not fair.

Thus the choice of the ray through the aggregate endowment yields a well-defined, fair, and canonic arbitration scheme for two-person exchange economies with convex preferences. This scheme (i.e., selecting the PEEEA corresponding to this canonic ray) can be applied to any number of agents and commodities. In the case of more than two agents, the resulting PEEEA need not be fair; however, the scheme still yields a determinate distribution of welfare levels even when the preferences are not convex (in which case fair and efficient allocations may fail to exist altogether). An additional justification for the concept of PEEEA with an arbitrary number of agents is provided in the next section that discusses its natural maximin interpretations. There again, the special role of the ray through aggregate endowments will be brought out.

Finally, note also that the arbitration scheme presented here remains well defined when considering economies with production; the resulting PEEEA, however, need not be fair even in the two-person case, since fair and efficient allocations may not exist in production economies.

IV. Maximin Properties of PEEEA

In light of the interest in the maximin criterion for social welfare generated by Rawls's *A Theory of Justice* (1971), we turn now to some natural maximin interpretations of the notion of PEEEA.

Consider a ray in commodity space. For each individual choose the utility function representing his preferences so that the utility level of a point on the ray is equal to its Euclidean distance from the origin. Due to our assumptions on preferences (total, transitive, reflexive, continuous, and monotonic binary relations over the commodity space), this is a well-defined utility function. Applying the Rawlsian maximin social welfare function to these particular utility representations yields a Pareto-efficient allocation (which is unique if strict convexity of preferences is also assumed). This allocation is egalitarian-equivalent; all the individuals obtain equal utility levels, and a corresponding egalitarian bundle is located on the ray at a distance from the origin equal to the common utility level. Thus, the application of Rawls's maximin principle leads to a Pareto-efficient allocation that for each individual is indifferent preference-wise to the egalitarian allocation in some hypothetical economy. Stated somewhat differently, the PEEEA notion is consistent with the maximin principle.

A different way to relate the concept of PEEEA to Rawls's theory of justice is the following. Suppose that individuals in a Rawlsian original position have to decide about the distribution of some vector of aggregate endowments. As shown in Pazner and Schmeidler (1976), these individuals will unanimously agree on the egalitarian distribution of resources. One problem with this solution is that it usually will not be Pareto efficient. But if it is assumed that people in the original position accept the principle of Pareto efficiency, the egalitarian contract can be taken to mean that only Pareto-efficient allocations that could have been generated utility-wise by egalitarian economies (i.e., only PEEEA) are admissible. This suggests that the notion of PEEEA is consistent with social choices in the original position. We shall return to this point later on.

Returning to the method of calibrating the utilities in order to apply the maximin criterion, we believe that some remarks seem appropriate. First, whenever a particular ray is considered, the choice of the Euclidean distance as a measure of utility is immaterial, since if the same monotone-increasing transformation is applied to all the utility functions, the maximin solution remains the same. The essential restriction is that given a consumption bundle along this ray, all the agents are supposed to assign an identical utility level to this consumption bundle. Thus, the choice of a ray, together with this "common utility" supposition, implies a particular method of interpersonal utility comparisons. It should be clear that the entire set of PEEEA is generated by letting the ray along which utilities are calibrated vary across all positive directions. So, the choice of the ray dictates the final outcome.

The ray passing through the aggregate endowment is of particular significance as it also passes through the average (egalitarian) bundle in any allocation. In view of the previously mentioned result that individuals in a Rawlsian original position will agree on the egalitarian distribution where each gets the vector of average endowments, the choice of the ray passing through this vector can be rationalized in terms of the revealed preferences of individuals in the original position. The welfare expectations of all the parties to the social contract being precisely reflected by the vector of average endowments makes it appropriate to calibrate preferences in this particular way. Agreeing on the PEEEA corresponding to this choice of ray (i.e., applying the maximin criterion to this specific way of comparing utilities) is then a natural solution from the viewpoint of the Rawlsian framework. This also completes our discussion of the previous section where we suggested this particular arbitration scheme for general n-person economies.

It may be worthwhile to note that the argument of the previous paragraph regarding the calibration of utilities by the average bundle can be used to rationalize the common practice to base welfare evaluations on economic indexes (cost of living, per capita consumption, or income, etc.) that are constructed on the basis of average economic bundles. To the pragmatic reasons underlying this implicit choice of "utility numéraire" one can add the analytical justification provided above.

We conclude this section by noting that while the maximin criterion cannot by itself rule out any particular Pareto-efficient allocation on a priori grounds, the major appeal of the PEEEA concept lies in its doing so in a plausible way. In the absence of explicit interpersonal utility comparisons, the maximin criterion is entirely inoperative. The notion of PEEEA, on the other hand, is devoid of any such comparisons, since the set of allocations induced by it is invariant under any admissible representation of the agents' preferences. This last remark should not be confused with the fact illustrated above that under a particular method of interpersonal comparisons any PEEEA can be generated by means of the maximin criterion.

V. PEEEA IN ECONOMIES WITH PRODUCTION

As mentioned earlier, the present paper is partly motivated by the fact that fair and Pareto-efficient allocations may fail to exist in standard Arrow-Debreu production economies. Since Pareto efficiency is too convincing a criterion to be given up lightly, the basic purpose of the paper is to present a new notion of equity that does not conflict with efficiency in production economies as well. In this section our sole intention is to show that the concept of PEEEA is well defined even in the presence of production. All the results that were presented for exchange economies (with the obvious exception of the fairness result of Proposition 6 in the Mathematical Appendix to Section III) carry over to production economies.

As usual, describe the production possibilities of the economy by the set of technologically feasible production plans. This set, combined with the aggregate initial endowments (including labor services, capital goods, etc.) defines the set of feasible final aggregate bundles (assumed to be closed and bounded). Due to the presence of different labor (leisure) services and nonfinal-consumption goods, it is customary and reasonable to weaken the monotonicity assumption on preferences to weak monotonicity (and local nonsatiation). However, we add an assumption of utility connectedness across the agents. Specifically, we assume that whenever an agent is not at his minimum welfare position, it is possible to increase the level of welfare of any other agent (by direct transfer of commodities or indirectly via production). Thus, existence and maximin properties of the PEEEA are maintained.

Two special points concerning the choice of a ray are worth noting. First, rays along which the labor services (leisure) are not in equal proportions could be excluded. This may be rationalized by the implicit assumption that each agent derives utility only from his own labor services (which is interpreted as leisure) so that the quantities of the labor services with which he is not endowed are immaterial to him; (this observation is also part of the reason for dropping the strict monotonicity assumption on preferences). And if we want the egalitarian-reference-bundles to be truly egalitarian, we would like them to contain equal amounts of labor services (i.e., equal consumption of leisure) as well as equal amounts of final commodities.

The second point concerning the choice of a ray is the following. If we choose a ray passing through a particular feasible final aggregate bundle, the average bundle of a corresponding PEEEA need not lie on the same ray. Thus the existence of a canonic ray (i.e., a ray passing through both the egalitarian-reference-bundle and the average bundle of a corresponding PEEEA) is not obvious. However, an application of a fixed point argument (as shown in the Appendix) yields the existence of the desired canonic ray. But if only rays along which labor services are in equal proportions are considered, such a canonic ray may not exist.

MATHEMATICAL APPENDICES

Mathematical Appendix to Section II

Let T denote the (finite) set of agents in the economy and let R_+^l denote the nonnegative orthant of the Euclidean space of dimension l, the set of commodity bundles. Each t in T has a preference relation $\gtrsim_t$ on R_+^l, which is assumed to be *total, transitive, continuous,* and *monotonic* (i.e., for all x, y, z in R_+^l the following hold: $x \gtrsim_t y$ or $y \gtrsim_t x$; $x \gtrsim_t y$ and $y \gtrsim_t z$ imply that $x \gtrsim_t z$; the sets $\{x' \in R_+^l | x' \gtrsim_t x\}$ and $\{x' \in R_+^l | x \gtrsim_t x'\}$ are closed in R_+^l, and $x \geqq y$ and $x \neq y$ imply that $x >_t y$, where inequalities between vectors in R_+^l hold coordinate-wise by definition, and the relations $>_t$ and $\sim_t$ are induced by $\gtrsim_t$ in the usual way).

The *economy* is formally defined as the vector $(T, R_+^l, \{\gtrsim_t\}_{t \in T}, w)$ where w is an aggregate, initial commodity vector, $w > 0, w \in R_+^l$. The equity problem is that of dividing w among the members of T. An *allocation* is a T-list of elements of R_+^l whose sum is smaller than or equal to w. An allocation is denoted by $\{x_t\}_{t \in T}$ or simply $\{x_t\}$. An allocation $\{x_t\}$ is *Pareto-efficient* if for any other allocation $\{y_t\}$ the implication,

$$(\forall_t \in T, y_t \gtrsim_t x_t) \Rightarrow (\forall_t \in T, y_t \sim_t x_t)$$

holds.

DEFINITION. An allocation $\{x_t\}$ is said to be *egalitarian equivalent* if there is a bundle z in R_+^l so that for all t in T, $z \sim_t x_t$; such a vector z is called an *egalitarian-reference-bundle.*

In particular, the constant allocation $\{y_t\}$ where for all t, $y_t = w/|T|$ is egalitarian equivalent. First, the following results are established.

PROPOSITION 1. For every $\bar{x} > 0$ in R_+^l there is a positive real number $\bar{r}$ so that there exists a Pareto-efficient egalitarian-equivalent allocation $\{x_t\}$ with $\overline{rx}$ being the egalitarian-reference-bundle (i.e., for all t in T, $\overline{rx} \sim_t x_t$).

For the next result the *strict convexity* assumption is used: For all t in T, all $x, y \in R_+^l$ and all $0 < r < 1$, if $x \neq y$ and $x \gtrsim_t y$, then $rx + (1 - r)y >_t y$. We then have the straightforward result.

PROPOSITION 2. Under the assumption of strict convexity, for every egalitarian bundle there is at most one Pareto-efficient allocation.

The following technical result is needed for the sequel:

PROPOSITION 3. Let there be given a convergent sequence $(\bar{x}_n)$ in R^l_+ with a limit $\bar{x} > 0$ and suppose also that $\bar{x}_n > 0$ for all n. Denote by $\bar{r}_n$ (and $\bar{r}$) the real number corresponding to $\bar{x}_n$ (and $\bar{x}$) via Proposition 1. Then one has $\bar{r}_n \to \bar{r}$.

Denote by P the set

$$\left\{x \in R^l_+ | x > 0 \qquad \text{and} \qquad \sum_{i=1}^{l} x^i = 1\right\}$$

and denote by RP the set of egalitarian-reference-bundles: $\{\overline{rx} \in R^l_+ | \bar{x} \in P$ and $\bar{r}$ corresponds to $\bar{x}$ via Proposition 1$\}$. As a simple consequence of Proposition 3, one has that RP is homeomorphic to P. Furthermore, if RP is a bounded set, then the homeomorphism, as well as Propositions 1 and 3, can be extended to $\bar{P}$, the closure of P in R^l_+, and $R\bar{P}$ correspondingly. Next denote by ARP the set of allocations, each of them being Pareto-efficient and egalitarian-equivalent to some bundle in RP, By Propositions 1, 2, and 3 we have

PROPOSITION 4. Under the strict convexity assumption the correspondence that applies Pareto-efficient $\bar{x}$-equivalent allocations to each $\bar{x}$ in RP is a well-defined continuous function from RP onto ARP.

However, note that the function of Proposition 4 is not one to one; one may have two distinct bundles in RP yielding the same Pareto-efficient-egalitarian-equivalent allocation in ARP. As an example, consider a two-person, two-commodity economy with aggregate initial endowment. (3,3) and a Pareto-efficient allocation $\{(1, 1), (2, 2)\}$. The corresponding preferences are represented by symmetric utility functions, $u_1(x) = x^1x^2$, $u_2(x) = x^1 + x^2$. The symmetry of the utility functions implies that any allocation with values on the diagonal is Pareto-efficient and that if two indifference curves meet off-diagonal, they meet twice. As the consequence of this example, it is not true that ARP is homeomorphic to P. Nevertheless, it is intuitively obvious that the set of Pareto-efficient-egalitarian-equivalent allocations is of dimension $l - 1$ at most, whereas the set of Pareto-efficient allocations is homeomorphic to the simplex in $R^{|T|}$ (i.e., of dimension $|T| - 1$). This last assertion appears in the book by Arrow-Hahn (1971, Ch. 5), which also includes all the technical tools needed for following the present discussion; the topological characterization of ARP is outside the scope of this paper. In any event, when the number of agents is large relative to the number of commodities, the restriction of the Pareto-set induced by the egalitarian-equivalent requirement is significant; relatively few Pareto-efficient allocations are not ruled out as being inequitable.

Mathematical Appendix to Section III

An allocation $\{x_t\}$ is *fair*, by definition, if for all t and t' in T: $x_t \gtrsim_t x_{t'}$. In the case of a two-person economy, the following results hold:

PROPOSITION 5. In a two-person economy a fair allocation is egalitarian equivalent.

The preference relation of agent t is said to be *convex* if in the definition of strict convexity in the previous Mathematical Appendix, the relation $\gtrsim_t$ is substituted for by $>_t$.

PROPOSITION 6. In a two-person economy with convex preferences, if the egalitarian-reference-bundle lies on the ray through the aggregate endowment, the corresponding PEEEA is fair.

Mathematical Appendix to Section IV

Given $\bar{x} > 0$ in R^l_+ and t in T, we define the utility function u_t representing the preferences $\gtrsim_t$ of agent t as follows: for any x in R^l_+, let $s\bar{x}$ be the unique point on the ray through $\bar{x}$ indifferent to x according to the preferences of agent t (i.e., $s\bar{x} \sim_t x$). $u_t(x)$ is then equal, by definition, to the Euclidean norm of $s\bar{x}$. It is obvious that this u_t is a well-defined continuous utility function representing the preferences $\gtrsim_t$.

PROPOSITION 7. Given $\bar{x} > 0$ in R^l_+, let $\{u_t\}_{t \in T}$ be the list of utility functions defined above. The problem of maximizing over allocations $\{y_t\}$ the minimum over t in T of $u_t(y_t)$ has as the solution of PEEEA of Proposition 1.

Mathematical Appendix to Section V

The set of technologically feasible production plans is denoted by K, $(K \subset R^l)$. As usual, if $z \in K$, then the negative coordinates of z denote inputs, and the positive coordinates of z denote outputs. Let $W = \{w + K\} \cap R^l_+$ denote the feasible aggregate final consumption vectors. It is assumed that W is a compact set with a nonempty interior. As previously, free disposal is also assumed; i.e., $x \leqq y \in W$ implies that $x \in W$.

In this section we relax the monotonicity assumption on preferences, and it is only assumed that for all t in T and for all x, y in R^l_+: $x > y$ implies that $x >_t y$.

We further assume that the economy satisfies a so-called *utility connectedness* assumption: for any allocation $\{x_t\}$ and agent t' in T, if $x_{t'} \neq 0$, then there is an allocation $\{y_t\}$ so that for all $t \neq t'$, $y_t >_t x_t$.

PROPOSITION 8. Propositions 1, 2, 3, 4, 5, and 7 carry over to an economy with production as defined above; strict convexity for an economy with production means strict convexity of the set W in addition to strict convexity of preferences.

Note that in Propositions 1 and 7 convexity of preferences is not assumed. Hence the corresponding results for an economy with production do not require convexity either.

PROPOSITION 9. If the preference relation of each agent t in T is convex and the set W is strictly convex, then there is a PEEEA for which the average bundle and the egalitarian-reference-bundle lie on the same ray.

Remember that W is strictly convex in R^l_+ if for all x and y in W, $x \neq y$, $(x + y)/2$ is in the interior of W (relative to R^l_+).

Mathematical Appendix: The Proofs

This section includes proofs, schemes of proofs, and hints for proofs of the propositions stated in the previous subsections. The notations are those of the previous appendices.

Proof of Proposition 1. Set $C = \{r > 0 \mid \text{there is an } r\bar{x}\text{-equivalent allocation}\}$. The set C is nonempty, since $x_t = \bar{x}r/|T|$, for all t in T, defines an egalitarian-equivalent allocation whenever $r\bar{x} \leqq w$ and $r > 0$. (Existence of such an r is obvious, since $w > 0$.) It is bounded, since monotonicity of preferences implies that there is no $r\bar{x}$-equivalent allocation when $r\bar{x} > w$. (Here the fact that $\bar{x} > 0$ is used to assert the existence of such an r.) Let $\bar{r}$ be the l.u.b. of C (sup C). Because of the compactness of the set of allocations and the continuity of preferences $\bar{r} \in C$.

To complete the proof, one has to show that an $\overline{rx}$-equivalent allocation is Pareto-efficient. Denote by $\{x_t\}$ an $\overline{rx}$-equivalent allocation, and suppose, per absurdum, that there is an allocation $\{y_t\}$ with $y_t \succsim_t x_t$ for all t and $y_t \succ_t x_t$ for some t. Because of our monotonicity assumption there is another allocation, say $\{z_t\}$, so that $z_t \succ_t x_t$ for all t in T. By continuity (and monotonicity) there is, for each t in T, a positive number s_t so that $z_t \sim (\bar{r} + s_t)x$. Setting $\bar{s} = \min_t s_t$ and applying once again monotonicity and continuity, we get an $(\bar{r} + \bar{s})\bar{x}$-equivalent allocation—a contradiction. Q.E.D.

The proof of Proposition 2 is well-known. Existence of two distinct Pareto-efficient allocations that have identical utility representation (same reference bundle) contradicts strict convexity. The proof of Proposition 3 is similarly straightforward. It only requires the continuity of agents' preferences and the compactness of the set of allocations. As mentioned in the Mathematical Appendix to Section II the combination of Propositions 1, 2, and 3 yields Proposition 4.

The proof of Proposition 5 is also immediate. If the two indifference surfaces $\{x \in R^l_+ \mid x \sim_i x_i\} i = 1, 2$ (where $\{x_1, x_2\}$ is the fair allocation of the Proposition) do not meet, then one is above the other—a contradiction to the assumption that $\{x_1, x_2\}$ is envy-free.

Proof of Proposition 6. Denote by x the reference point of the allocation (on the ray from the origin through w). Except in the trivial case when $x_1 = x_2 = x = w/2$ (which is fair) the inequality $x > w/2$ holds, which implies, in turn, the relations $x_i \sim_i x \succ_i w/2$ for $i = 1, 2$. If, per absurdum $x_1 \succ_2 x_2$ (or $x_2 \succ_1 x_1$), then by convexity we get $x_1/2$

$+ x_2/2 = w/2 >_2 x_2$ (or $w/2 >_1 x_1$)—a contradiction. Q.E.D.

The proofs of Propositions 7 and 8 are very simple and do not require any new ideas; hence, they are omitted.

Proof of Proposition 9. Set $S = \{\Sigma_{t \in T} x_t | \{x_t\}$ is a Pareto-efficient allocation$\}$, $\bar{S} = \{x/|x| |$ for some x in $S\}$, and $\hat{S}$ is the convex hull of $\bar{S}$. Since S is compact, so is $\hat{S}$ where $\bar{S}$ denotes the set of rays to which the average bundles of efficient allocations belong. We define an upper-semicontinuous correspondence F: $\hat{S} \rightarrow \hat{S}$ a fixed point of which is a ray that satisfies the conclusion of Proposition 9. For x in $\hat{S}$ we define $F(x)$ to be the set of average bundles corresponding to PEEEA's with reference-bundle on the ray through x. It is obvious that the existence of a fixed point of F concludes the proof of the Proposition. We shall check the conditions of Kakutani's fixed-point theorem. The set $\hat{S}$ is compact and convex and $F(\hat{S}) \subset \hat{S}$. It is equally obvious that $F(x)$ is a convex set for all $x \in \hat{S}$ and that Proposition 1 (and the corresponding part of Proposition 7) can be extended in our case to upper-semicontinuous correspondence instead of the continuous function in case of strictly convex preferences. (Again, Arrow and Hahn, 1971, Ch. 5, can be consulted for the technical details.) Q.E.D.

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REFERENCES

Arrow, K. J., *Social Choice and Individual Values,* second ed. (New Haven and London: Yale University Press, 1963).

———, and F. Hahn, *General Competitive Analysis* (San Francisco: Holden-Day, Inc., 1971).

Bergson, A., "A Reformulation of Certain Aspects of Welfare Economics," this *Journal,* LII (1938), 310–34.

Foley, D. K., "Resource Allocation in the Public Sector," *Yale Economic Essays,* VII (1967), especially 73–76.

Graaff, J. de V., *Theoretical Welfare Economics* (Cambridge: Cambridge University Press, 1957).

Harsanyi, J. C., "Approaches to the Bargaining Problem Before and After the Theory of Games: A Critical Discussion of Zeuthen's, Hicks', and Nash's Theories," *Econometrica,* XXIV (1956), 144–57.

Hicks, J. R., "The Foundations of Welfare Economics," *Economic Journal,* XLIX (1939), 696–700, 711–12.

Kalai, E., and M. Smorodinsky, "Other Solutions to Nash's Bargaining Problem," *Econometrica,* XLIII (1975), 513–18.

Kaldor, N., "Welfare Propositions of Economics and Interpersonal Comparisons of Utility," *Economic Journal,* XLIX (1939), 549–52.

Musgrave, R. A., *The Theory of Public Finance* (New York: McGraw-Hill, 1959), p. 160.

Nash, J. F., "The Bargaining Problem," *Econometrica,* XVIII (1950), 155–62.

Pazner, E., and D. Schmeidler, "A Difficulty in the Concept of Fairness," *Review of Economic Studies,* XLI (1974), 441–43.

———, "Social Contract Theory and Ordinal Distributive Equity," *Journal of Public Economics,* V (1976), 246–57.

Rawls, J., *A Theory of Justice* (Cambridge: Harvard University Press, 1971). (On envy, see pp. 530–41.)

Samuelson, P. A., *Foundations of Economic Analysis* (Cambridge: Harvard University Press, 1947), Ch. 8.
———, "Evaluation of Real National Income," *Oxford Economic Papers,* N.S. II (1950), 1–29.
———, "Social Indifference Curves," this *Journal,* LXX (1956), 1–22.
Scitovsky, T., "A Note on Welfare Propositions in Economics," *Review of Economic Studies,* IX (1941), 77–88.
Varian, H., "Envy, Equity and Efficiency," *Journal of Economic Theory,* IX (1974), 63–91.