UNIVERSITY OF DELHI
M.A. Economics: Summer Semester 2014

Course 004: Macroeconomic Theory

Problem Set 3: R-C-K Model & OLG Model

1. Consider the **centralized version** of the standard R-C-K model where the economy consists of H identical inifinitely households. Life-time utility of a representative household is given by:

$$\sum_{t=0}^{\infty} \beta^t \log(c_{ht})\,;\; 0 < \beta < 1.$$

Population within each household grows at a constant n and hence so does the total population is the economy. Technology is represented by a Cobb-Douglas production function:

$$Y_t = (K_t)^{\alpha} (A_t N_t)^{1-\alpha}\,;\; 0 < \alpha < 1,$$

where K_t and N_t are the total aggregate capital stock and the aggreage labour force at the disposal of the social planner at any point of time t. Capital depreciates fully upon production (100% depreciation). The labour productivity term A_t increases as an exogenous constant rate m.

The economy starts with a given stock of capital (K_0), a given initial population (N_0), and a given initial labour productivity index $(A_0 = 1)$.

The social planner is benevolent; his objective function is the same as that of the households, except that he cares for the **per capita consumption** $\left(c_t \equiv \frac{C_t}{N_t}\right)$ instead of c_{ht}.

(i) Write down the dynamic optimization problem of the social planner subject to his aggregate resource constraint.

(ii) Convert the entire optimization problem in terms of consumption **per unit of effective labour** $\left(\hat{c}_t \equiv \frac{C_t}{A_t N_t}\right)$ and capital **per unit of of effective labour** $\left(\hat{k}_t \equiv \frac{K_t}{A_t N_t}\right)$. [Hint: Since the social planner's objective function is defined in terms of c_t, in order to convert it in terms of $\hat{c}_t$, one has to use the relationship: $c_t = \hat{c}_t A_t = \hat{c}_t(1+m)^t$ and write the utility function accordingly.]

(iii) Write down the corresponding Bellman equation.

(iv) Use the FONC of the Bellman equation and the corresponding Enevelope theorem to arrive at a $2X2$ system of difference equations in $\hat{c}_t$ and $\hat{k}$.

(v) Identify the steady state of this system. Argue that along the staedy state per capita consumption as well as per capita output in the centrally planned economy grows at a constant rate m.

2. Consider a standard two period overlapping generations framework where all agents have identical preferences; each young agent has one unit of labour endowment and no capital endowment. A representative agent belonging to generation t works only in the first period of his life. Out of his first period wage income, he consumes at part (c_t^1) and saves (and invests) the rest (s_t). His savings in period t gets converted into capital stock in period $t+1$ one-to-one, which generates some (expected) interest income in period $t+1$. The agent consumes his entire (expected) interest earnings in period $t+1$, along with the capital stock. Suppose the representative agent of generation t has the following life-time utility function:

$$u(c_t^1, c_{t+1}^2) = \frac{\left(c_t^1\right)^{1-\sigma}}{1-\sigma} + \frac{\left(c_{t+1}^2\right)^{1-\sigma}}{1-\sigma}\,;\; \sigma \neq 1.$$

Aggregate production function in the economy is Cobb-Douglas:

$$Y_t = (K_t)^\alpha (A_t N_t)^{1-\alpha}\,;\ 0 < \alpha < 1.$$

Population in successive generation grows at a constant rate n and there is no depreciation of capital stock.

(i) Write down the optimization problem of the representative agent belonging to generation t subject to his respective budget constraints in period 1 and period 2.

(ii) Derive the optimal values of c_t^1, s_t and c_{t+1}^2 as a function of current wage rate (w_t) and expected future wage rate (r_{t+1}^e).

(iii) Assuming that agents have perfect foresight such that $r_{t+1}^e = r_{t+1}$, derive the dynamic equation representing the evolution of capital-labour ratio (k_t) for this economy.

(iv) Derive a parametric restriction on σ which will ensure that the phase path is upward sloping. What is the economic significance of this restriction (in terms of income and substitution effect)?

(v) Assuming that the above parametric restriction is satisfied, derive the equation which identifies the steady state value of capital-labour ratio in this economy. From this equation, show that a unique non-trivial steady state exists. [Hint: Unless the parameter values are specified, you would not be able to solve the equation explicitly. Use diagrammatic technique to prove that the LHS and the RHS have a unique non-zero (positive) intersection point].

(vi) Can you comment on the local stability property of this non-trivial steady state? [Hint: Recall that local stability property requires that $0 < \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} < 1$.]

3. Show that in the OLG framework, the golden rule value of capital-labour ratio not only maximizes the steady state value of **'average' (or per capita) consumption**, but it also maximizes the steady state value of **life-time utility** of any agent.

4. Consider a standard two period overlapping generations framework where all agents have identical preferences; each young agent has one unit of labour endowment and no capital endowment. A representative agent belonging to generation t works only in the first period of his life. Out of his first period wage income, he consumes at part (c_t^1) and saves (and invests) the rest (s_t). His savings in period t gets converted into capital stock in period $t+1$ one-to-one, which generates some (expected) interest income in period $t+1$. The agent consumes his entire (expected) interest earnings in period $t+1$, along with the capital stock. Suppose the representative agent of generation t has the following life-time utility function:

$$u(c_t^1, c_{t+1}^2) = \log c_t^1 + \log c_{t+1}^2.$$

Aggregate production function in the economy is Cobb-Douglas:

$$Y_t = (K_t)^\alpha (A_t N_t)^{1-\alpha}\,;\ 0 < \alpha < 1.$$

Population in successive generation grows at a constant rate n and there is no depreciation of capital stock.

(a) Assume that agents have 'static expectation' about future interest rate, i.e., $r_{t+1}^e = r_t.$.

(i) Derive aggregate savings in this economy at any period t.

(ii) Derive the dynamic equation for the capital-labour ratio and draw the corresponding phase diagram. On the basis of the phase diagram, argue that there exists a unique nontrivial steady state k^*, which is stable.

(iii) Calculate the precise value of k^* in terms of the parameters.

(b) Now assume that agents have 'perfect foresight' about future interest rate, i.e., $r^e_{t+1} = r_{t+1}$..

(i) Derive aggregate savings in this economy at any period t.

(ii) Derive the dynamic equation for the capital-labour ratio and draw the corresponding phase diagram. On the basis of the phase diagram, argue that there exists a unique nontrivial steady state k^*, which is stable.

(iii) Calculate the precise value of k^* in terms of the parameters

(iv) Compare the steady state value of k^* derived in case (a) vis-a-vis case (b). Which one is higher and why?