Two Dimensional Phase Diagram For Solow Model

Consider the two dynamic equations of the Solow model:

The corresponding $\triangle K$ and $\triangle N$ functions are given by:

$$\begin{aligned}\triangle K &\equiv K_{t+1} - K_t = sF(K_t, N_t) - \delta K_t;\\ \triangle N &\equiv N_{t+1} - N_t = nN_t.\end{aligned}$$

From above it is clear that when we plot N_t on the horizontal axis and K_t on the vertical axis , the level curve $\triangle N = 0$ is represented by the vertical axis ($N_t = 0$). For any positive value of N_t, $\triangle N > 0$. Thus the horizontal arrows depicting the movement of N_t over time will always point rightwards (as shown in the diagram below).

In plotting the level curve $\triangle K = 0$, notice that

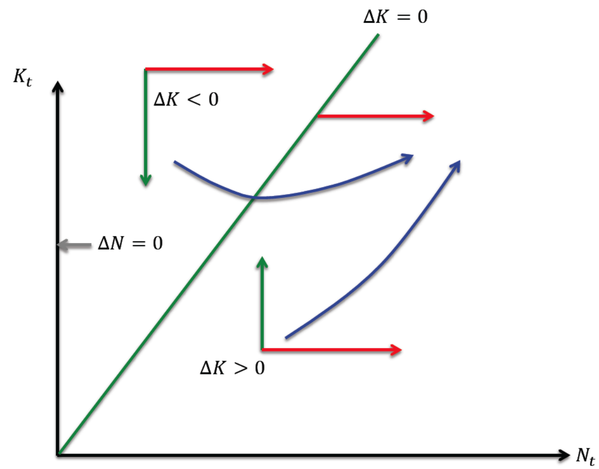
$$\begin{aligned}\triangle K &= 0 \Rightarrow sF(K_t, N_t) = \delta K_t\\ &\Rightarrow \frac{sF(K_t, N_t)}{K_t} = \delta\\ &\Rightarrow \frac{sF(K_t, N_t)}{N_t}\frac{N_t}{K_t} = \delta\\ &\Rightarrow \frac{sf(k_t)}{k_t} = \delta\\ &\Rightarrow k_t = \bar{k} : \frac{sf(\bar{k})}{\bar{k}} = \delta\\ &\Rightarrow \frac{K_t}{N_t} = \bar{k}\\ &\Rightarrow K_t = \bar{k}N_t. \qquad (1)\end{aligned}$$

Hence when we plot N_t on the horizontal axis and K_t on the vertical axis , the level curve $\triangle K = 0$ is represented by a straight line passing through the origin such that $K_t = \bar{k}N_t$ (where $\bar{k}$ is a known constant).

Note that for any point above this line, $\frac{K_t}{N_t} > \bar{k} \Rightarrow \frac{sf(k_t)}{k_t} < \delta \Rightarrow sF(K_t, N_t) < \delta K_t \Rightarrow \triangle K < 0$.

Similarly for any point below this line, $\frac{K_t}{N_t} < \bar{k} \Rightarrow \frac{sf(k_t)}{k_t} > \delta \Rightarrow sF(K_t, N_t) > \delta K_t \Rightarrow \triangle K > 0$.

This generates the following phase diagram:



The phase diagram shows that if the economy starts at the region above the $K_t = \bar{k}N_t$ line, then K_t falls while N_t rises; so the economy move in the south-east direction until it crosses over to the region below the $K_t = \bar{k}N_t$ line. In this region, both K_t and N_t are perpetually increasing.

Is that all that we can say in this phase diagram? It turns out that we can say something more. In particular we can say whether in the region below the $K_t = \bar{k}N_t$ line, K_t is increasing faster than N_t or not. Notice that if K_t increases faster than N_t then the corresponding trajectory (shown by the blue lines in the diagram) will be convex (depicting that $\frac{K_t}{N_t}$ is increasing over time). On the other hand if K_t increases at a slower rate than N_t then the corresponding trajectory will be concave (depicting that $\frac{K_t}{N_t}$ is decreasing over time). Finally if K_t increases at the same rate as N_t then the corresponding trajectory will once again be a straight line passing through the origin (depicting that $\frac{K_t}{N_t}$ remains constant over time).

Now from the dynamic equation for K_{t+1}, the rate of growth of K_t is given by:

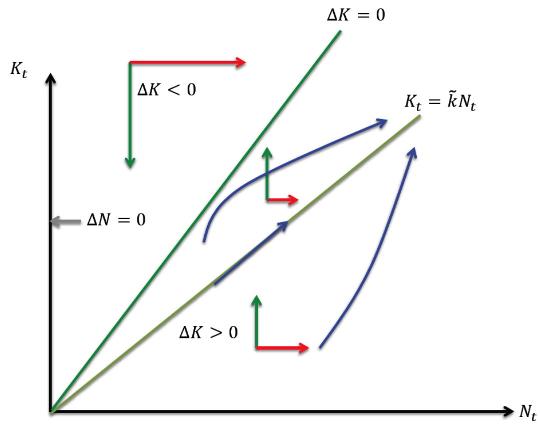
$$\begin{aligned}\frac{K_{t+1} - K_t}{K_t} &= \frac{sF(K_t, N_t) - \delta K_t}{K_t} \\ &= \frac{sf(k_t)}{k_t} - \delta > 0 \text{ in the region below } \triangle K = 0.\end{aligned}$$

On the other hand, the rate of growth of N_t is given by:

$$\frac{N_{t+1} - N_t}{N_t} = n > 0.$$

Now in the region below $\triangle K = 0$,

$$\begin{aligned}\frac{K_{t+1} - K_t}{K_t} &\gtreqless \frac{N_{t+1} - N_t}{N_t} \Rightarrow \frac{sf(k_t)}{k_t} - \delta \gtreqless n \\ &\Rightarrow \frac{sf(k_t)}{k_t} \gtreqless \delta + n \\ &\Rightarrow k_t \lesseqgtr \tilde{k} : \frac{sf(\tilde{k})}{\tilde{k}} = \delta + n \text{ (Note that } \tilde{k} < \bar{k}).\end{aligned}$$



Thus we get another straight line passing through the origin in the region below the $\triangle K = 0$, denoting $K_t = \tilde{k}N_t$ (as depicted by the pale green line in the diagram below). For any point above this line, $\tilde{k} < k_t < \bar{k} \Rightarrow \frac{sf(\tilde{k})}{\tilde{k}} < \delta + n \Rightarrow \frac{K_{t+1}-K_t}{K_t} < \frac{N_{t+1}-N_t}{N_t}$. Thus the trajectory will be concave here. For any point below this line, $k_t < \tilde{k} \Rightarrow \frac{sf(\tilde{k})}{\tilde{k}} > \delta + n \Rightarrow \frac{K_{t+1}-K_t}{K_t} > \frac{N_{t+1}-N_t}{N_t}$. Thus the trajectory will be convex here. Finally, for any point on this line, $k_t = \tilde{k} \Rightarrow \frac{sf(\tilde{k})}{\tilde{k}} = \delta + n \Rightarrow \frac{K_{t+1}-K_t}{K_t} = \frac{N_{t+1}-N_t}{N_t}$. Thus the trajectory will be a straight line here. These trajectories are shown in the diagram above.

The shapes of the trajectories further tell us the if the economy starts below the $K_t = \tilde{k}N_t$ line then it will be convex and approach the $K_t = \tilde{k}N_t$ line in the long run; if the economy above below the $K_t = \tilde{k}N_t$ line then it will be concave and once again approach the $K_t = \tilde{k}N_t$ line in the long run; finally, if the economy starts exactly at the $K_t = \tilde{k}N_t$ line then it will stays there. In other word, the $K_t = \tilde{k}N_t$ line represents the long run steady state of this two dimensional system such that the economy attains a constant capital-labour ratio $\tilde{k}$ in the long run.

(Verify that this $\tilde{k}$ and the steady state value k^* derived eralier are exactly the same.