202: Dynamic Macroeconomics

Dynamic Inefficiency & Technical Progress in Solow Model

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Solow Model: Growth Implications

- **In the long run (at steady state)**,
  - per capita income of the economy does not grow;
  - aggregate income grows at a constant rate - given by the **exogenous** rate of growth of population (n);
  - government policies to increase the savings ratio has no long run growth affect.
- **In the short run (during transition)**,
  - per capita income of the economy grows due to capital accumulation - but the growth rate keeps declining;
  - aggregate income grows due to both capital accumulation and population growth;
  - the further away the economy is from its steady state, the higher is its growth rate;
  - government policies to increase the savings ratio **temporarily** increases the growth rate, since it raises the steady state value of k as well as the transitional growth rate.

Critique of Solow Model: does it explain 'growth'?

- The fundamental critisism of the Solow 'growth' model is that **it fails to explain long run growth**:
  - The per capita income does not grow at all in the long run;
  - The aggregate income grows at an exogenously given rate n, which the model does not attempt to explain.
- The short run growth of per capita income is explained by capital accumulation.
- But capital accumulation happens in the Solow model because, **by assumption**, **households perpetually save a fixed proportion of their income - irrespective of the rate of return**.
- It is not clear why households will keep accumulating capital - in the face of a falling rate of return!! (Recall that as k increases $r \equiv f'(k)$ keeps declining.)
- This seems irrational. Surely there is some cost to savings! If nothing else, the households might be better off consuming the extra output rather than saving it - especially if the rate of return of capital is too low!

Critique of Solow Model (Contd.):

- In fact the assumed constancy of the savings ratio in the Solow model generates another problem - that of **dynamic inefficiency**.
- What is dynamic efficinecy/inefficiency?
- It is a conecpt similar to Pareto efficiency/inefficiency - but used in an intertemporal sense.
- An economic situation is ‘dynamically inefficient’ if we can improve the welfare of the agents in all future dates, without reducing their current welfare. The economy is ‘dynamically efficient’ otherwise. (Remember agents are identical here; so improving welfare of one imples improving welfare of all.)

Solow Model: Golden Rule & Dynamic Inefficiency

- To understand concept of dynamic efficiency in the Solow model, let us now got back to long run steady state of Solow.
- Recall that for given values of δ and n, the savings rate in the economy uniquely pins down the corresponding steady state capital-labour ratio:

$$k^*(s) : \frac{f(k^*)}{k^*} = \frac{n+\delta}{s}.$$

- We have already seen that a higher value of s is associated with a higher k^*, and therefore, a higher **level** of steady state per capita income $(f(k^*))$.
- How about the corresponding level of consumption?
- Notice that in this model, per capita consumption is defined as:

$$\frac{C_t}{N_t} \equiv \frac{Y_t - S_t}{N_t}$$
$$\Rightarrow \quad c_t = f(k_t) - sf(k_t)$$

Golden Rule & Dynamic Inefficiency in Solow Model (Contd.)

- Accordingly, for given values of δ and n, steady state level of per capita consumption is related to the savings ratio of the economy in the following way:

$$
\begin{aligned}
c^*(s) &= f(k^*(s)) - sf(k^*(s)) \\
&= f(k^*(s)) - (n+\delta)\, k^*(s). \quad \text{[Using the definition of } k^*\text{]}
\end{aligned}
$$

- We have already noted that if the government tries to manipulate the savings ratio (by imposing an appropriate tax on consumption/savings), then such a policy has no long run **growth** effect.
- Can such a policy still generate a higher **level** of steady state per capita consumption at least?
- If yes, then such a policy would still be desirable, even if it does not impact on growth.

Golden Rule & Dynamic Inefficiency in Solow Model (Contd.)

- Taking derivative of $c^*(s)$ with respect to s :

$$\frac{dc^*(s)}{ds} = \left[f'(k^*(s)) - (n+\delta) \right] \frac{dk^*(s)}{ds}.$$

- Since $\frac{dk^*}{ds} > 0$,

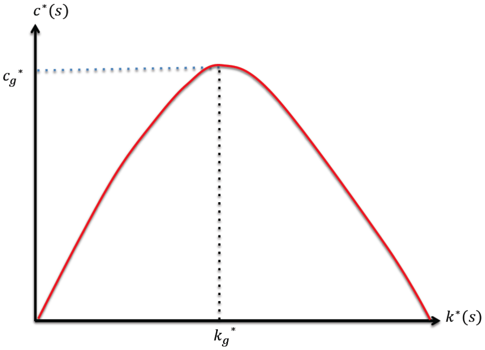
$$\frac{dc^*(s)}{ds} \gtreqqless 0 \text{ according as } f'(k^*(s)) \gtreqqless (n+\delta).$$

- In other words, steady state value of per capita consumption, $c^*(s)$, is maximised at that level of savings ratio and associated $k^*(s)$ where

$$f'(k^*) = (n+\delta).$$

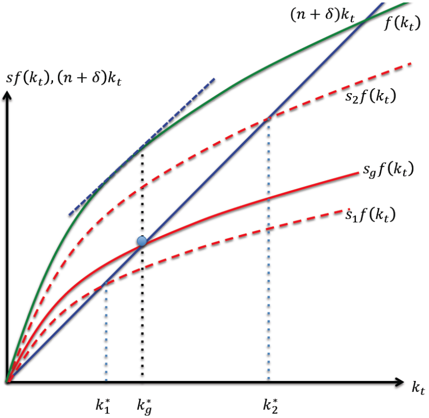
- We shall denote this savings ratio as s_g and the corresponding **steady state** capital-labour ratio as k_g^* - where the subscript 'g' stand for **golden rule.**

Digrammatic Representation of the Golden Rule Steady State:



- The point (k_g^*, c_g^*) in some sense represents the 'best' or the 'most desirable' steady state point (although in the absence of an explicit utility function, such qualifications remain somewhat vague).

Alternative Digrammatic Representation of the Golden Rule Steady State:



- There are many possible steady states to the left and to the right of k_g^* - associated with various other savings ratios.

Golden Rule & the Concept of 'Dynamic Inefficiency'

- Importantly, all the steady states to the **right** of k_g^* are called '**dynamically inefficient**' steady states.
- From any such point one can '**costlessly**' move to the left - to a lower steady state point - and in the process enjoy a higher level of current consumption as well as higher levels of future consumption at all subsequent dates. (**How?**)
- Notice however that the steady states to the **left** of k_g^* are **not** 'dynamically inefficient'. (**Why not?**)

Cause of 'Dynamic Inefficiency' in Solow Model

- Notice that dynamic inefficiency occurs because people oversave.
- This possibility arises in the Solow model because the savings ratio is exogenously given - it is not chosen through households' optimization process.
- If the steady state of an economy indeed happens to be dynamically inefficient, then it justifies an active, interventionist role of the government in the Solow model - even though government cannot affect the long run rate of growth of the economy.

Limitations of the Solow Growth Model:

- Even though the Solow model is supposed to be a growth model - it cannot really explain long run growth:
  - The per capita income does not grow at all in the long run;
  - The aggregate income grows at an exogenously given rate n, which the model does not attempt to explain.
- The steady state in the Solow model might be dynamically inefficient. It is not clear why households will not correct this inefficinency by choosing their savings ratio optimally. But this latter possibility is simply not allowed in the Solow model.

Extensions of Solow Growth Model:

- The basic Solow growth model has subsequently been extended to counter some of these critisisms.
- The primary challenge is to retain the basic result of the Solow model (namely, existence of a unique and globally stable steady state) while relaxing various restrictive assumptions.
- We shall look at two such extensions:
  1. Solow Model with Technological Progress: This extension allows the per capita income to grow in the long run; developed by Solow himself (Solow (1957)).
  2. Neoclassical Growth Model with Optimizing Households: This extension allows the households to choose their consumption/savings behaviour optimally over infinite horizon; developed independently by Cass (1965) & Koopmans (1965).

Solow Model with Exogenous Technological Progress:

- Let us now introduce a productivity-specific parameter into our Solovian firm-specific production function:

$$Y_{it} = F(K_{it}, N_{it}, A_t); F_A > 0,$$

  where F satisfies all the standard Neoclassical properties specified earlier.
- The term A_t captures the state of the technology at time t. Since all firms have access to identical technology, this technology-specific parameter is the same for all firms (hence no i-subscript here).
- The assumption of identical firms and CRS implies that the aggregate production function will also take similar form:

$$Y_t = F(K_t, N_t, A_t); F_A > 0.$$

- Technological progress implies that the productivity-specific term, A_t, increases in value over time. Thus with the same amount of labour and capital, the economy can now produce greater amount of output.

Different Types of Technological Progress:

- Technological progress can be of three types:
  1. **Labour Augmenting or Harrod-Neutral:** Technical improvement enhances the productivity of labour alone:
  $$Y_t = F(K_t, A_t N_t).$$
  2. **Capital Augmenting or Solow-Neutral:** Technical improvement enhances the productivity of capital alone:
  $$Y_t = F(A_t K_t, N_t).$$
  3. **Capital & Labour Augmenting or Hicks-Neutral:** Technical improvement enhances the productivity of both capital and labour in equal proportion and thus augments total factor productivity:
  $$Y_t = F(A_t K_t, A_t N_t) = A_t F(K_t, N_t).$$
- Notice that with Cobb-Douglas Production Function, all the three notions of technical progress are equivalent. (**Prove this.**)

Technological Progress and Balanced Growth:

- Modern Growth Theory often focuses on **Balanced Growth Path**: a scenario when every variable in the economy grows at some constant rate (not necessarily equal for all variables).
- Recall that steady state is also a special case of balanced growth (when the growth rate is constant at 0).
- It can be shown that unless the production function is Cobb-Douglas, **only Harrod-neutral technological progress is consistent with a balanced growth path**. (Proof follows a few slides later.)
- Henceforth, we shall therefore restrict our analysis only to Harrod-neutral technological progress.

Solow Model with Harrod-Neutral Technological Progress:

- Suppose all assumptions of the original Solow model remain unchanged, except that we now have a firm-specific production technology, given by:

$$Y_{it} = F(K_{it}, A_t N_{it}).$$

- The corresponding aggregate production technology is given by

$$Y_t = F(K_t, A_t N_t) \equiv F(K_t, \hat{N}_t),$$

where we denote the productivity-augmented labour term: $A_t N_t \equiv \hat{N}_t$ (effective labour).

- Labour productivity increases automatically over time (like 'manna from heaven') at an exogenous rate m:

$$\frac{1}{A_t}\frac{dA}{dt} = m.$$

- Each firm now equates the marginal product of 'effective labour' with the wage rate per unit of 'effective labour' ($\hat{w}$) and the marginal product of capital with the rental rate of capital (r).

Solow Model with Harrod-Neutral Technological Progress (Contd.):

- CRS and identical firms implies that the firm-specific marginal product and the (social) marginal product for the aggregate economy are the same.
- Thus we get the aggregate demand functions for 'effective labour' and capital as:

$$\begin{aligned} F_{\hat{N}}(K_{it}, \hat{N}_{it}) &= F_{\hat{N}}(K_t, \hat{N}_t) = \hat{w}_t; \\ F_K(K_{it}, \hat{N}_{it}) &= F_K(K_t, \hat{N}_t) = r_t. \end{aligned}$$

- At the beginning of any time period t, the economy starts with a given stock of population (N_t), a given stock of capital (K_t) and a given level of labour productivity (A_t).
- The wage rate per unit of effective labour and rental rate adjust so that that there is full employment of the given endowment of effective labour and capital stock at every point of time t.

Capital- Effective Labour Ratio & Output per unit of Effective Labour:

- Using the CRS property, we can write:

$$\hat{y}_t \equiv \frac{Y_t}{\hat{N}_t} = \frac{F(K_t, A_t N_t)}{A_t N_t} = F\left(\frac{K_t}{A_t N_t}, 1\right) \equiv f(\hat{k}_t),$$

where $\hat{y}_t$ represents output per unit of effective labour, and $\hat{k}_t$ represents the capital-effective labour ratio in the economy at time t.

- Using the relationship that $F(K_t, \hat{N}_t) = \hat{N}_t f(\hat{k}_t)$, we can easily show that:

$$\begin{aligned} F_{\hat{N}}(\hat{N}_t, K_t) &= f(\hat{k}_t) - \hat{k}_t f'(\hat{k}_t) = \hat{w}_t; \\ F_K(\hat{N}_t, K_t) &= f'(\hat{k}_t) = r_t. \end{aligned}$$

[Derive these two expressions yourselves].

Properties of the Reduced Form Production Function:

- Once again, given the properties of the aggregate production function, one can derive the following properties of the reduced form production function (in terms of effective labour) $f(\hat{k})$:

$$
\begin{aligned}
\text{(i) } f(0) &= 0; \\
\text{(ii) } f'(\hat{k}) &> 0;\ f''(\hat{k}) < 0; \\
\text{(iii) } \underset{\hat{k}\to 0}{Lim} f'(\hat{k}) &= \infty;\ \underset{\hat{k}\to\infty}{Lim} f'(\hat{k}) = 0.
\end{aligned}
$$

- Finally, using the definition that $\hat{k}_t \equiv \frac{K_t}{A_t N_t}$, we can write

$$
\begin{aligned}
\frac{1}{\hat{k}_t}\frac{d\hat{k}}{dt} &= \frac{1}{K_t}\frac{dK}{dt} - \frac{1}{N_t}\frac{dN}{dt} - \frac{1}{A_t}\frac{dA}{dt} \\
&= \frac{sF(K_t, A_t N_t) - \delta K_t}{K_t} - (n+m) \\
\Rightarrow & \frac{d\hat{k}}{dt} = sf(\hat{k}_t) - (n+m+\delta)\hat{k}_t \equiv \tilde{g}(\hat{k}_t). \qquad (1)
\end{aligned}
$$

Dynamics of Capital-Effective Labour Ratio:

- Equation (1) represents the basic dynamic equation in the Solow model with technological progress. Once again we use the phase diagram technique to analyse the dynamic behaviour of $\hat{k}_t$.
- In plotting the $\tilde{g}(\hat{k}_t)$ function, note:

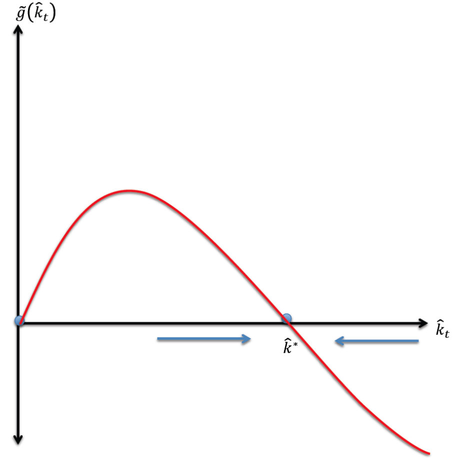
$$
\begin{aligned}
\tilde{g}(0) &= 0; \\
\tilde{g}'(\hat{k}_t) &= \left[sf'(\hat{k}_t) - (n+m+\delta)\right] \gtreqless 0 \text{ according as} \\
\hat{k}_t &\gtreqless \underline{\hat{k}} : sf'(\underline{\hat{k}}) = n+m+\delta; \\
\tilde{g}''(\hat{k}_t) &= sf''(\hat{k}_t) < 0.
\end{aligned}
$$

- Moreover, as before,

$$
\begin{aligned}
\underset{\hat{k}_t \to 0}{Lim}\, \tilde{g}'(\hat{k}_t) &= \infty; \\
\underset{\hat{k}_t \to \infty}{Lim}\, \tilde{g}'(\hat{k}_t) &= -(n+m+\delta) < 0.
\end{aligned}
$$

Dynamics of Capital-Effective Labour Ratio (Contd.):

- We can now draw the phase diagram for $\hat{k}_t$:



Dynamics of Capital-Effective Labour Ratio (Contd.):

- From the phase diagram we can identify two possible steady states:

$$\begin{aligned} \text{(i) } \hat{k} &= 0 \text{ (Trivial Steady State);} \\ \text{(ii) } \hat{k} &= \hat{k}^* > 0 \text{ (Non-trivial Steady State).} \end{aligned}$$

- As before, ignoring the non-trivial steady state, we get a unique non-trivial steady state, $\hat{k}^*$, which is globally asymptotically stable: starting from any initial capital-effective labour ratio $\hat{k}_0 > 0$, the economy would always move to $\hat{k}^*$ in the long run.
- Implication:
  - In the long run, output per unit of effective labour: $\hat{y}_t \equiv f(\hat{k}_t)$ will be constant at $f(\hat{k}^*)$.

Long Run Growth Implications of Solow Model with Technological Progress:

- However now in the long run per capita income grows at a constant rate m, while aggregate income grows at a constant rate $(m + n)$ (and so does aggregate capital stock).
- Thus incorporating technical progress in the Solow model indeed allows us to counter the critisism that per capita income does not grow in the long run.
- Notice however that **both the growth rates are still exogenous**; we still do not know what determines m and n. Thus Solow model still does not tell us what determines long run economic growth!!

Proof that along a Balanced Growth Path, Technological Progress must be Harrod-Neutral:

- Assume a production function that allows for both labour-augmenting and capital-augmenting technical progress:

$$Y_t = F(B_t K_t, A_t N_t),$$

where $A_t = \exp^{mt}$ and $B_t = \exp^{qt}$; m and q are non-negative constants.

- From above:

$$\frac{Y_t}{K_t} = \frac{F(B_t K_t, A_t N_t)}{K_t} = F\left(\exp^{qt}, \exp^{mt}\frac{N_t}{K_t}\right)$$

$$\text{i.e., } \frac{Y_t}{K_t} = \exp^{qt}.F\left(1, \exp^{(m-q)t}\frac{N_t}{K_t}\right) = \exp^{qt}.\phi\left(\exp^{(m-q)t}\frac{N_t}{K_t}\right).$$

- Now, population grows at a constant rate n : $\frac{1}{N_t}\frac{dN}{dt} = n$
- While $\frac{1}{K_t}\frac{dK}{dt} = s.\frac{Y_t}{K_t} - \delta \equiv \gamma$ (say)

Technological Progress must be Harrod-Neutral: (Contd.)

- If γ is a constant (as it should be along a balanced growth path), then

$$\frac{Y_t}{K_t} = \exp^{qt} .\phi \left(\frac{N_0}{K_0} \exp^{(m-q+n-\gamma)t} \right)$$

- But γ would be constant if and only if $\frac{Y_t}{K_t}$ is a constant.
- On the other hand $\frac{Y_t}{K_t}$ would be a constant if:

1. either $q = 0$ **and** $m + n = \gamma$
2. or $q > 0$ **and** the two terms $\exp^{qt}$ and $\phi(.)$ grow at equal but opposite rate, i.e.,

$$\frac{1}{\phi_t} \frac{d\phi}{dt} = -q.$$

Technological Progress must be Harrod-Neutral: (Contd.)

- Case 1 implies that technological progress is indeed Harrod-Neutral.
- Case 2 implies

$$\begin{aligned} \frac{1}{\phi_t}\frac{d\phi}{dt} &= -q \\ \Rightarrow \quad \frac{x_t}{\phi_t}\frac{d\phi_t}{dx_t}\left(\frac{1}{x_t}\frac{dx}{dt}\right) &= -q; \text{ where } x_t \equiv \frac{N_0}{K_0}\exp^{(m-q+n-\gamma)t}. \end{aligned}$$

In other words,

$$\begin{aligned} \frac{x_t}{\phi_t}\frac{d\phi_t}{dx_t}(m-q+n-\gamma) &= -\beta \\ \text{i.e., } \frac{x_t}{\phi_t}\frac{d\phi_t}{dx_t} &= \frac{q}{(q+\gamma-m-n)} = \epsilon \text{ (say).} \end{aligned}$$

Technological Progress must be Harrod-Neutral: (Contd.)

- Integrating,

$$\begin{aligned} \log \phi_t &= \epsilon \log x_t + (\text{constant}) \\ &\Rightarrow \phi_t = (\text{constant}) . x_t^{\epsilon} \end{aligned}$$

- Given the definition of ϕ_t, substituting,

$$\begin{aligned} Y_t &= K_t \exp^{qt} . (\text{constant}) . x_t^{\epsilon} \\ &= K_t \exp^{qt} . (\text{constant}) . \left(\frac{N_0}{K_0} \exp^{(m-q+n-\gamma)t} \right)^{\epsilon} \\ &= (\text{constant}) . K_t \exp^{qt} \left(\frac{N_t \exp^{mt}}{K_t \exp^{qt}} \right)^{\epsilon} \\ &= (\text{constant}) . \left(K_t \exp^{qt} \right)^{1-\epsilon} \left(N_t \exp^{mt} \right)^{\epsilon} \end{aligned}$$

- In other words, the production function is Cobb-Douglas!

Long Run Growth of Per Capita Income without Technological Progress?

- Exogenous technological progress is not a very satisfactory way to generate long run growth of per capita income in the Solow model.
- Without a proper theory to explain this phenomenon, it remains a mere technical exercise.
- Moreover the fact that only Harrod-neutral technical progress is consistent with the Solow-type long run steady state also seems to limit the applicability of the model.
- Can we have long run growth of per capita income in the Solow model - even without exogenous technical progress?
- The answer is: yes, but only if you allow some of the Neoclassical properties of the production function to be relaxed.

Growth of Per Capita Income without Technical Progress: (Contd.)

- Recall that the long run constancy of the per capita income in the Solow model arises due to the strong uniqueness and stability property of the steady state - which in turn depends on two key assumptions: the property of **diminishing returns** and the **Inada Conditions.**
- We can have long run growth of per capita income in the Solow model even without technological progress only if we are willing to relax at least one of these two key assumptions.

Growth of Per Capita Income without Technical Progress: (Contd.)

- Let us first relax one of the Inada conditions, namely that $\lim_{k\to\infty} f'(k) = 0$.
- Examples:
  - Jones-Manuelli Production Function:

$$Y_t = K_t^{\alpha} N_t^{1-\alpha} + \beta K_t; 0 < \alpha < 1;\ \beta > 0.$$

In this case, $f(k_t) = k_t^{\alpha} + \beta k_t$; and $\lim_{k\to\infty} f'(k) = \beta \neq 0$.

  - CES Production Function:

$$Y_t = \left[\alpha K_t^{\rho} + (1-\alpha) N_t^{\rho}\right]^{\frac{1}{\rho}}; 0 < \alpha < 1;\ \rho > 0.$$

In this case, $f(k_t) = \left[\alpha k_t^{\rho} + (1-\alpha)\right]^{\frac{1}{\rho}}$; and $\lim_{k\to\infty} f'(k) = \alpha^{\frac{1}{\rho}} \neq 0$.
(Notice that all other Neocalssical assumptions are satisfied by these production functions)

Growth of Per Capita Income without Technical Progress (Contd.):

- While there is no natural justification for the Inada conditions, (and most well-known production functions, except the Cobb-Douglas, typically violate one of these), relaxing only the Inada conditions may not generate a balanced growth path. (Recall our obsession with balanced growth!!)
- Another way to generate long run growth of per capita income in the Solow model without technological progress is to do away with the assumption of diminishing returns altogether (i.e., $f''(k) \nless 0$).
- An example of non-dinimishing returns but CRS production function is the linear technology case:

$$Y_t = AK_t + BN_t; \ A, B > 0.$$

- A special case of linear technology - AK production function:

$$Y_t = AK_t; \ A > 0.$$

Growth of Per Capita Income without Technical Progress (Contd.):

- Replacing the Neoclassical production function in the Solow model by the AK production technology indeed generates a balanced growth path.
- Rate of growth of per capita output in this case (in long as well as short run) is:
$$sA - (n + \delta)$$
- Notice that now the government can directly affect the growth rate of the economy by influencing the savings ratio!
- There are several interesting economic justifications for this kind of AK production technology.
- We shall examine some of these justifications later in the course.

Solow Model with Exogenous Technical Progress & AK Technology : Reference

- Robert Barro & Xavier Sala-i-Martin: Economic Growth, 2004 (2nd Edition), The MIT Press, Chapter 1.