Lecture 4: The Revelation Principle

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Contracts under Adverse Selection I

Examples of Contracts:

$$\begin{aligned}
C1: &\quad (q_1^*, \theta_1 u(q_1^*)), (q_2^*, \theta_2 u(q_2^*)) \\
C2: &\quad (0, 0), (q_2^*, \theta_2 u(q_2^*)) \\
C3: &\quad (q_1^*, \theta_1 u(q_1^*)), (q_1^*, \theta_1 u(q_1^*)) \\
C4: &\quad (q_1^{SB}, \theta_1 u(q_1^{SB})), (q_2^{SB}, \theta_2 u(q_2^{SB}) - \Delta\theta u(q_1)),
\end{aligned}$$

where q_1^{SB} and q_2^{SB} are as above.

Question

- *What are the actions available to agents under each of the above contracts?*
- *What are the outcomes of the above contracts?*
- *For P, which of the above contracts is the best?*

More General Schemes I

Up to this point, Principal has solved:

$$\max_{(q_1,T_1),(q_2,T_2)} \{\nu[T_1 - cq_1] + (1-\nu)[T_2 - cq_2]\}$$

Question

Can the principal do better for herself by offering more general/complicated contracts?

Suppose: Principal offers wider choice set $[q, T_i(q)]$, for $i = 1, 2$, where $q \in Q \subset \Re_+$ and $T_i(q)$ is some function

$$T_i : Q \mapsto \Re_+.$$

Principal can offer even a wider choice set $[q, T(q)]$, where $q \in Q \subset \Re_+$ and $T(q)$ is **any** function, i.e.,

$$T : Q \mapsto \Re_+.$$

More General Schemes II

Under this more general scheme, Principal solves:

$$\max_{(T_1(q),T_2(q))} \{\nu[T_1(q_1) - cq_1] + (1-\nu)[T_2(q_2) - cq_2]\}$$

Question

Does this more general scheme lead to a different outcome? Is the outcome better for the Principal?

Contract as Mechanism I

In the above context, an outcome is a pair (q, T).

- Outcome: pair $(q, T) \in \Re^2_+$
- Utility/payoff of both parties depend on the outcome realized
- $\mathcal{O}$ be the set of outcomes; $\mathcal{O} \subset \Re^2_+$.
- a an action (message/signal) that can be taken (sent) by the agent
- $\mathcal{A}$ be the set of feasible actions/messages; $a \in \mathcal{A}$.

Definition

Mechanism: A mechanism M is a pair $(\mathcal{A}, g)$, where $g(.) : \mathcal{A} \mapsto \mathcal{O}$, s.t.

$$(\forall a \in \mathcal{A})[g(a) = (q(a), T(a))]$$

Contract as Mechanism II

Contracts as Mechanisms:

1. $\mathcal{A} = \{\theta_1, \theta_2\}$; $g(\theta_1) = (q_1^*, \theta_1 u(q_1^*))$ and $g(\theta_2) = (q_2^*, \theta_2 u(q_2^*))$
2. $\mathcal{A} = \{a_1, a_2\}$; $g(a_1) = (q_1^*, \theta_1 u(q_1^*))$ and $g(a_2) = (q_2^*, \theta_2 u(q_2^*))$
3. $\mathcal{A} = \{\theta_1, \theta_2\}$; $g(\theta_1) = (0, 0)$ and $g(\theta_2) = (q_2^*, \theta_2 u(q_2^*))$
4. $\mathcal{A} = \{\theta_1, \theta_2\}$; $g(\theta_1) = (q_1, \theta_1 u(q_1))$ and $g(\theta_2) = (q_2^{SB}, \theta_2 u(q_2^{SB}) - \Delta\theta u(q_1))$, where q_2^{SB} and q_1^{SB} are as above.
5. $\mathcal{A} = \{a', a''\}$; $g(a') = (q_1, \theta_1 u(q_1))$ and $g(a'') = (q_2^{SB}, \theta_2 u(q_2^{SB}) - \Delta\theta u(q_1))$, where q_2^{SB} and q_1^{SB} are as above.
6. $\mathcal{A} = \{a', a'', a'''\}$; $g(a') = (q_1, \theta_1 u(q_1))$ and $g(a'') = (q_2^{SB}, \theta_2 u(q_2^{SB}) - \Delta\theta u(q_1))$, and $g(a''') = (0, T > 0)$.

Contract as Mechanism III

Question

Under each of the above mechanisms

- *What is the equilibrium ?*
- *What is the outcome ?*

Remark

- Each of the above mechanisms generates a Bayesian game
- Each equilibrium of the game (defined in terms of action taken by players) induces an outcome.
- That is, if σ_M is an equilibrium, then the mechanism induces an outcome allocation mapping $o \equiv g \circ \sigma_M : \Theta \mapsto \mathcal{O}$

Direct Vs Indirect Mechanisms I

Indirect: Principal offers wider choice set $[q, T(q)]$, where $q \in Q \subset \Re_+$ and

$$T : Q \mapsto \Re_+.$$

Under this approach,

- $\mathcal{A} = Q \subset \Re_+$;
- $g(q) = (q, T(q))$

Now, the agent of type θ_i will choose

$$q^*(\theta_i) = \arg\max_{q \in Q}\{U(\theta_i, q, T(q)) \equiv \arg\max_{q \in Q}\{\theta_i u(q) - T(q)\}$$

Let

$$q^*(\theta_1) = q_1, \text{ and } T(q_1^*) = t_1. \tag{1}$$

and

$$q^*(\theta_2) = q_2, \text{ and } T(q_2^*) = t_2. \tag{2}$$

Direct Vs Indirect Mechanisms II

Note the following will hold: For all $i, j = 1, 2$

$$\begin{aligned} U(\theta_i, q_i, t_i) = \theta_i u(q_i) - t_i &\geq \theta_i u(q_j) - t_j = U(\theta_i, q_j, t_j) \\ U(\theta_i, q_i, t_i) = \theta_i u(q_i) - t_i &\geq 0 \end{aligned}$$

That is, we have

$$\theta_1 u(q_1) - t_1 \geq \theta_1 u(q_2) - t_2 \quad (3)$$

$$\theta_2 u(q_2) - t_2 \geq \theta_2 u(q_1) - t_1 \quad (4)$$

$$\theta_1 u(q_1) - t_1 \geq 0, \quad (5)$$

$$\theta_2 u(q_2) - t_2 \geq 0. \quad (6)$$

Direct Vs Indirect Mechanisms III

Direct: The principal offers the following contract: $\{(q_1, t_1), (q_2, t_2)\}$, where

$$q_i = q^*(\theta_i), \text{ and } t_i = T(q_i^*),$$

as defined in (1). Under this approach,

- $\mathcal{A} = \{\theta_1, \theta_2\}$;
- $g(\theta_1) = (q_1, T_1)$ and $g(\theta_2) = (q_2, T_2)$

Question

What are the outcomes under the above contracts?

Question

- *The first approach is a general (indirect) mechanism*
- *The second approach is a direct revelation mechanism*
- *The second approach is a direct and 'truthful revelation' mechanism*

Direct Vs Indirect Mechanisms IV

Proposition

For every mechanism there exists a direct truthful revelation mechanism.

Remark

- An indirect mechanism can be replaced with a direct mechanism which attains the same outcome
- Optimization using direct mechanism is simpler

Under the general approach, P solves:

$$\max_{(T(q))} \sum \{\nu[T_1 - cq_1] + (1-\nu)[T_2 - cq_2]\},$$

s.t.

$$q_i = \arg\max_{q \in Q} \{\theta_i u(q) - T(q)\}$$

and $\theta_i u(q_i) - T_i \geq 0$.

Direct Vs Indirect Mechanisms V

Under the direct approach, P solves:

$$\max_{(q_1,T_1),(q_2,T_2)} \sum \{\nu[T_1 - cq_1] + (1-\nu)[T_2 - cq_2]\}$$

s.t.

$$\begin{aligned} \theta_1 u(q_1) - t_1 &\geq 0, & (7) \\ \theta_2 u(q_2) - t_2 &\geq 0. & (8) \end{aligned}$$

$$\begin{aligned} \theta_1 u(q_1) - t_1 &\geq \theta_1 u(q_2) - t_2 & (9) \\ \theta_2 u(q_2) - t_2 &\geq \theta_2 u(q_1) - t_1 & (10) \end{aligned}$$

The Revelation Principle I

Definition

Mechanism: A mechanism M is a pair $(\mathcal{A}, g)$, where $g(.) : \mathcal{A} \mapsto \mathcal{O}$, s.t.

$$(\forall a \in \mathcal{A})[g(a) = (q(a), T(a))]$$

Definition

A Direct Revelation Mechanism (DRM): A mechanism M is direct if $\mathcal{A} = \Theta$.

Definition

Direct Truthful Revelation Mechanism: A mechanism M is direct and truthful if $\mathcal{A} = \Theta$, and for all $\theta_i, \theta_j \in \Theta$

$$U(\theta_i, g(\theta_i)) = \theta_i u(q(\theta_i)) - T(\theta_i) \geq \theta_i u(q(\theta_j)) - T(\theta_j) = U(\theta_i, g(\theta_j)) \quad (11)$$

$$U(\theta_j, g(\theta_j)) = \theta_j u(q(\theta_j)) - T(\theta_j) \geq \theta_j u(q(\theta_i)) - T(\theta_i) = U(\theta_j, g(\theta_i)) \quad (12)$$

The Revelation Principle II

Suppose, the principal adopts a general mechanism $M = (\mathcal{A}, g)$.
Agent with θ_i will choose $a^*(\theta_i) \in \mathcal{A}$ s.t. for all $a \in \mathcal{A}$

$$\theta_i u(q(a^*(\theta_i))) - T(a^*(\theta_i)) \geq \theta_i u(q(a)) - T(a) \tag{13}$$

Remark

Note mechanism a $M = (\mathcal{A}, g)$ induces an *outcome* mapping/rule $o(.) : \Theta \mapsto \mathcal{O}$ such that

$$o(\theta) = g(a^*(\theta_i)) = (q(a^*(\theta_i)), T(a^*(\theta_i))).$$

Proposition

For every a mechanism $M = (\mathcal{A}, g)$, there exists a DTRM that implements the same allocation.

The Revelation Principle III

Proof: Take any mechanism, say, $M = (\mathcal{A}, g)$. Let

$$g(a) = (q(a), T(a)).$$

Suppose it induces output allocation rule $o(.) : \Theta \mapsto \mathcal{O}$.
If the principle adopts such a mechanism, agent with θ_i will choose $a^*(\theta_i) \in \mathcal{A}$ s.t. for all $a \in \mathcal{A}$

$$U(\theta_i, g(a^*(\theta_i))) = \theta_i u(q(a^*(\theta_i))) - T(a^*(\theta_i)) \geq \theta_i u(q(a)) - T(a) = U(\theta_i, g(a))$$

In particular, for all $\theta_j \in \Theta$ and $a^*(\theta_j)$, the following holds:

$$\theta_i u(q(a^*(\theta_i))) - T(a^*(\theta_i)) \geq \theta_i u(q(a^*(\theta_j))) - T(a^*(\theta_j)). \qquad (14)$$

The Revelation Principle IV

Define a mapping $\tilde{g}(.) : \Theta \mapsto \mathcal{A}$, s.t. for all $\theta_i, \theta_j \in \Theta$

$$\begin{aligned} \tilde{g}(\theta_i) &= (\tilde{q}(\theta_i), \tilde{T}(\theta_i)) \\ &= (q(a^*(\theta_i)), T(a^*(\theta_i))) = g(a^*(\theta_i)) \end{aligned}$$

$$\begin{aligned} \tilde{g}(\theta_j) &= (\tilde{q}(\theta_j), \tilde{T}(\theta_j)) \\ &= (q(a^*(\theta_j)), T(a^*(\theta_j))) = g(a^*(\theta_j)) \end{aligned}$$

Now, $(\Theta, \tilde{g}(.))$ is a DRM.

Moreover, in view of definition of $\tilde{g}(.)$, (14) implies: for all $\theta_i, \theta_j \in \Theta$

$$U(\theta_i, \tilde{g}(\theta_i)) = \theta_i u(\tilde{q}(\theta_i)) - \tilde{T}(\theta_i) \geq \theta_i u(\tilde{q}(\theta_j)) - \tilde{T}(\theta_j) = U(\theta_i, \tilde{g}(\theta_j)), i.e.,$$

$(\Theta, \tilde{g}(.))$ is a DTRM.