1 Mixed strategy

In this section we shall introduce strategic form games that include mixed actions.
As usual, set of players is denoted by N. Player i's 'pure strategy/action set' is denoted by A_i. We assume that players have finite number of pure actions. $A_i = \{a_i^1, a_i^2, \ldots, a_i^{m_i}\}$. Thus a_i^j denotes $j-th$ pure action of player i; i has m_i pure actions.
If a player randomizes over her pure actions then it is called a 'mixed action/strategy'. Therefore for player i, a mixed strategy is a probablity distribution over A_i. A typical mixed strategy for player i can be denoted by $p_i = \{p_i^1, p_i^2, \ldots, p_i^{m_i}\}$, where p_i^j denotes the probability attached to pure action a_i^j. Since this is a probability distribution, $0 \leq p_i^j \leq 1$ for all $j = 1, 2, \ldots, m_i$ and $\sum_{j=1}^{m_i} p_i^j = 1$.
Set of all mixed strategies of player i is denoted by Δ_i. Note that pure strategies also belong to Δ_i. These are probability distributions which put probability 1 on one pure strategy and 0 on the rest. Players choose strategies simultaneously and independently.
We assume that all agents are expected utility maximizer. When player i has chosen pure action a_i^j and the rest have chosen $(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$, expected utility of player i is computed by

$$E[\pi_i(a_i^j, p_{-i})] = \sum_{\{a_k^l\}_{k \neq i} \in A_{-i}} \left(\Pi_{k \neq i} p_k^l\right) \pi_i \left(a_i^j, \{a_k^l\}_{k \neq i}\right)$$

Expected utility of player i from p_i when the rest are playing $(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$ is given by

$$E[\pi_i(p_i, p_{-i})] = \sum_{j=1}^{m_i} p_i^j E[\pi_i(a_i^j, p_{-i})]$$

A strategic form game (with mixed strategies) can be written as

$$< N, \{\Delta_i\}_{i \in N}, \{E[\pi_i]\}_{i \in N} >$$

.

2 Mixed strategy Nash equilibrium

Best response of player i when other players are playing p_{-i} is a set of mixed strategies that maximizes i's expected payoff (given p_{-i})

$$B_i(p_{-i}) = \{\bar{p}_i \in \Delta_i \mid E[\pi_i(\bar{p}_i, p_{-i})] \geq E[\pi_i(p_i, p_{-i})] \text{ for all } p_i \in \Delta_i\}$$

$(p_1^*, \ldots, p_n^*)$ is a Nash equilibrium of $< N, \{\Delta_i\}_{i \in N}, \{E[\pi_i]\}_{i \in N} >$, if strategies are mutually best response, that is $p_i^* \in B_i(p_{-i}^*)$ for all $i \in N$.
Following results characterizes all mixed strategy Nash equilibrium.

Result 1: Let $G =< N, \{\Delta_i\}_{i \in N}, \{E[\pi_i]\}_{i \in N} >$ is a strategic form game. $(p_1^*, \ldots, p_n^*)$ is a Nash equilibrium of G implies that for all players $i \in N$, every pure strategy (a_i^j) that is played with strict positive probability $(p_i^{*j} > 0)$ is a best response to p_{-i}^*.
[Note that the above result does not exclude the possibility that a pure strategy which gets 0 probability in Nash equilibrium (that is $p_i^{*j} = 0$) can also be a best response to p_{-i}^*.]
Proof of Result 1: We are going to use the following identity

$$E[\pi_i(p_i, p_{-i})] = \sum_{j=1}^{m_i} p_i^j E[\pi_i(a_i^j, p_{-i})] = \sum_{j \in \{k | p_i^k > 0\}} p_i^j E[\pi_i(a_i^j, p_{-i})]$$

which holds for all p_i and p_{-i}. At (p_i^*, p_{-i}^*),

$$E[\pi_i(p_i^*, p_{-i}^*)] = \sum_{j \in \{k | p_i^{*k} > 0\}} p_i^{*j} E[\pi_i(a_i^j, p_{-i}^*)]$$

That is the expected payoff from playing p_i^* is a weighted average of expected payoffs from all pure strategies, which are getting positive probability at p_i^*. Now suppose that one of those pure strategies (say a_i^k) is not a best response to p_{-i}^*. Then i can increase her payoff by reducing p_i^{*k} and transferring probability weight to a pure strategy that is a best response to p_{-i}^*. However, such possibility implies that p_i^* itself is not a best response to p_{-i}^*. But

$(p_1^*, \dots, p_n^*)$ is a Nash equilibrium. Therefore our assumption that a_i^k is not a best response to p_{-i}^* is wrong. [Proved]

An important implication of this result is: All pure strategies of player i, that are played with positive probability in Nash equilibrium, have equal expected payoff.

The converse is also true.

Result 2: Let $G =< N, \{\Delta_i\}_{i \in N}, \{E[\pi_i]\}_{i \in N} >$ is a strategic form game. Let $(p_1^*, \dots, p_n^*)$ be a profile of mixed strategies with the following property. For all players $i \in N$, every pure strategy (a_i^j) that is played with strict positive probability $(p_i^{*j} > 0)$ is a best response to p_{-i}^*. Then $(p_1^*, \dots, p_n^*)$ is a Nash equilibrium of G.

Proof of Result 2: Since each a_i^j, which is played with strict positive probability $(p_i^{*j} > 0)$, is a best response to p_{-i}^*, all of them fetch equal expected payoff (say R). Expected payoff from p_i^* is

$$E[\pi_i(p_i^*, p_{-i}^*)] = \sum_{j \in \{k | p_i^{*k} > 0\}} p_i^{*j} E[\pi_i(a_i^j, p_{-i}^*)] = R \sum_{j \in \{k | p_i^{*k} > 0\}} p_i^{*j} = R$$

Since R is the highest expected payoff of i against p_{-i}^*, p_i^* is also a best response to p_{-i}^*. This is true for any $i \in N$. Therefore $(p_1^*, \dots, p_n^*)$ is a Nash equilibrium of G. [Proved]

3 Existence of Nash equilibrium [Optional]

Result 3: Every strategic game with finite pure strategies has a (mixed-strategy) Nash Equilibrium.

We are going to use Brouwer's fixed point theorem.

Brouwer's fixed point theorem: Let Δ be a compact (that is closed and bounded) subset of real space. Further assume that Δ is convex. If $f : \Delta \to \Delta$ is a continuous map, then there must exist $x^* \in \Delta$ such that $f(x^*) = x^*$. x^* is a fixed point of f.

Proof of Result 3: Let Δ be Cartesian product of players' strategy set. That is $\Delta = \Pi_{i \in N} \Delta_i$. Take any $p = (p_1, \ldots, p_n) \in \Delta$.
For all $i \in N$, $a_i^j \in A_i$, let us define,

$$g_i^j(p) = \max\left\{\left[E[\pi_i(a_i^j, p_{-i})] - E[\pi_i(p_i, p_{-i})]\right], 0\right\}$$

$g_i^j(p)$ is the gain for player i from switching to the pure strategy a_j^i from p_i, when rest of the players are playing p_{-i}. We already know that

$$E[\pi_i(p_i, p_{-i})] = \sum_{j=1}^{m_i} p_i^j E[\pi_i(a_i^j, p_{-i})] = \sum_{j \in \{k | p_i^k > 0\}} p_i^j E[\pi_i(a_i^j, p_{-i})]$$

that is $E[\pi_i(p_i, p_{-i})]$ is a weighted sum of $E[\pi_i(a_i^j, p_{-i})]$ for all pure strategies with $p_i^j > 0$. Since $E[\pi_i(p_i, p_{-i})]$ is a weighted sum, it can not be the case that $g_i^j(p) > 0$ for all $p_i^j > 0$.
Let us now define a map f as follows. For all $1 \leq i \leq n$ and $1 \leq j \leq m_i$,

$$f_{ij}(p) = \frac{\left[p_i^j + g_i^j(p)\right]}{\left[1 + \sum_{j=1}^{m_i} g_i^j(p)\right]}$$

Check that $0 \leq f_{ij}(p) \leq 1$ and $\sum_{j=1}^{m_i} f_{ij}(p) = 1$. Thus f is a continuous map from $\Delta \to \Delta$. Δ is a compact, convex set. By Brouwer's fixed point theorem, f has a fixed point; there exist a $p^* \in \Delta$ such that $f(p^*) = p^*$. For all $1 \leq i \leq n$ and $1 \leq j \leq m_i$,

$$f_{ij}(p^*) = \frac{\left[p_i^{*j} + g_i^j(p^*)\right]}{\left[1 + \sum_{j=1}^{m_i} g_i^j(p^*)\right]} = p_i^{*j}$$

Thus, $p_i^{*j}\left[\sum_{j=1}^{m_i} g_i^j(p^*)\right] = g_i^j(p^*)$. This equality can hold under two possibilities
(i) At least one $g_i^j(p^*)$ is greater than 0, which implies $\left[\sum_{j=1}^{m_i} g_i^j(p^*)\right]$ is greater than 0. Then from the above equality, $g_i^j(p^*) > 0$, whenever $p_i^{*j} > 0$. However we have already argued that it is not possible by definition of $g_i^j(p)$.
So only possibility is

(ii) All $g_i^j(p^*) = 0$. This means, there is no gain for player i from switching to any pure strategy a_j^i from p_i^*, when rest of the players are playing p_{-i}^*. Since there is no gain from switching to any pure strategy, there is also no gain from switching to any other mixed strategy (a weighted average of returns from pure strategies). Thus p_i^* is a best response to p_{-i}^* for all i. Therefore $p^* = (p_1^*, \ldots, p_n^*)$ is a Nash equilibrium. [Proved]