1 Extensive form game and its subgames

In this section, we introduce a few notations describing an extensive form games and its subgames.

Let $\Gamma =< N, H, P, \{\pi_i\}_{i \in N} >$ be an extensive form game.

N: Set of players

H: Set of histories

P: Player function (describes which player is making a choice at history h for all $h \in H$)

π_i: payoff of player i at all terminal histories

Action: Take a history h, let i be the player making a choice at h (that is $P(h) = i$). $A(h)$ denotes the set of actions available to player i at h.

Strategy: A strategy is a complete plan of action. A strategy of player i (say s_i) describes i's choice of action at every history where i is the decision maker. Formally, s_i is a function; $s_i(h) \in A(h)$ at all $h \in H$, such that $P(h) = i$. S_i denotes strategy set of player i.

Intuitively, a **subgame** starting at history h is the residual game that follows h. Formally, it can be denoted by $\Gamma(h) =< N, H_h, P_h, \{\pi_i\}_{i \in N} >$

$H|_h$: Set of histories in the residual game. $H|_h = \{h' \mid \{(h, h')\} \in H\}$

$P|_h$: Player function of the residual game. $P|_h(h') = P(\{h, h'\})$

Given a strategy s_i of player i in Γ, restriction of s_i in the subgame $\Gamma(h)$ is simply the relevant portion of the plan of action for the residual game. Formally, $s_i|_h(h') = s_i(\{h, h'\})$ for all $h' \in H|_h$.

2 Subgame Perfect Nash equilibrium and One Deviation property

Definition

A strategy profile $(s_1^*, \ldots, s_n^*)$ is a **Subgame Perfect Nash Equilibrium** of a game Γ implies that $(s_1^*|_h, \ldots, s_n^*|_h)$ is a Nash equilibrium of subgame

$\Gamma(h)$ for all $h \in H$ (Note that this includes the main game, that is history $\emptyset$). Thus for all $h \in H$ and for $i \in N$, $s_i^*|_h$ is a best response to $s_{-i}^*|_h$.

Definition

A strategy profile $(s_1^*, \ldots, s_n^*)$ satisfies **One Deviation Property** if for all $h \in H$ (suppose $P(h) = i$),

$$\pi_i\left(s_i^*|_h, s_{-i}^*|_h\right) \geq \pi_i\left(s_i', s_{-i}^*|_h\right)$$

where (i) s_i' is a strategy of player i in the subgame $\Gamma(h)$ and (ii) s_i' and $s_i^*|_h$ only differ at the start of the subgame (that is at history h).
In each subgame, the player who makes the first move cannot benefit by changing her initial action.

Result

For finite horizon extensive form games:
A strategy profile $(s_1^*, \ldots, s_n^*)$ is a Subgame Perfect Nash Equilibrium of a game $\Gamma \Leftrightarrow (s_1^*, \ldots, s_n^*)$ satisfies One Deviation Property.
Proof: If $(s_1^*, \ldots, s_n^*)$ is a Subgame Perfect Nash Equilibrium then it trivially satisfies One Deviation Property.

We shall show that if $(s_1^*, \ldots, s_n^*)$ satisfies One Deviation Property then it must be a SPNE. (Proof by contradiction) If this is not correct then we must have a strategy profile $(s_1^*, \ldots, s_n^*)$ which satisfies ODP but is not a SPNE. If $(s_1^*, \ldots, s_n^*)$ is not a SPNE then (by definition of SPNE) there is a history h and player $k \in N$ such that $s_k^*|_h$ is NOT a best response to $s_{-k}^*|_h$ in $\Gamma(h)$.

Step 1: Thus k has at least one profitable deviation in the subgame $\Gamma(h)$. From among all profitable deviations, choose $\hat{s}_k$ for which the number of histories h' where $s_k^*|_h$ differs from $\hat{s}_k$ (that is, $s_k^*|_h(h') \neq \hat{s}_k(h')$) is minimal. Since Γ has a finite horizon, so does $\Gamma(h)$ and this minimal number is finite.

Let $\hat{h}$ be the longest history such that $s_k^*|_h(\hat{h}) \neq \hat{s}_k(\hat{h})$.

Step 2: Now, consider the strategy profile $\left(\hat{s}_k, s_{-k}^*|_h\right)$ for subgame $\Gamma(h)$. The game path of this strategy profile must be passing through $\hat{h}$. Otherwise, the difference of choice of action at $\hat{h}$ between $\hat{s}_k$ and $s_k^*|_h$ is immaterial for profitability of $\hat{s}_k$. In which case, such difference is redundant and one can have a profitable deviation with fewer differences with $s_k^*|_h$. This is not possible because by choice $\hat{s}_k$ has the minimal difference with $s_k^*|_h$ among all profitable deviations (refer to Step 1). Therefore the game path of strategy profile $\left(\hat{s}_k, s_{-k}^*|_h\right)$ must be passing through $\hat{h}$. Now, $\hat{s}_k(\hat{h})$ must be responsible for the profitability of $\hat{s}_k$, otherwise we can have a profitable deviation with fewer differences with $s_k^*|_h$.

Step 3: Now, consider $\Gamma(\hat{h})$ (the subgame following $\hat{h}$) and the strategy profile $\left(\hat{s}_k|_{\hat{h}}, s_{-k}^*|_{\hat{h}}\right)$. Since $\hat{h}$ was the longest history such that $s_k^*|_h(\hat{h}) \neq \hat{s}_k(\hat{h})$, $\hat{s}_k|_{\hat{h}}$ and $s_k^*|_{\hat{h}}$ differ only at the initial history of the subgame under consideration. By choice of $\hat{s}_k$ (refer to Step 1) and Step 2, $\hat{s}_k|_{\hat{h}}$ is a profitable deviation from $s_k^*|_{\hat{h}}$ when the rest are playing $s_{-k}^*|_{\hat{h}}$. However this violates ODP of $(s_1^*, \ldots, s_n^*)$ at history $\hat{h}$.

Hence what we assumed to start with (that there is strategy profile which satisfies ODP but is not a SPNE) is incorrect. [Proved]