Linear Contracts

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SB: Linear Contracts I

Assumptions:

- $q(e, \epsilon) = e + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$.
- Principal is risk-neutral. $V(q, w) = q - w$
- Agent is risk-averse. $u(w, e) = -e^{-r(w-\psi(e))}$, $r > 0$, where $\psi(e)$ is the (money) cost of effort e.
- $r = -\frac{u''}{u'} > 0$, i.e., CARA
- $\psi(e) = \frac{1}{2}ce^2$, $c > 0$.
- Contract: $w(q) = t + sq$, where $s > 0$.
- $\bar{w}$ = Certainty equivalent of the reservation (outside) wage

SB: Linear Contracts II

Note $u(w, e)$ is increasing in w and decreasing in e.

The First Best: The first best is solution to

$$\max_{e,t,s} E(q - w)$$

s.t. $-e^{-r(w-\psi(e))} = -e^{-r\bar{w}}$, i.e., $w - \psi(e) = \bar{w}$, i.e., $w = \bar{w} + \psi(e)$.

Therefore, the first best is solution to

$$\max_{e} E(e + \epsilon - \bar{w} - \psi(e)), i.e.,$$

$$\max_{e}\{e - \frac{1}{2}ce^2\},$$

since $E(\epsilon) = 0$. Therefore, the first best effort level is given by the following foc

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$$ce^* = 1, i.e., e^* = \frac{1}{c}. \tag{1}$$

When e contractible, the following contract can achieve the first best:

$$w = \bar{w} + \tfrac{1}{2c} \text{ if } e = \tfrac{1}{c};$$
$$w = -\infty \text{ otherwise.}$$

Second Best: e is not contractible but q is. The principal solves

$$\max_{e,t,s} E(q - w)$$

s.t.

$$\begin{aligned} E(u(w,e)) = E(-e^{-r(w-\psi(e))}) &\geq -e^{-r(\bar{w})} = u(\bar{w}) && (IR) \\ e &= \arg\max_{\hat{e}} E(-e^{-r(w-\psi(\hat{e}))}) && (IC) \end{aligned}$$

SB: Linear Contracts IV

Note that $-r(w-\psi(e)) = -r(t+sq-\psi(e)) = -r(t+s(e+\epsilon)-\psi(e))$, i.e., $-r(w-\psi(e)) = -r(t+se-\psi(e)) - rs\epsilon$. Therefore,

$$E(-e^{-r(w-\psi(e))}) = -E(e^{-r(t+se-\psi(e))-rs\epsilon}), i.e.$$

$$E(-e^{-r(w-\psi(e))}) = -E(e^{-r(t+se-\psi(e))}.e^{-rs\epsilon}), i.e.$$

$$E(-e^{-r(w-\psi(e))}) = -e^{-r(t+se-\psi(e))}E(e^{-rs\epsilon}).$$

Since for a random variable x is such that $x \sim N(0,\sigma_x^2)$, so

$$E(e^{\gamma x}) = e^{\gamma^2 \frac{\sigma_x^2}{2}}.$$

Therefore, we have

$$E(-e^{-r(w-\psi(e))}) = -e^{-r(t+se-\psi(e))}.e^{r^2 s^2 \frac{\sigma^2}{2}}, i.e.,$$

SB: Linear Contracts V

$$E(-e^{-r(w-\psi(e))}) = -e^{-r(t+se-\psi(e))+r^2s^2\frac{\sigma^2}{2}}. \quad (2)$$

Remark

Let's define

$$-e^{-r\hat{w}(e)} = E(-e^{-r(w-\psi(e))}) \quad (3)$$

From (2) *and* (3)

$$r\hat{w}(e) = -r(t+se-\psi(e)) + r^2s^2\frac{\sigma^2}{2}, i.e.,$$

$$\underbrace{\hat{w}(e)}_{certainty-equivalent\ \ wage} = \underbrace{t+se}_{expected\ \ wage} - \frac{1}{2}ce^2 - \underbrace{rs^2\frac{\sigma^2}{2}}_{risk-premium}$$

SB: Linear Contracts VI

Therefore, the agent will choose e to solve

$$\max_{\hat{e}}\{\hat{w}(e) = r(t + se - \psi(e)) - r^2 s^2 \frac{\sigma^2}{2}\}.$$

the foc for which is $s - ec = 0$, i.e.,

$$e^{SB} = \frac{s}{c} \tag{4}$$

Therefore, the Principal's problem can be written as

$$\max_{e,t,s} E(q - w),\ i.e.,\ \max_{e,t,s} E(e + \epsilon - (t + sq)),\ i.e.,$$

$$\max_{e,t,s} E(e + \epsilon - t - s(e + \epsilon)),\ i.e.,$$

$$\max_{e,t,s}(e - t - se)$$

s.t.

SB: Linear Contracts VII

$$\hat{w}(e) = t + se - \psi(e) - rs^2\frac{\sigma^2}{2} \quad \geq \quad \bar{w} \qquad (IR)$$

$$e \quad = \quad \frac{s}{c} \qquad (IC)$$

That is,

$$\max_{t,s}\{\frac{s}{c} - t - s\frac{s}{c}\}$$

s.t.

$$t + s\frac{s}{c} - \frac{c}{2}\frac{s^2}{c^2} - rs^2\frac{\sigma^2}{2} = \bar{w}$$

That is,

$$\max_{s}\{\frac{s}{c} - \frac{s^2}{c} + \frac{s^2}{2c} + rs^2\frac{\sigma^2}{2} - \frac{s^2}{c}\}$$

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The foc w.r.t. s is

$$s = \frac{1}{1 + rc\sigma^2} \tag{5}$$

Remark

$r > 0 \Rightarrow s < 1$, *and* $s < 1 \Rightarrow e^{SB} < e^*$.
$r = 0 \Rightarrow s = 1$, *i.e.*, $e^{SB} = e^*$.
$s \propto \frac{1}{r}$, $s \propto \frac{1}{c}$ *and* $s \propto \frac{1}{\sigma}$.

Linear Contracts: Sharecropping I

Model:

- $q =$ output; $q = q(e, \epsilon)$; $q \in \{q_L, q_H\}$, $q_L < q_H$.
- Monetary worth of $q = q$ (assume price is 1)
- $\epsilon =$ a random variable, a noise term;
- $e =$ effort level opted by the agent; $e \in \{0, 1\}$.
- $\psi(0) = 0$ and $\psi(1) = \psi$.
- $p_H = Pr(q = q_H | e = 1)$ is the probability of the realized output being q_H; and $p_L = Pr(q = q_H | e = 0)$.
- $w =$ wage paid by the principal to the agent; $w(.) = w(q)$.
- Let the wage contract $w(q) = sq$ be linear; say, $0 \leq s \leq 1$.

Linear Contracts: Sharecropping II

Assume that both parties are risk-neutral. So

Payoff functions are:

- Principal: $V(x) = x$, $V' > 0$, $V'' = 0$;
- Agent: $u(w, e) = u(w) - \psi(e)$, where $u' > 0$, $u'' = 0$.

Optimum Linear Contract:
Suppose the P wants to induce $e = 1$. Then, risk-neutral P will solve

$$\max_s \{(1 - s)[p_H q_H + (1 - p_H) q_L]\}$$

s.t.

$$s[p_H q_H + (1 - p_H) q_L] - \psi \geq 0 \quad (6)$$
$$s[p_H q_H + (1 - p_H) q_L] - \psi \geq s[p_L q_H + (1 - p_L) q_L] \quad (7)$$

Note $s > 0$ and (7) implies (6).

Linear Contracts: Sharecropping III

Let $\Delta p = p_H - p_L$ and $\Delta q = q_1 - q_0$.

Exercise:

- Ignoring IR, show that IC binds
- the foc w.r.t. s is
$$s^{SB} = \frac{\psi}{\Delta p \Delta q}$$
- Find out whether IR finds

Linear Contracts: Sharecropping IV

Second Best: Suppose the P wants to induce $e = 1$. Then, risk-neutral P will solve

$$\max_{w_L, w_H} \{p_H[q_H - w_H] + (1 - p_H)[q_L - w_L]\}$$

s.t.

$$p_H w_H + (1 - p_H) w_L - \psi \geq 0 \quad (8)$$

$$p_H w_H + (1 - p_H) w_L - \psi \geq p_L w_H + (1 - p_L) w_L \quad (9)$$

Exercise:

- The SB contract is superior to the sharecropping; that is linear contract is NOT Second Best
- Compared to the SB, the agent is better-off under sharecropping contract
- Find out whether IR finds

Sub-optimality of Linear Contracts I

Suppose:

- $q = q(e, \epsilon) = e + \epsilon$
- The error term $\epsilon \in [-k, k]$, where $0 < k < \infty$
- For instance, assume ϵ has uniform distribution over $[-k, k]$
- Principal is risk-neutral. $V(q, w) = q - w$
- Agent is risk-averse. $u(w, e) = u(w) - \psi(e)$, where $u' > 0$, $u'' < 0$ and $\psi(e)$, is the dis-utility of effort e; $\psi'(e) > 0$, $\psi''(e) > 0$
- Let $e^{FB} = e^*$
- Let w^* solve $u(w^*) = \psi(e^*)$.

Sub-optimality of Linear Contracts II

Note since $q = q(e, \epsilon) = e + \epsilon$,

$$q \in [e^* - k, e^* + k] \text{ if } e = e^*.$$

$$q < e^* - k \text{ only if } e < e^*.$$

So, when the output has bounded support which depends on the effort, q can sever as a perfectly informative about e.

Recall w^* solves $u(w^*) = \psi(e^*)$.

Now consider the following contract:

$$w(q) = \begin{cases} w^*, & \text{if } q \in [e^* - k, e^* + k]; \\ -\infty, & \text{if } q \notin [e^* - k, e^* + k. \end{cases}$$

This contract ensures the FB outcome; it implements e^* as well, and provides full insurance to the risk-averse agent.

Sub-optimality of Linear Contracts I

Now we consider unbounded support for the output.
Mirrlees (1975, 1999 RES) showed that even with unbounded support, output can be sufficiently informative about effort.
Assumptions:

- $q(e, \epsilon) = e + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$.
- $f(q, e) = \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{(q-e)^2}{2\sigma^2}}$
- Principal is risk-neutral. $V(q, w) = q - w$
- Agent is risk-averse. $u(w, e) = u(w) - \psi(e)$, $u' > 0$, $u'' < 0$ where $\psi(e)$, is the (money) cost of effort e; $\psi'(e) > 0$, $\psi''(e) > 0$.

Sub-optimality of Linear Contracts II

Note that

$$f_e(q,e) = -\frac{1}{\sqrt{2\pi\sigma}} \exp^{-\frac{(q-e)^2}{2\sigma^2}} \times \frac{-(q-e)}{\sigma^2}$$

$$\frac{f_e(q,e)}{f(q,e)} = \frac{q-e}{\sigma^2}$$

Therefore, for given effort level e,

$$q \to \infty \Rightarrow \frac{f_e(q,e)}{f(q,e)} \to \infty \tag{10}$$

$$q \to -\infty \Rightarrow \frac{f_e(q,e)}{f(q,e)} \to -\infty \tag{11}$$

That is, for given e, the likelihood ratio $\frac{f_e(q,e)}{f(q,e)}$ is increasing in q, without bounds.

Sub-optimality of Linear Contracts III

First-Best: The P solves

$$\max_{e,w(q)} \{E(q-w)\}$$

s.t.

$$E(u(w(q),e)) = \int u(w(q))f(q,e)dq - \psi(e) \geq \bar{u} = 0$$

Let (e^*, w^*) be the solution.

Clearly, in the FB the risk-averse agent is fully insured. The FB wage w^* is given by the binding IR, i.e.,

$$E(u(w^*,e^*)) = \int u(w^*)f(q,e)dq - \psi(e^*) = u(w^*) - \psi(e^*) = 0, i.e.,$$

$$\int_{-\infty}^{\underline{q}} u(w^*)f(q,e^*)dq + \int_{\underline{q}}^{\infty} u(w^*)f(q,e^*)dq - \psi(e^*) = 0 \qquad (12)$$

Sub-optimality of Linear Contracts IV

In view of (11) for any $M > 0$, however large, $\exists \underline{q}$ such that

$$(\forall q < \underline{q})[\frac{f_e(q, e^*)}{f(q, e^*)} < -M], i.e.,$$

$$(\forall q < \underline{q})[f(q, e^*) < \frac{-1}{M} f_e(q, e^*)] \tag{13}$$

Now, consider the following contract

$$w(q) = \begin{cases} w^*, & \text{if } q \geq \underline{q}; \\ K, & \text{if } q < \underline{q}. \end{cases}$$

$w(q)$ will induce the FB effort e^* if

$$e^* = \arg\max\{E(u(w(q), e)) = \int u(w(q))f(q, e)dq - \psi(e)\}, i.e.,$$

if K is such that

Sub-optimality of Linear Contracts V

$$e^{*} = \arg\max\{\int_{-\infty}^{\underline{q}} u(K) f(q,e) dq + \int_{\underline{q}}^{\infty} u(w^{*}) f(q,e) dq - \psi(e)\}, i.e.,$$

$$\int_{-\infty}^{\underline{q}} u(K) f_e(q,e^{*}) dq + \int_{\underline{q}}^{\infty} u(w^{*}) f_e(q,e^{*}) dq = \psi'(e^{*}), i.e., \tag{14}$$

$$\int_{\underline{q}}^{\infty} u(w^{*}) f_e(q,e^{*}) dq - \psi'(e^{*}) = -\int_{-\infty}^{\underline{q}} u(K) f_e(q,e^{*}) dq \tag{15}$$

Suppose, K in the above contract satisfies (14), i.e., (15). Under the contract the agent's payoff is

$$\int_{-\infty}^{\underline{q}} u(K) f(q,e^{*}) dq + \int_{\underline{q}}^{\infty} u(w^{*}) f(q,e^{*}) dq - \psi(e^{*}) \tag{16}$$

Now (14) – (16) give us

Sub-optimality of Linear Contracts VI

$$\int_{-\infty}^{\underline{q}} [u(w^*) - u(K)]f(q, e^*)dq = I \text{ say} \tag{17}$$

Note the above contract fails to meet IR only by the term in the LHS of (14), i.e., by I. But (17), in view of (13), implies

$$I \leq \frac{-1}{M} \int_{-\infty}^{\underline{q}} [u(w^*) - u(K)]f_e(q, e^*)dq$$

This in view of (15) gives

$$I \leq \frac{-1}{M} \int_{-\infty}^{\infty} u(w^*)f_e(q, e^*)dq - \psi'(e^*)$$

But, RHS tends to zero as $M \to \infty$.
Therefore, the above contract *almost* satisfies IR for sufficiently large M.

Sub-optimality of Linear Contracts VII

From (13), note that

- $\underline{q} \to -\infty$ as $M \to \infty$.
- $\underline{q} \to -\infty$ implies the penalty captured by K increases

Therefore

- FB can be approximated arbitrarily through sever punishment by the following contract

$$w(q) = \begin{cases} w^* + \epsilon, & \text{if } q \geq \underline{q}; \\ K, & \text{if } q < \underline{q}; \underline{q} \to -\infty. \end{cases}$$

- The agent is almost fully insured
- As the size of punishment grows, the frequency of its use falls
- However, existence of unbounded punishment is critical to the above claims.

References: Mirlees (1999 RES) and BD