Lecture 6: Screening with continuum of types

Ram Singh

Department of Economics

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Questions

We are ready to address the following questions:

Question

- How can we solve the problems involving many (possibly infinite) types of agents?
- Do the previous results -on rent-extraction, allocative inefficiency, and efficiency-rent trade-off - hold for more complex settings?
- Does allocative efficiency and rent extraction increase with the type?
- Do contracts always satisfy the monotonicity (of consumption or production)?
- When is bunching likely to emerge?

Continuum of Types I

Suppose,

- θ is drawn from the support $[\underline{\theta}, \overline{\theta}]$
- $F(\theta)$ and $f(\theta)$ are cdf and density functions, respectively.
- Payoff function of an agent (buyer) is $u(\theta) = \theta v(q(\theta)) - T(\theta)$.
- The menu offered contracts is $(q(\theta), T(\theta))$

From the revelation principle it follows that the principal's problem is

$$\max_{q(\theta), T(\theta)} \{ \int_{\underline{\theta}}^{\overline{\theta}} [T(\theta) - cq(\theta)] f(\theta) d\theta \}$$

s.t.

$$(\forall \theta, \hat{\theta} \in [\underline{\theta}, \overline{\theta}]) \ [\theta v[q(\theta)] - T(\theta) \geq \theta v[q(\hat{\theta})] - T(\hat{\theta})] \qquad \text{(ICs)}$$

$$(\forall \theta \in [\underline{\theta}, \overline{\theta}]) \ [\theta v[q(\theta)] - T(\theta) \geq 0] \qquad (IRs)$$

Continuum of Types II

From the previous analysis we know that (*IR*) will bind only for the lowest type. It is straight to check that *IC* implies that for all

$$(\forall \theta > \underline{\theta})[\theta v[q(\theta)] - T(\theta) > 0].$$

Therefore, the above *IRs* can be replaced with

$$\underline{\theta} v[q(\underline{\theta})] - T(\underline{\theta}) \geq 0. \qquad (IR')$$

Implementable Allocations: Prosperities I

Definition : An allocation $(q(\theta), T(\theta))$ is implementable iff it satisfies (IC), i.e., iff

$$(\forall \theta, \hat{\theta} \in [\underline{\theta}, \overline{\theta}]) \; [\theta v[q(\theta)] - T(\theta) \geq \theta v[q(\hat{\theta})] - T(\hat{\theta})].$$

Proposition

Assuming that $U(.)$ satisfies the single crossing condition, an allocation $(q(\theta), T(\theta))$ is implementable iff

$$(\forall \theta \in [\underline{\theta}, \overline{\theta}]) \; [\theta v'[q(\theta)] \frac{dq(\theta)}{d\theta} - T'(\theta) = 0] \tag{0.1}$$

and

$$\frac{dq(\theta)}{d\theta} \geq 0. \tag{0.2}$$

The proposition says that IC above is equivalent to (0.1) and (0.2), where (0.1) called the local incentive compatibility constraint and (0.2), of course if the monotonicity condition.

Implementable Allocations: Prosperities II

Proof: Suppose $(q(\theta), T(\theta))$ is implementable, i.e., (IC) hold. We prove the claim for our specification of the utility function for the agent (the buyer). Let

$$W(\theta, \hat{\theta}) = \theta v[q(\hat{\theta})] - T(\hat{\theta}), i.e.,$$

$W(\theta, \hat{\theta})$ denotes the buyer's payoff when his actual type is θ but he announces/pretends his type to be $\hat{\theta}$.

Assuming that $q(\theta)$ and $T(\theta)$ are differentiable functions of θ, and $W(.)$ is differentiable function of q and T, the buyer's problem can be looked at as choosing $\hat{\theta}$ to maximize $W(.)$. Now,

$(q(\theta), T(\theta))$ is implementable implies that

$$\frac{dW(\theta, \hat{\theta})}{d\hat{\theta}}|_{\hat{\theta}=\theta} = \frac{dW(\theta, \theta)}{d\hat{\theta}} = 0, i.e.,$$

Implementable Allocations: Prosperities III

$$\{\theta v'[q(\hat{\theta})]\frac{dq(\hat{\theta})}{d\hat{\theta}} - T'(\hat{\theta})\}_{\hat{\theta}=\theta} = \theta v'[q(\theta)]\frac{dq(\theta)}{d\theta} - T'(\theta) = 0$$

This is foc. The soc will gives us

$$\frac{d^2 W(\theta, \hat{\theta})}{d\hat{\theta}^2}|_{\hat{\theta}=\theta} \leq 0, i.e.,$$

$$\{\theta v^{''}[q(\hat{\theta})](\frac{dq(\hat{\theta})}{d\hat{\theta}})^2 + \theta v'[q(\hat{\theta})](\frac{d^2 q(\hat{\theta})}{d\hat{\theta}^2}) - T^{''}(\hat{\theta})\}_{\hat{\theta}=\theta} \leq 0, i.e.,$$

$$\theta v^{''}[q(\theta)](\frac{dq(\theta)}{d\theta})^2 + \theta v'[q(\theta)](\frac{d^2 q(\theta)}{d\theta^2}) - T^{''}(\theta) \leq 0. \tag{0.3}$$

Differentiating (foc) w.r.t θ, we get

$$\theta v''[q(\theta)](\frac{dq(\theta)}{d\theta})^2 + v'[q(\theta)]\frac{dq(\theta)}{d\theta} + \theta v'[q(\theta)](\frac{d^2 q(\theta)}{d\theta^2}) - T''(\theta) = 0 \tag{0.4}$$

Implementable Allocations: Prosperities IV

From (0.3) and (0.4),

$$v'[q(\theta)]\frac{dq(\theta)}{d\theta} \geq 0, i.e., \frac{dq(\theta)}{d\theta} \geq 0,$$

in view of the fact that $v'(.) > 0$. That is, (0.2) holds.

Remark

For our specification of the agent's payoff function we have shown that (IC) implies (0.1) and (0.2). More generally, (0.1) and (0.2) follow from the single-crossing condition.

Now suppose (0.1) and (0.2) hold.
Assume the contrary to (IC), i.e., assume that

$$(\exists\theta, \hat{\theta} \in [\underline{\theta}, \overline{\theta}])\ [\theta v[q(\theta)] - T(\theta) < \theta v[q(\hat{\theta})] - T(\hat{\theta})], i.e.,$$

$$\exists\theta, \hat{\theta} \in [\underline{\theta}, \overline{\theta}][W(\theta, \hat{\theta}) > W(\theta, \theta)].$$

Implementable Allocations: Prosperities V

Let $\theta < \hat{\theta}$.

$$W(\theta,\hat{\theta}) - W(\theta,\theta) = \int_{\theta}^{\hat{\theta}} \frac{\partial W(\theta,x)}{\partial x} dx = \int_{\theta}^{\hat{\theta}} \left[\theta v'[q(x)]\frac{dq(x)}{dx} - T'(x)\right] dx$$

By (0.2), $\frac{dq(x)}{dx} \geq 0$, and

$$x > \theta \Rightarrow xv'[q(x)] > \theta v'[q(x)].$$

Therefore, we have

$$\int_{\theta}^{\hat{\theta}} \left[\theta v'[q(x)]\frac{dq(x)}{dx} - T'(x)\right] dx < \int_{\theta}^{\hat{\theta}} \left[xv'[q(x)]\frac{dq(x)}{dx} - T'(x)\right] dx$$

and

$$\int_{\theta}^{\hat{\theta}} \left[xv'[q(x)]\frac{dq(x)}{dx} - T'(x)\right] dx = 0,$$

Implementable Allocations: Prosperities VI

since by (0.1), $(\forall x)\left[xv'[q(x)]\frac{dq(x)}{dx} - T'(x) = 0\right]$. Therefore,

$$\int_{\theta}^{\hat{\theta}}\left[\theta v'[q(x)]\frac{dq(x)}{dx} - T'(x)\right]dx < 0, i.e.,$$

$$(\forall\theta < \hat{\theta})[W(\theta,\hat{\theta}) - W(\theta,\theta) < 0].$$

Case $\theta > \hat{\theta}$ is analogous. That is,

$$\underline{\theta},\overline{\theta}][W(\theta,\hat{\theta}) < W(\theta,\theta).$$

Therefore, we get a contradiction. Q.E.D.

Second Best: Solution I

Coming back to the principal's problem, in view of Proposition 1, the principal's problem can be written as

$$\max_{q(\theta),T(\theta)} \left\{ \int_{\underline{\theta}}^{\overline{\theta}} [T(\theta) - cq(\theta)]f(\theta)d\theta \right\}$$

s.t.

$$[\underline{\theta}v[q(\underline{\theta})] - T(\underline{\theta}) \geq 0] \tag{0.5}$$

$$\frac{dq(\theta)}{d\theta} \geq 0 \tag{0.6}$$

$$(\forall \theta \in [\underline{\theta}, \overline{\theta}]) \left[\theta v'[q(\theta)] \frac{dq(\theta)}{d\theta} - T'(\theta) = 0 \right] \tag{0.7}$$

Note that at the optimum, (0.5) will bind. Also, (0.6) and (0.7) are equivalent to (IC), i.e.,

$$\theta v[q(\theta)] - T(\theta) = \max_{\hat{\theta}} \theta v[q(\hat{\theta})] - T(\hat{\theta}).$$

Second Best: Solution II

Also note that

$$\max_{\hat{\theta}} \theta v[q(\hat{\theta})] - T(\hat{\theta}) = \theta v[q(\theta)] - T(\theta) = W(\theta, \theta) = W(\theta).$$

Therefore, by envelop theorem,

$$\frac{dW(\theta)}{d\theta} = v[q(\theta)] + \theta v'[q(\theta)] \frac{dq(\theta)}{d\theta} - T'(\theta)$$

In view of (0.7),

$$\frac{dW(\theta)}{d\theta} = v[q(\theta)].$$

Now, note that

$$\int_{\underline{\theta}}^{\theta} v[q(x)]dx = \int_{\underline{\theta}}^{\theta} \frac{dW(x)}{dx} dx = W(x)|_{\underline{\theta}}^{\theta} = W(\theta) - W(\underline{\theta}).$$

Second Best: Solution III

Therefore,

$$W(\theta) = \int_{\underline{\theta}}^{\theta} v[q(x)]dx + W(\underline{\theta}), i.e.,$$

$$W(\theta) = \int_{\underline{\theta}}^{\theta} v[q(x)]dx, \tag{0.8}$$

since (0.5) binds, i.e., $W(\underline{\theta}) = 0$.
Therefore, from $W(\theta) = \theta v[q(\theta)] - T(\theta)$ it follows that

$$T(\theta) = \theta v[q(\theta)] - \int_{\underline{\theta}}^{\theta} v[q(x)]dx.$$

Ignoring (0.6) for the time being, the principal's problem is

$$\max_{q(\theta)} \left\{ \int_{\underline{\theta}}^{\overline{\theta}} [\theta v[q(\theta)] - \int_{\underline{\theta}}^{\theta} v[q(x)]dx - cq(\theta)] f(\theta) d\theta \right\}.$$

Second Best: Solution IV

Integrating the second term by parts, we get[1]

$$\max_{q(\theta)} \left\{ \int_{\underline{\theta}}^{\overline{\theta}} \left([\theta v[q(\theta)] - cq(\theta)] f(\theta) - v[q(\theta)][1 - F(\theta)] \right) d\theta \right\}.$$

This implies point-wise maximization of the integrand w.r.t. $q(\theta)$, the foc for which is

$$\theta v'[q(\theta)] = c + \frac{1 - F(\theta)}{f(\theta)} v'[q(\theta)]. \tag{0.9}$$

Remember, the first best requires

$$\theta v'[q^{FB}(\theta)] = c.$$

Therefore,

- $\theta = \overline{\theta} \Rightarrow q(\theta) = q^{FB}(\theta)$. But,
- $(\forall \theta < \overline{\theta})[q(\theta) < q^{FB}(\theta)]$, i.e., under consumption for every type except $\overline{\theta}$.

Second Best: Solution V

- Moreover, *ceteris paribus*, $q^{FB}(\theta) - q^{SB}(\theta)$ is proportional to $1 - F(\theta)$.

The last inference is true for uniform densities. However, may not hold in general.

Exercise

Show that

- The information rent enjoyed by agent with $\overline{\theta}$ depends on $q^{SB}(\theta)$, $\theta < \overline{\theta}$
- In general, the information rent enjoyed by agent with θ, depends on $q^{SB}(\hat{\theta})$, $\hat{\theta} < \theta$.

Hint: It immediately follows from one of the expressions above. Find out the expression.

Second Best: Solution VI

Rewriting (0.9), we get

$$[\theta - \frac{1 - F(\theta)}{f(\theta)}]v'[q(\theta)] = c. \tag{0.10}$$

Let

$$h(\theta) = \frac{f(\theta)}{1 - F(\theta)}.$$

Definition

$h(\theta)$

- is called the **Hazard Rate.**
- It is the conditional probability that the consumer's type belongs to $[\theta, \theta + d\theta]$ given that he belongs to $[\theta, \overline{\theta}]$.

Second Best: Solution VII

(0.10) can be written as

$$g(\theta)v'[q(\theta)] = c, \tag{0.11}$$

where

$$g(\theta) = \theta - \frac{1}{h(\theta)}.$$

Differentiating (0.11) w.r.t. θ we get

$$\frac{dq}{d\theta} = -\frac{g'(\theta)v'[q(\theta)]}{v''[q(\theta)]g(\theta)}$$

Since by assumption $v''(.) < 0$ and $g(\theta) > 0$,

if $h'(\theta) \geq 0$ then

$$\frac{dq(\theta)}{d\theta} \geq 0.$$

Second Best: Solution VIII

However, $h'(\theta) < 0$ can hold if $f(\theta)$ decreases rapidly with θ.

Remark

Non-monotonicity (of consumption or production) will occur when

- $h'(\theta) \geq 0$ does not hold for a range of $\theta \in [\underline{\theta}, \overline{\theta}]$; or
- θ directly affects the payoff for the principal.

In such cases, some bunching will emerge.

[1] We know that $\int_{\underline{\theta}}^{\overline{\theta}} UV' = UV|_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} U'V$. Let $U = \int v[q(x)]dx$ and $v' = f(\theta)$. Therefore, $\int_{\underline{\theta}}^{\overline{\theta}} (\int_{\underline{\theta}}^{\theta} v[q(x)]dx) f(\theta) d\theta = [\int_{\underline{\theta}}^{\theta} v[q(x)]dx F(\theta)]_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} v[q(\theta)]F(\theta)d\theta = \int_{\underline{\theta}}^{\overline{\theta}} v[q(\theta)]d\theta - \int_{\underline{\theta}}^{\overline{\theta}} v[q(\theta)]F(\theta)d\theta = \int_{\underline{\theta}}^{\overline{\theta}} v[q(\theta)](1 - F(\theta))d\theta$, since $F(\underline{\theta}) = 1$ and $\int_{\underline{\theta}}^{\theta} v[q(\theta)]d\theta F(\theta) = 0$