Lecture 7: Ex-ante Vs Ex-post Contracts

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February 4, 2015

Ex-ante contracting?

Proposition

When principal does not observe θ but can offer contract ex-ante, the FB allocation can be implemented.

Returning to the basic model, let

- the cost of production function be $C(q, \theta) = \theta q + F$, where $\theta \in \{\theta_1, ..., \theta_n\}$, where $\theta_1 < \theta_2 ... < \theta_n$ and $Pr(\theta = \theta_i) = \nu_i$.
- $\theta \in \{\theta_1, \theta_2\}$ and $Pr(\theta = \theta_1) = \nu$.
- The benefit function for principal be $V(q)$, where $V'(q) > 0$ and $V''(q) < 0$.

Ex-ante contracting with risk neutrality II

Under ex-post contracting, a menu of contracts $\{(q_1, t_1), (q_2, t_2)\}$ is incentive compatible and feasible if

$$\begin{aligned} U_1 = t_1 - \theta_1 q_1 &\geq 0 \\ U_2 = t_2 - \theta_2 q_2 &\geq 0 \end{aligned}$$

and

$$\begin{aligned} t_1 - \theta_1 q_1 &\geq t_2 - \theta_1 q_2 \\ t_2 - \theta_2 q_2 &\geq t_1 - \theta_2 q_1, i.e., \end{aligned}$$

$$U_1 \geq U_2 + \Delta\theta q_2 \tag{0.1}$$

$$U_2 \geq U_1 - \Delta\theta q_1 \tag{0.2}$$

Ex-ante contracting with risk neutrality III

Under ex-ante contracting, a menu of contracts $\{(q_1, U_1), (q_2, U_2)\}$ is incentive compatible and feasible if it satisfies (0.1) and (0.2) and is such that

$$\nu U_1 + (1-\nu)U_2 \geq 0 \tag{0.3}$$

That is, at the time of signing of the contract, the agent should get non-negative utility from it.

Example

Example 1: Consider $\{(q_1^*, U_1^*), (q_2^*, U_2^*)\}$, where $U_1^* = (1-\nu)\Delta\theta q_2^*$ and $U_2^* = -\nu\Delta\theta q_2^*$

Ex-ante contracting with risk neutrality IV

Example

Example 2: Let

$$W^* = \nu(V(q_1^*) - \theta_1 q_1^*) + (1-\nu)(V(q_2^*) - \theta_2 q_2^*).$$

Consider the contract $\{(q_1^*, t_1^*), (q_2^*, t_2^*)\}$, where

$$t_1^* = V(q_1^*) - W^*$$

and

$$t_2^* = V(q_2^*) - W^*.$$

Ex-ante contracting with risk neutrality V

Exercise

1. Show that both of the above contracts satisfy (0.1) – (0.2) and implement the FB. Check whether (0.3) binds for both.
2. Find out the rent enjoyed by the principal under the above contracts.

Ex-ante contracting with Risk-averse Agent I

Assume the agent is risk-averse. Now, an incentive feasible contract will satisfy

$$\nu u(U_1) + (1-\nu)u(U_2) \geq 0 \tag{0.4}$$

and the following ICs:

$$\begin{aligned} u(U_1) &\geq u(U_2 + \Delta\theta q_2) \\ u(U_2) &\geq u(U_1 - \Delta\theta q_1), i.e., \end{aligned}$$

The ICs can be written as

$$U_1 \geq U_2 + \Delta\theta q_2 \tag{0.5}$$

$$U_2 \geq U_1 - \Delta\theta q_1 \tag{0.6}$$

Now the principal's optimization problem can be rewritten as

$$\max_{(U_1,q_1),(U_2,q_2)} \{\nu(V(q_1) - \theta_1 q_1 - U_1) + (1-\nu)(V(q_2) - \theta_2 q_2 - U_2)\}$$

Ex-ante contracting with Risk-averse Agent II

s.t., (0.4) and (0.5) as constraints. The Lagrangian

$$\begin{aligned}\mathcal{L}(U_1, U_2, q_1, q_2, \lambda, \mu) &= \nu(V(q_1) - \theta_1 q_1 - U_1) + (1-\nu)(V(q_2) - \theta_2 q_2 - U_2) \\ &+ \lambda(U_1 - U_2 - \Delta\theta q_2) + \mu(\nu u(U_1) + (1-\nu)u(U_2))\end{aligned}$$

foc w.r.t. to U_1 and U_2 are

$$-\nu + \lambda + \mu\nu u'(U_1^{SB})) = 0 \tag{0.7}$$

$$-(1-\nu) - \lambda + \mu(1-\nu)u'(U_2^{SB})) = 0 \tag{0.8}$$

(0.7) and (0.8) give

$$\mu[\nu u'(U_1^{SB})) + (1-\nu)u'(U_2^{SB}))] = 1 \tag{0.9}$$

i.e., $\mu > 0$. Note from (0.4) and (0.5), when $q_2^{SB} > 0$, $U_2^{SB} < 0 < U_1^{SB}$. Now, (0.7) and (0.9) give us

Ex-ante contracting with Risk-averse Agent III

$$\lambda = \frac{\nu(1-\nu)[u'(U_2^{SB}) - u'(U_1^{SB})]}{\nu u'(U_1^{SB}) + (1-\nu)u'(U_2^{SB})} > 0.$$

That is, both (0.4) and (0.5) bind.
The foc w.r.t. to q_1 and q_2 are

$$\begin{aligned} V'(q_1^{SB}) &= \theta_1 && (0.10)\\ V'(q_2^{SB}) &= \theta_2 + \frac{\nu(u'(U_2^{SB}) - u'(U_1^{SB}))}{\nu u'(U_1^{SB}) + (1-\nu)u'(U_2^{SB})}\Delta\theta, i.e., && (0.11)\\ V'(q_2^{SB}) &= \theta_2 + \frac{\lambda}{1-\nu}\Delta\theta \end{aligned}$$

That is, $q_1^{SB} = q_1^*$ and $q_2^{SB} < q_2^* < q_1^*$.

Ex-ante contracting with Risk-averse Agent IV

Question

1. What does the FB require in this context, in terms of production levels and the risk-sharing?
2. Is the contract offered by the Principal efficient on either of the above counts?

Example

Suppose agent has CARA preference, represented by the following utility function

$$u(.) = \frac{1 - e^{-rx}}{r} = \frac{1}{r}\left(1 - \frac{1}{e^{rx}}\right).$$

Now, the foc (0.11) will become

$$V'(q_2^{SB}) = \theta_2 + \frac{\nu}{1-\nu}\Delta\theta(1 - \frac{1}{\nu + (1-\nu)e^{r\Delta\theta q_2^{SB}}}) \tag{0.12}$$

Ex-ante contracting with Risk-averse Agent V

That is, the level of q_2^{SB} depends on r. Moreover, it can be seen that

$$U_1^{SB} = \Delta\theta q_2^{SB} + \frac{1}{r}\ln(1 - \nu + \nu e^{-r\Delta\theta q_2^{SB}}) > 0 \quad (0.13)$$

$$U_2^{SB} = \frac{1}{r}\ln(1 - \nu + \nu e^{-r\Delta\theta q_2^{SB}}) < 0 \quad (0.14)$$

Exercise

Find out

$$\lim_{r\to\infty} q_2^{SB}, \ \& \ \lim_{r\to\infty} U_1^{SB}, \ \& \ \lim_{r\to\infty} U_1^{SB}$$

Ex-ante contracting with Risk-averse Agent VI

Remark

- In presence of Risk-neutrality (0.12) implies $V'(q_2^{SB}) = \theta_2$, *i.e.*, as before, $q_2^{SB} = q_2^*$.
- From (0.12), infinite risk-aversion implies q_2^{SB} solves
$$V'(q_2^{SB}) = \theta_2 + \frac{\nu}{1-\nu}\Delta\theta.$$
- Therefore, ex-post contracting is equivalent to *Ex-ante contracting with infinitely risk-averse agents*
- In presence of Risk-aversion there is trade off b/w allocative efficiency (which demands wedge b/w U_1 and U_2) and efficient insurance (which demands equality of U_1 and U_2).