Lecture 10: Decision Making Under Uncertainty

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Lotteries I

Consider the outcomes/states of nature that follow from tossing of a coin:

- Outcomes belong to the set $\{H, T\}$;
- Outcomes may be equiprobable;
- Outcomes may not be equiprobable.

If outcomes are equiprobable, we can denote the experiment as a lottery/risky-alternative by

$$(p_H, p_T) = (\frac{1}{2}, \frac{1}{2}).$$

If the coin is biased, it will give us a different lottery say $(\frac{1}{3}, \frac{2}{3})$.

Different coins, in principle, will generate different lotteries.

Different experiments, will generate different lotteries.

Lotteries II

Examples of lotteries:

- Output/profit from a project; Low or High
- Result of an Exam; Pass or Fail
- Outcome of a career-path
- Outcome of the Placement Process at DSE; Selected or Not

In general let

$$\Omega = \{s_1, s_2, ..., s_S\}$$

be the set of outcomes. For this setting we define

Definition

Simple Lottery: is a vector $L = (p_1, ..., p_S)$, such that $p_s \geq 0$ and $\sum_s (p_s) = 1$. p_s is the probability of the occurrence of outcome s.

Lotteries III

Let $\mathbb{L}$ be the set of simple lotteries.

A typical element of $\mathbb{L}$ is $L_k = (p_1, ..., p_S)$; k^{th} lottery.

Example

Suppose,

$$\mathbb{L} = \{(p_1, p_2, p_3)|p_i \geq 0 and \sum p_i = 1\}$$

Simple Lotteries; $L_1 = (1, 0, 0)$, $L_2 = (0, 1, 0)$, $L_3 = (0, 0, 1)$, $L_4 = (\frac{1}{2}, \frac{1}{2}, 0)$, $L_5 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}) \in \mathbb{L}$.

Probability Spaces and Lotteries

Consider the experiment of tossing of a coin:
Let,

- $\Omega = \{H, T\}$
- $\vartheta = \{\phi, \{H\}, \{T\}, \{H, T\}\}$
- $\mu : \vartheta \mapsto [0, 1]$

$$(\forall A \in \vartheta)[0 \leq \mu(A) \leq 1]$$
$$\mu(\Omega) = 1.$$

μ is a measure of (objective) probability over the elements of ϑ, e.g.,

$$\mu(\phi) = 0,$$
$$\mu(\{H\}) = \frac{1}{2} = \mu(\{T\}),$$
$$\mu(\{H, T\}) = 1.$$

We call (Ω, ϑ, μ) to be a probability space.

Probability Spaces and Lotteries

Consider another experiment: Casting of dice.
Let

- $\Omega = \{s_1, s_2, s_3, s_4, s_5, s_6\}$
- $\vartheta = \{\phi, \{s_1\}, \{s_2\}, ..., \{s_1, s_2, s_3, s_4, s_5, s_6\}\}$
- $\mu : \vartheta \mapsto [0, 1]$

$$(\forall A \in \vartheta)[0 \leq \mu(A) \leq 1]$$

$$\mu(\Omega) = 1., e.g.,$$

$$\mu(\phi) = 0,$$

$$(\forall i)\ \mu(\{s_i\}) = \frac{1}{6}., etc$$

(Ω, ϑ, μ) is another probability space.

Lottery Types I

Consider three lotteries: $L_1 = (1, 0, 0)$, $L_2 = (0, 1, 0)$, $L_3 = (0, 0, 1)$.

Now, consider a lottery that gives you L_1 with probability $\frac{1}{2}$,
L_2 with probability $\frac{1}{4}$,
and L_3 with probability $\frac{1}{4}$.

Definition

Compound Lottery:

- Take any $L_k \in \mathbb{L}$, $k = 1, ..., K$, lotteries defined over Ω.
- Let $(\alpha_1, ..., \alpha_K)$ be such that $\alpha_k \geq 0$ and $\sum_k \alpha_k = 1$.
- Then, a lottery that yields L_k with probability α_k is a compound lottery denoted by $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$.

Lottery Types II

Consider a compound lottery denoted by $(L_1, L_2, L_3; \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, where $L_1 = (1, 0, 0)$, $L_2 = (0, 1, 0)$, $L_3 = (0, 0, 1)$.

Now, consider

$$\frac{1}{2}L_1 + \frac{1}{4}L_2 + \frac{1}{4}L_3 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$$

Definition

Reduced Lottery: Take any compound lottery denoted by $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$. The lottery $\alpha_1 L_1 + ... + \alpha_K L_K$ is called the reduced form lottery for these lotteries.

Illustrations

Let $\mathbb{L} = \{(p_1, p_2, p_3)|p_i \geq 0 and \sum p_i = 1\}$

Example

Simple Lotteries; $L_1 = (1, 0, 0)$, $L_2 = (0, 1, 0)$, $L_3 = (0, 0, 1)$, $L_4 = (\frac{1}{2}, \frac{1}{2}, 0)$, $L_5 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}) \in \mathbb{L}$.

Example

Compound Lotteries: $(L_1, L_2, L_3; \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $(L_4, L_5; \frac{1}{4}, \frac{3}{4})$.

both of compound lotteries produce

Example

Reduced Lotteries:

$$\frac{1}{2}L_1 + \frac{1}{4}L_2 + \frac{1}{4}L_3 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) = \frac{1}{4}L_4 + \frac{3}{4}L_5$$

Preferences over lotteries I

If the decision maker has complete, transitive and continuous preference relation over $\mathbb{L}$, then

$$\exists U(.) : \mathbb{L} \mapsto R \text{ such that}$$

$$L \succeq L' \Leftrightarrow U(L) \geq U(L'),$$
$$L \succ L' \Leftrightarrow U(L) > U(L').$$

Definition

Expected Utility Form: $U(.)$ has expected utility form if there exist $u_s \in R$, $s = 1, ..., S$ such that for every $L = (p_1, ..., p_S) \in \mathbb{L}$

$$U(L) = u_1 p_1 + ... + u_S p_S.$$

For example, $U(1, 0, ..., 0) = u_1.1 + 0 + ... + 0 = u_1$.

Preferences over lotteries II

Definition

von Neumann-Morgenstern (v.N-M) expected utility function: $U(.) : \mathbb{L} \mapsto R$ is v.N-M expected utility function if it has an expected utility form.

Take any $\mathbb{L}$ defined over S, and any $U(.) : \mathbb{L} \mapsto R$.

Proposition

$U(.) : \mathbb{L} \mapsto R$ has an expected utility form iff for any K lotteries $L_k \in \mathbb{L}$, $k = 1, ..., K$ and any $\alpha_1, ..., \alpha_K \in R$ such that $\alpha_K \geq 0$ and $\sum_k \alpha_k = 1$,

$$U(\sum_k \alpha_k L_k) = \sum_k \alpha_k U(L_k).$$

Preferences over lotteries III

Definition

Independence Axiom: $\succeq$ defined on $\mathbb{L}$ satisfies IA if for any $L, L', L'' \in \mathbb{L}$ and any $\alpha \in (0, 1)$,

$$L \succeq L' \Leftrightarrow [\alpha L + (1 - \alpha)L'' \succeq \alpha L' + (1 - \alpha)L''].$$

That is, when $L \succeq L'$ the ranking $\alpha L + (1 - \alpha)L'' \succeq \alpha L' + (1 - \alpha)L''$ is independent of L''.

The Expected Utility Theorem I

Theorem

Suppose $\succeq$ on $\mathbb{L}$ is rational, continuous and satisfies the IA, then $\succeq$ can be represented by a utility function that has an expected utility form, i.e., $\exists U() : \mathbb{L} \mapsto R$ *and* $\exists u_1, ..., u_S \in R$ *such that for any* $L = (p_1, ..., p_S)$, $L' = (p'_1, ..., p'_S) \in \mathbb{L}$,

$$L \succeq L' \Leftrightarrow U(L) \geq U(L'), i.e,$$

$$L \succeq L' \Leftrightarrow \sum_1^S u_s p_s \geq \sum_1^S u_s p'_s.$$

von-Neumann and Morgenstern (1944, Chapter 3)

The Expected Utility Theorem II

Proposition

Suppose $U(.) : \mathbb{L} \mapsto R$ *is a v.N-M utility function that represents* $\succeq$ *on* $\mathbb{L}$*, then* $\tilde{U}(.) : \mathbb{L} \mapsto R$ *represents* $\succeq$ *on* $\mathbb{L}$ *iff there exist* $\beta > 0$ *and* $\gamma \in R$ *such that*

$$\tilde{U}(.) = \beta U(.) + \gamma.$$