Lecture 11: Decision Making Under Uncertainty

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February 11, 2015

Money Lottery I

Take a probability space (Ω, ϑ, μ), where $\Omega = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, etc. as above. Let

$$a : \Omega \mapsto \{x_1, ..., x_6\}, \; a(s_i) = x_i.$$

We can interpret $x \in R$ as wealth. Let

$$X = \{x_1, ..., x_6\} = \{0, 10, 100, 50, 20, 200\}.$$

We may represent the random variable $a : \Omega \mapsto X$ simply by the vector $(0, 10, 100, 50, 20, 200)$.

Money Lottery is (Ω, ϑ, μ) along with $a : \Omega \mapsto X$.

ζ is set of subsets of X. Let,

$$a^{-1} : \zeta \mapsto \vartheta.$$

Money Lottery II

Now consider a function $F(.)$ such that

$$F(0) = \mu[s_1] = \mu \circ a^{-1}\{0\} = \frac{1}{6},$$

$$F(50) = \mu[s_1, s_2, s_4, s_5] = \mu \circ a^{-1}\{x \in X | x \leq 50\} = \frac{4}{6},$$

$$F(100) = \mu[s_1, s_2, s_3, s_4, s_5] = \mu \circ a^{-1}\{x \in X | x \leq 100\} = \frac{5}{6}$$

That is,

$$F(.) : X \mapsto [0, 1]. \; F(x) = \mu \circ a^{-1}(-\infty, x]$$

The above lottery can be represented by the random variable $a : \Omega \mapsto X$ along with the distribution function $F(.) : X \mapsto [0, 1]$.

Money Lottery III

Consider another lottery spanned by random variable $\hat{a} : \Omega \mapsto \hat{X}$, where

$$\hat{X} = \{x_1, ..., x_6\} = \{20, 10, 100, 50, 40, 45\}.$$

(Ω, ϑ, μ) along with $\hat{a}$ is a different Money Lottery.

For $\hat{a}$, let the distribution function be $\hat{F}(.) : R \mapsto [0, 1]$.

$\hat{F}(50) = \mu[s_1, s_2, s_4, s_5, s_6] = \frac{5}{6}$, $\hat{F}(100) = \mu[s_1, s_2, s_3, s_4, s_5, s_6] = 1$.

Note that $F(100) = \frac{5}{6}$ and $\hat{F}(100) = 1$, i.e, different different combinations of space and random variable, i.e., different lotteries generate different Distribution Functions.

Therefore, we can study this second lottery with the help of the relevant random variable and the distribution function $\hat{F}(.)$.

Money Lottery IV

When Ω and therefore X are finite, $F(x) = \sum_{\{s:a(s)\leq x\}} \mu(s) = \sum_{\{s:a(s)\leq x\}} p_s$, where $p_s = \mu(s)$.

When x is continuous, under certain conditions $F(x)$ has a density function $f(.)$ such that,

$$(\forall x \in [x_0, \infty))[F(x) = \int_{x_0}^{x} f(t)dt]$$

Definition

Money Lottery: is a distribution fn $F(.) : [x_0, \infty) \mapsto [0, 1]$ where $x_0 \in R$ generally.

Let

$$\mathbb{L} = \{F(.)|F(.) : [x_0, \infty) \mapsto [0, 1]\}.$$

Money Lotteries and Expected Utility I

Assumptions:

- The decision maker has $\succeq$ defined over $\mathbb{L}$
- Expected Utility Theorem holds.

That is, for $L = (p_1, ..., p_S)$

$$U(L) = p_1 u_1 + ... + p_S u_S = \sum_{i=1}^{S} p_i u_i$$

When x is continuous, there exists a function $u : [x_0, \infty) \mapsto R$ such that

$$(\forall F(.) \in \mathbb{L})[U(F) = \int u(x) dF(x)].$$

$$F(.) \succeq G(.) \Leftrightarrow U(F) \geq U(G), i.e.,$$

$$F(.) \succeq G(.) \Leftrightarrow \int u(x) dF(x) \geq \int u(x) dG(x).$$

Money Lotteries and Expected Utility II

We will assume u to be continuous, increasing in x and *bounded above.*

It is easy to see that

$$U(F) = \int u(x)dF(x) = \int u(x)f(x)dx.$$

Moreover, $U(.)$ is linear in distributions $F(.)$.
Since,

$$U(\gamma F(x) + G(x)) = \int u(x)d[\gamma F(x) + G(x)] =$$

$$\gamma \int u(x)dF(x) + \int u(x)dG(x) = \gamma U(F(x)) + U(G(x)).$$

Risk Aversion I

Definition

Degenerate Lottery: A lottery represented by $F(.)$ is degenerate if $\mu \circ a^{-1}(x') = 1$ for some $x' \in R$. In that case,

$(\forall x < x')[F(x) = 0]$ and $(\forall x \geq x')[F(x) = 1]$.

Example: Let $a(s_i) = x_i$. Now, lotteries $(1, 0), (0, 1)$ over $\Omega = \{s_1, s_2\}$ and $(1, 0, 0)$ over $\hat{\Omega} = \{s_1, s_2, s_3\}$ are Degenerate Lotteries.
The money lottery $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ over $\hat{\Omega} = \{s_1, s_2, s_3\}$ such that

$$a(s_i) = 100$$

is also a degenerate money lottery.

Definition

Non-Degenerate Lottery: A lottery represented by $F(.)$ is no-degenerate if it is not degenerate.

Risk Aversion II

Let $\bar{x}$ be the initial wealth level of the decision maker with $\succeq$. Assume $\bar{x} = 0$.

Take a lottery $F(.) \in \mathbb{L}$. The expected monetary value of (return from) this lottery is

- $\sum p_i x_i$, if x takes only finite values
- $\int x dF(x)$, if x is a continuous variable

Let,

$F_d(.)$ be a (degenerate) lottery that yields $\int x dF(x)$ with probability 1.

Example

Let $\Omega = X = \{0, 1, 2\}$. If we take lottery $(\frac{1}{2}, 0, \frac{1}{2})$, then the corresponding degenerate lottery is $(0, 1, 0)$.

Risk Aversion III

Definition

Risk Aversion: A decision maker with $\succeq$ is risk averse for the lottery $\hat{F}(.)$ if

$$\hat{F}_d(.) \succeq \hat{F}(.).$$

Definition

Risk Aversion: A decision maker with $\succeq$ is risk averse if

$$(\forall F(.) \in \mathbb{L})[F_d(.) \succeq F(.)]., i.e.,$$

$$(\forall F(.) \in \mathbb{L})[u(\int x dF(x)) \geq \int u(x) dF(x)].$$

Risk Aversion IV

Definition

Strict Risk Aversion: A decision maker with $\succeq$ is strictly risk averse if for all non-degenerate lotteries

$$(\forall F(.) \in \mathbb{L})[F_d(.) \succ F(.)]., i.e.,$$

$$(\forall F(.) \in \mathbb{L})[u(\int x dF(x)) > \int u(x) dF(x)].$$

By definition, if u exhibits (strict) risk aversion then u is (strictly) concave. (Jensen's inequality).

Risk Aversion V

Definition

Risk Neutrality: A decision maker with $\succeq$ is risk-neutral if

$$(\forall F(.) \in \mathbb{L})[F_d(.) \sim F(.)]., i.e.,$$

$$(\forall F(.) \in \mathbb{L})[u(\int xdF(x)) = \int u(x)dF(x)].$$

Example

Let $\Omega = X = \{0, 1, 2\}$. A risk-neutral agent is indifferent between the lottery $(\frac{1}{2}, 0, \frac{1}{2})$ on one hand and the corresponding degenerate lottery is $(0, 1, 0)$, on the other hand.

Certainty Equivalent I

Consider a decision maker with u, and the initial wealth level $\bar{x}$. Now this person's utility is given by

- $\int u(\bar{x}+\tilde{z})dF(\tilde{z})$, if s/he gets lottery $F(\tilde{z})$
- $u(\bar{x}+c(F,u,\bar{x}))$, if s/he gets amount $c(F,u,\bar{x})$ with certainty.

Certainty Equivalent: $c(F,u,\bar{x})$ is the certainty equivalent of the lottery $F(\tilde{z})$ if

$$u(\bar{x}+c(F,u,\bar{x})) = \int u(\bar{x}+\tilde{z})dF(\tilde{z}). \tag{1}$$

Proposition

The following statements are equivalent:
u is concave;
u exhibits risk-aversion;
$(\forall F(.) \in \mathbb{L})[c(F,u,\bar{x}) \leq \int \tilde{z}dF(\tilde{z})]$

Risk Premium I

Consider a decision maker with u, and the initial wealth level $\bar{x}$. Now this person's utility is given by

- $\int u(\bar{x}+\tilde{z})dF(\tilde{z})$, if s/he gets lottery $F(\tilde{z})$
- $u(\bar{x}+\int \tilde{z}dF(\tilde{z}))$, if s/he gets the expected value of the lottery $F(\tilde{z})$ with certainty

Definition

Risk Premium: Consider a decision maker with u at wealth level $\bar{x}$. Now, $\rho(\bar{x},\tilde{z})$ is the risk premium for risk/lottery $\tilde{z}$ with distribution $F(\tilde{z})$ if

$$\int u(\bar{x}+\tilde{z})dF(\tilde{z}) = u(\bar{x}+\int \tilde{z}dF(\tilde{z}) - \rho(\bar{x},\tilde{z})). \quad (2)$$

That is, at the wealth level $\bar{x}$, the decision maker is indifferent b/w bearing the risk $\tilde{z}$ and having a sure amount of $\int \tilde{z}dF(z) - \rho(\bar{x},\tilde{z})$.

Risk Premium II

From (1) and (2),

$$c(F,u,\bar{x}) = \int \tilde{z}dF(\tilde{z}) - \rho(\bar{x},\tilde{z}), i.e., \ \rho(\bar{x},\tilde{z}) = \int \tilde{z}dF(\tilde{z}) - c(F,u,\bar{x}). \quad (3)$$

When u exhibits risk-aversion, i.e., $(\forall F(.) \in \mathbb{L})[c(F,u,\bar{x}) \leq \int \tilde{z}dF(\tilde{z})]$,

$$\rho(\bar{x},\tilde{z}) \geq 0.$$

Definition

Insurance Premium: For given wealth level $\bar{x}$, let's add risk $\tilde{z}$ with distribution $F(\tilde{z})$. Insurance Premium $c_I(F,u,\bar{x})$ is given by

$$u(\bar{x} - c_I(F,u,\bar{x})) = \int u(\bar{x}+\tilde{z})dF(\tilde{z}). \quad (4)$$

the insurance premium, $c_I(F,u,\bar{x})$ is the amount that makes the decision maker indifferent b/w accepting the risk $\tilde{z}$ and a payment of $c_I(F,u,\bar{x})$.

Risk Premium III

From (1) and (4),

$$c_I(F,u,\bar{x}) = -c(F,u,\bar{x}) = \rho(\bar{x},\tilde{z}) - \int \tilde{z}dF(\tilde{z}). \tag{5}$$

When the risk is actuarially fair, i.e., $\int \tilde{z}dF(\tilde{z}) = 0$,

$$c_I(F,u,\bar{x}) = -c(F,u,\bar{x}) = \rho(\bar{x},\tilde{z}).$$

Since, $\rho(\bar{x},\tilde{z}) \geq 0$ the decision maker will pay a non-negative amount to get rid of the risk.

Exercise: Show that when u is strictly concave and $\int \tilde{z}dF(\tilde{z}) \leq 0$, $c_I(F,u,\bar{x}) > 0$.

Measuring Risk Aversion I

Can $u''(x)$ measure risk-aversion?

Definition

Arrow-Pratt Coefficient: Arrow-Pratt Coefficient of Absolute Risk-aversion at wealth level $\bar{x} \in R$ is

$$r_A(\bar{x}, u) = -\frac{u''(\bar{x})}{u'(\bar{x})}.$$

r_A is a local measure and is defined only when $u'(\bar{x}) \neq 0$;

$r_A(\bar{x}, u) > 0$ implies aversion toward risk.
$r_A(\bar{x}, u) < 0$ implies love for risk.
$r_A(\bar{x}, u) = 0$ implies risk neutrality.

Suppose, $v(\bar{x}) = \beta u(\bar{x}) + \gamma$, where $\beta > 0$, then $r_A(\bar{x}, v) = r_A(\bar{x}, u)$.
$r_A(\bar{x}, u)$ is invariant to affine transformations of u.

Measuring Risk Aversion II

From $r_A(\bar{x}, u)$ we can recover u up to two constants of integration. In fact, we can recover the preference relation fully.

Let $r_A(\bar{x}, u) = -\frac{u''(\bar{x})}{u'(\bar{x})} = t$, $t > 0$. Integrating, $r_A(\bar{x}, u)$ gives us

$$u(\bar{x}) = -\beta e^{-tx} + \gamma \text{ for some } \beta > 0.$$

$-\beta e^{-tx} + \gamma$ represents the same preference relation, regardless of $\beta > 0$ and γ.
As a special case, we get $u(x) = -e^{-tx}$.

In general, we can write

$$u(x) = \int e^{-\int r(x)dx} dx.$$

Measuring Risk Aversion III

It is possible to demonstrate that for 'small' risks

$$r_A(\bar{x}, u) = 2\frac{\rho(\bar{x}, \tilde{z})}{\sigma^2_{\tilde{z}}}, i.e.,$$

$r_A(\bar{x}, u)$ and $\rho(\bar{x}, \tilde{z})$ have the same sign and are proportional to each other.

Definition

Coefficient of Relative Risk Aversion: For a u, at $\bar{x}$, the CRRA is

$$r_R(u, \bar{x}) = -\bar{x}\frac{u''(\bar{x})}{u'(\bar{x})}.$$

Suppose the risk $\tilde{z}$ is proportional in that an arbitrary realization takes value $z\bar{x}$.

Measuring Risk Aversion IV

Example

- for $u(x) = \log x$ and $u(x) \sim \log x$, $r_R(u, x) = 1$
- for $u(x) \sim x^{1-c}$, where $c < 1$, $r_R(u, x) = c$
- for $u(x) \sim -x^{(1+c)}$, where $c > 1$, $r_R(u, x) = -c$.

Clearly, $r_R(u, \bar{x}) = \bar{x} r_A(u, \bar{x})$.

Comparing Risk Aversion I

Proposition

An individual with $u_2(.)$ is more risk averse than the individual with $u_1(.)$ if any of the following holds:

- $(\forall x)([r_A(x, u_2) \geq r_A(x, u_1)]$;
- *There exists an increasing and concave function $\psi(.)$ such that* $(\forall x)[u_2(x) = \psi(u_1(x))]$;
- $u_2 \circ u_1^{-1}(.)$ *is concave;*
- $(\forall x)(\forall F(.))[c(F, u_2, x) \leq c(F, u_1, x)]$;
- $\forall F(.)[\int u_2(x + \tilde{z})dF(x) \geq u_2(\bar{x}) \Rightarrow \int u_1(x + \tilde{z})dF(x) \geq u_1(\bar{x})]$.