Decision Making Under Uncertainty*

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1 Expected Utility Theory

In many real world contexts, a decision maker has make a choice from the set of 'risky' alternatives. As common sense will suggest, an alternative or an act is risky if under it several outcomes are possible - some of the outcomes may be less desirable than the others. For instance, taking an exam is risky act. The possible outcomes can be 'pass' and 'fail'. Choice of a career path is risky in that it can lead to various possible wealth or utility levels over the lifetime. Most of the industrial and commercial projects are also risk. A project may fail totally. Even if it succeeds, the resulting profits can take various possible values.

At a basic level, tossing of a coin is a risky act. There are two possible outcomes, i.e., the set of outcomes is $\{H, T\}$. Plausibly, risky alternatives are also described as lotteries. The act/experiment of tossing a coin can be written as a lottery, (p_H, p_T), where p_H is the probability of Head and p_T the probability of Tail. For a fair coin the outcomes are equiprobable, i.e., $p_H = p_T = \frac{1}{2}$. So, tossing of a fair coin can be treated as a lottery denoted by $(p_H, p_T) = (\frac{1}{2}, \frac{1}{2})$. On the other hand, tossing of a biased coin will be a different lottery, say $(\frac{1}{3}, \frac{2}{3})$. Differently biased coins will generate different lotteries.

In general, suppose for a risk alternative the set of possible outcomes is known and finite. Such a risk/alternative can be described as a lottery denoted by a probability tuple/vector whose components are the probabilities. For instance, Let, s denote a state of nature or a possible outcome of a risky alternative. Let Ω denote the set of possible outcomes. Ω is assumed to be non-empty. When there are S possible outcomes, we have $\Omega = \{s_1, s_2, ..., s_S\}$. For the experiment involving tossing of a coin $\Omega = \{s_1, s_2\} = \{H, T\}$.

A 'Simple Lottery', L, is a vector $(p_1, ..., p_S)$, where p_s is the probability of the occurrence of outcome s. Moreover, $p_s \geq 0$ and $\sum_s (p_s) = 1$. For given set of possible outcomes, $\Omega = \{s_1, s_2, ..., s_S\}$, let $\mathbb{L}$ denote the set of simple lotteries, i.e.,

$$\mathbb{L} = \left\{ (p_1, p_2, ..., p_S) |\ p_i \geq 0 \text{ and } \sum_{i=1}^{S} p_i = 1. \right\}$$

For example, for a risky alternative of there are only three possible outcomes, s_1, s_2, and s_3, we have $\Omega = \{s_1, s_2, s_3\}$. A general lottery $L \in \mathbb{L}$ is a vector (p_1, p_2, p_3) such that $p_i \geq 0$ and $p_1 + p_2 + p_3 = 1$. Some of the specific simple lotteries are; $L_1 = (1, 0, 0)$, $L_2 = (0, 1, 0)$, $L_3 = (0, 0, 1)$, $L_4 = (\frac{1}{2}, \frac{1}{2}, 0)$, $L_5 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$. Note that $\Omega = \{s_1, s_2, s_3\}$, i.e., a three-component lottery can be represented as a point in equilateral triangle whose

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altitude is 1. In general, when $\Omega = \{s_1, s_2, ..., s_S\}$, a simple lottery will have have S components. A S component simple lottery can be represented geometrically as a point in $S-1$ dimensional simplex. A complete description of an experiment or a lottery requires probability assignment not only to each of the possible outcomes but also various possible combinations of the possible outcomes. Formally, the complete description requires working with what is called a probability space. To illustrate, consider the experiment of tossing of a coin. For the experiment we have $\Omega = \{H, T\}$. Let $\vartheta = \{\phi, \{H\}, \{T\}, \{H, T\}\}$, i.e., ϑ is the set of subsets of Ω. In another context, a subset of Ω is called an event. Consider the function: $\mu : \vartheta \mapsto [0, 1]$, such that:

$$(\forall A \in \vartheta)[0 \leq \mu(A) \leq 1]$$

$$\mu(\Omega) = 1.$$

That is, μ assign a probability to each subset of Ω. Further, μ assign a probability 1 to the entire set of outcomes, i.e., Ω. The function μ is called a measure of (objective) probability over the elements of ϑ. In the context of tossing of a fair coin, μ will assign probabilities as follows:$\mu(\phi) = 0$, $\mu(\{H\}) = \frac{1}{2} = \mu(\{T\})$, and $\mu(\{H, T\}) = 1$.

We call the tuple (Ω, ϑ, μ) to be a probability space. For a biased coin, the probability of different subsets of Ω will differ from the corresponding probabilities for a fair coin. Technically speaking, the functions μ will differ in the two cases. This means that the probability space associated with tossing of a biased coin will be different from the one corresponding to a fair coin.

Consider another experiment involving casting of unbiased dice. Here you can easily see that $\Omega = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, $\vartheta = \{\phi, \{s_1\}, \{s_2\}, ..., \{s_1, s_2, s_3, s_4, s_5, s_6\}\}$. The probability measure $\mu : \vartheta \mapsto [0, 1]$ will be such that: $(\forall A \in \vartheta)[0 \leq \mu(A) \leq 1]$, $\mu(\Omega) = 1$, $\mu(\phi) = 0$, $(\forall i)\ \mu(\{s_i\}) = \frac{1}{6}$, etc. This experiment has the probability space, (Ω, ϑ, μ), associated with it, where Ω, ϑ, and μ are as described here.

To sum up, different probability spaces will differ either in terms of Ω or ϑ, or μ, or all three. For more on probability space, read the following **optional** subsection.

Probability Spaces*

All starred section/subsections are optional readings. A probability space is defined by the tuple (Ω, ϑ, μ). The tuple represents the space of the possible outcomes or, alternatively put, the space of the possible states of nature pertaining to an experiment. Here, ϑ is a collection of subsets of Ω. Sets in ϑ are called events. When the set Ω is finite, ϑ is taken to be simply a set of all subsets of Ω. Otherwise, ϑ is $\sigma-$algebra of subsets of Ω. That is, the set ϑ is closed under complementation and countable unions of its member sets. Formally, ϑ satisfies the following properties: $\Omega \in \vartheta$; and $A \in \vartheta \Rightarrow A' \in \vartheta$. Moreover, $A_n \in \vartheta,\ n \geq 1 \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \vartheta$. When ϑ is $\sigma-$ algebra, you prove that it also satisfies the following properties: $\emptyset \in \vartheta$, $A, B \in \vartheta \Rightarrow A \cap B \in \vartheta$; and $A, B \in \vartheta \Rightarrow A - B \in \vartheta$; and $A_n \in \vartheta,\ n \geq 1 \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \vartheta$.

The last component of the above tuple is μ, which is a measure of (objective) probability over the elements of ϑ. Formally put, μ is a function from ϑ to the real numbers that assigns to each event a probability between 0 and 1: $\mu : \vartheta \mapsto [0,1]$; $(\forall A \in \vartheta)[0 \leq \mu(A) \leq 1]$, and $\mu(\Omega) = 1$.

1.1 Types of Lotteries

So far we have defined what can be called 'simple lotteries'. The following example lists five simple lotteries.

Example 1 Assume there are three possible outcomes, i.e., $\sharp\Omega = 3$. Some of possible simple lotteries are: $L_1 = (1,0,0)$, $L_2 = (0,1,0)$, $L_3 = (0,0,1)$, $L_4 = (\frac{1}{2}, \frac{1}{2}, 0)$, and $L_5 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$.

As should be clear by now, a simple lottery is a probability distribution over outcomes listed in the set Ω. If you play a simple lottery you will end up with one of the outcomes listed in Ω. That is, outcome of a simple lottery is some *certain* outcome or the state of nature. In several contexts, the outcome of a lottery can be another lottery, i.e, the outcomes of a lotteries can be risk alternatives. Such lotteries are called 'compound lotteries'. The following examples list two such lotteries.

Example 2 Assume there are three possible outcomes, i.e., $\sharp\Omega = 3$. Consider a lottery L whose outcomes are simple lotteries $L_1 = (1,0,0)$, $L_2 = (0,1,0)$, $L_3 = (0,0,1)$, with probability 1/2, 1/4 and 1/4, respectively. In that case, a compound lottery L can be described as $L \equiv (L_1, L_2, L_3; 1/2, 1/4, 1/4)$. The compound lottery L will give you lottery L_1 with probability 1/2, and so on.

Example 3 Assume there are three possible outcomes, i.e., $\sharp\Omega = 3$. Consider a lottery L' whose outcomes are simple lotteries $L_4 = (\frac{1}{2}, \frac{1}{2}, 0)$, and $L_5 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$, with probability 1/4, and 3/4, respectively. L' can be described as $L \equiv (L_4, L_5; 1/4, 3/4)$.

In general, a compound lottery is denoted by $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$, where $L_k \in \mathbb{L}$, $k = 1, ..., K$, are lotteries, and $\alpha_1, ..., \alpha_K$ are such that $\alpha_k \geq 0$ and $\sum_k \alpha_k = 1$. The lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$ yields lottery L_k with probability α_k, and so on.

Corresponding to every compound lottery there is what is called 'reduced form lottery. To illustrate, take any compound lottery, say $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$. Let $L_k = (p_1^k, ..., p_S^k)$ be the kth lottery, $k = 1, ..., K$. Now consider the expression $\alpha_1 L_1 + ... + \alpha_K L_K = \sum_{i=1}^{i=K} \alpha_i L_i$. It is easy to show that $\alpha_1 L_1 + ... + \alpha_K L_K \in \mathbb{L}$. In fact, $\alpha_1 L_1 + ... + \alpha_K L_K = (p^1, ..., p^S)$, where $p^s = \alpha_1 p_s^1 + ... + \alpha_K p_s^K = \sum_{i=1}^{i=K} \alpha_k p_s^k$. Moreover, $p^s \geq 0$ and $\sum_s p^s = 1$. That is, $(p^1, ..., p^S)$ is a simple lottery. The lottery $(p^1, ..., p^S)$ is called reduced form lottery associated with the compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$.

For the compound lottery in Example (2), $K = 3$ and it can easily be checked that $\sum_{i=1}^{i=3} \alpha_i L_i = (\sum_{i=1}^{i=3} \alpha_k p_1^k, \sum_{i=1}^{i=3} \alpha_k p_2^k, \sum_{i=1}^{i=3} \alpha_k p_3^k) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Therefore, $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ is the reduced form lottery associated with the compound lottery $(L_1, L_2, L_3; 1/2, 1/4, 1/4)$. It should be noted that two different compound lotteries may yield the same reduced form lottery. For example, the compound lotteries in Examples (2) and (3) both have same reduced form, which is $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. This means that o, there is a loss of information in moving to the reduced form of a compound lottery.

1.2 Preferences over Lotteries

It is standard to assume that while choosing among the compound lotteries, the decision maker considers only the relevant reduced form lotteries, e.g., a decision maker will be indifferent between compound lotteries defined in Examples (2) and (3) above. Further, the decision maker has a preference relation $\succeq$ defined over the set of lotteries, $\mathbb{L}$.

When $\succeq$ is a complete pre-ordering (complete and transitive) and continuous,[1] from Debreu (1966) we know that $\succeq$ can be represented by a utility function, say U, such that $U(.) : \mathbb{L} \mapsto R$, $L \succeq L' \Leftrightarrow U(L) \geq U(L')$, and $L \succ L' \Leftrightarrow U(L) > U(L')$. Note that the function U is defined up to a monotonically increasing transformation. That is, if take another function, say $\tilde{U}$, that is a monotonic transformation of U, then $\tilde{U}$ will also represent the same preference relation $\succeq$.

Definition 1 Expected Utility Form: A utility function $U(.) : \mathbb{L} \mapsto R$ is said to have an expected utility form if there exist $u_s \in \mathcal{R}$, $s = 1, ..., S$ such that for every $L = (p_1, ..., p_S) \in \mathbb{L}$

$$U(L) = u_1 p_1 + ... + u_S p_S.$$

That is, if a utility function $U(.)$ has an expected utility form then there will exist real numbers u_1, ..., u_S, such that utility of any lottery, $(p_1, ..., p_S)$ can be described as expected value of u_1, ..., u_S with weight being the respective probabilities, $p_1, ..., p_S$.

Definition 2 von Neumann-Morgenstern (v.N-M) expected utility function: A function $U(.) : \mathbb{L} \mapsto R$ is v.N-M expected utility function if it has an expected utility form.

Let $U(.) : \mathbb{L} \mapsto R$ be a v.N-M expected utility function. Note that $U(1, 0, ..., 0) = u_1$, ... , $U(0, 0, ..., 1) = u_S$, etc. Moreover, the value of $U(.)$ is the expected value of u_1, ..., u_S, i.e., $U(.)$ is linear in $p_1, ..., p_S$.

Definition 3 Independence Axiom: The preference relation $\succeq$ defined on $\mathbb{L}$ satisfies the Independence Axiom if for any $L, L', L'' \in \mathbb{L}$ and any $\alpha \in [0, 1]$,

$$L \succeq L' \Leftrightarrow [\alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''].$$

The Independence Axiom implies that when $L \succeq L'$ the ranking $\alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$ is independent of L''.

Now, we are ready to state what is called *The Expected Utility Theorem.*[2] (EUT).

Proposition 1 The Expected Utility Theorem: Suppose the preference relation $\succeq$ defined on $\mathbb{L}$ is rational, continuous and satisfies the independence axiom, then $\succeq$ can be represented by a utility function that has an expected utility form.

[1] $\succeq$ is continuous on $\mathbb{L}$ if for any $L, L' \in \mathbb{L}$, the sets $\{L : L \succeq L'\}$ and $\{L : L' \succeq L\}$ are closed.

[2] For details see the book by von-Neumann and Morgenstern (1944, Chapter 3)

We will assume that for the decision maker, the Expected Utility Theorem holds, i.e., s/he has preference relation that is defined on $\mathbb{L}$ and is rational, continuous and satisfies the independence axiom. This means that there exist non-negative real numbers $u_1, ..., u_S$ such that for any $L = (p_1, ..., p_S), L' = (p'_1, ..., p'_S) \in \mathbb{L}$,

$$L \succeq L' \Leftrightarrow \sum_1^S u_s p_s \geq \sum_1^S u_s p'_s.$$

Next, let us consider some other interesting features of a preferences for which the Expected Utility Theorem holds. Consider L_4, L_5, α_4 and α_5 as defined in Example 3. In view of the above, you can check that

$$\begin{aligned} U(\alpha_4 L_4 + \alpha_5 L_5) &= \frac{1}{2}u_1 + \frac{1}{4}u_2 + \frac{1}{4}u_3 \\ \alpha_4 U(L_4) + \alpha_5 U(L_5) &= \frac{1}{2}u_1 + \frac{1}{4}u_2 + \frac{1}{4}u_3 \end{aligned}$$

That is, $U(\alpha_4 L_4 + \alpha_5 L_5) = \alpha_4 U(L_4) + \alpha_5 U(L_5)$. Therefore, when EUT holds, the utility function is linear in lotteries. In fact, we can make the following claim.

Proposition 2 A utility function $U(.) : \mathbb{L} \mapsto R$ has an expected utility form iff for any K lotteries $L_k \in \mathbb{L}$, $k = 1, ..., K$ and any $\alpha_1, ..., \alpha_K \in R$ such that $\alpha_K \geq 0$ and $\sum_k \alpha_k = 1$, $U(\sum_k \alpha_k L_k) = \sum_k \alpha_k U(L_k)$, i.e., affine combination of lotteries is preserved.

However, the expected utility form is a cardinal property.

Proposition 3 Suppose $U(.) : \mathbb{L} \mapsto R$ is a v.N-M utility function that represents $\succeq$ on $\mathbb{L}$, then $\tilde{U}(.) : \mathbb{L} \mapsto R$ represents $\succeq$ on $\mathbb{L}$ iff there exist $\beta > 0$ and $\gamma \in R$ such that $\tilde{U}(.) = \beta U(.) + \gamma$. That is, only affine transformation is preserved.

Exercises*

Here are some **optional** exercises for you.

Exercise 1: Suppose the preference relation $\succeq$ satisfies the Independence Axiom. Prove that:

$$L \succ L' \Leftrightarrow [\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'']$$

$$L \sim L' \Leftrightarrow [\alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''].$$

Exercise 2: Suppose the preference relation $\succeq$ defined on $\mathbb{L}$ is representable by a utility function that has an expected utility form. Show that $\succeq$ is continuous and satisfies the independence axiom.

Hint: Take any $L, L', L'' \in \mathbb{L}$. Let $L = (p_1, ..., p_S)$, $L' = (p'_1, ..., p'_S)$ and $L = (p''_1, ..., p''_S)$. WLG, let $L \succeq L'$, i.e., $U(L) \geq U(L')$. $U(L) \geq U(L') \Rightarrow \alpha U(L) + (1-\alpha)U(L'') \geq \alpha U(L') + (1-\alpha)U(L'')$, since $\alpha \in [0,1]$. Since U has an expected utility form, $U(L) = u_1 p_1 + ... + u_S p_S$. That is, $\alpha[u_1 p_1 + ... + u_S p_S] + (1-\alpha)[u_1 p''_1 + ... + u_S p''_S] \geq \alpha[u_1 p'_1 + ... + u_S p'_S] + (1-\alpha)[u_1 p''_1 + ... + u_S p''_S]$, i.e., $u_1(\alpha p_1 + (1-\alpha)p''_1) + ... + u_S(\alpha p_S + (1-\alpha)p''_S) \geq$

$u_1(\alpha p'_1+(1-\alpha)p''_1)+...+u_S(\alpha p'_S+(1-\alpha)p''_S)$, i.e., $U(\alpha L+(1-\alpha)L') \geq U(\alpha L'+(1-\alpha)L')$, i.e., $\alpha L + (1-\alpha)L' \succeq \alpha L' + (1-\alpha)L'$.

Exercise 3: Suppose the preference relation $\succeq$ defined on $\mathbb{L}$ is representable by a utility function that has an expected utility form. Show that the indifference curves are the straight lines parallel to each other. In particular, if $L \sim L'$, then for all $\alpha \in [0,1]$, $\alpha L+(1-\alpha)L' \sim L$.

Hint: Note that $L \sim L' \Rightarrow U(L) = U(L') \Rightarrow \alpha U(L) = \alpha U(L') \Rightarrow U(\alpha L) = U(\alpha L') \Rightarrow \alpha L \sim \alpha L' \Rightarrow \alpha L + (1-\alpha)L' \sim \alpha L' + (1-\alpha)L'$. But, $L' \sim \alpha L' + (1-\alpha)L'$. Therefore, $\alpha L + (1-\alpha)L' \sim L'$. In view of $L \sim L'$, this implies $\alpha L + (1-\alpha)L' \sim L$.

2 Money Lotteries

As discussed earlier, complete description of lotteries requires specification of the underlying probability space. Take a probability space (Ω, ϑ, μ). Generally, the outcomes listed in the set Ω are not quantities of money. A money lottery is different from a general lottery in that under a money lottery the final outcome can be described in term of money. Alternatively, money lottery can be though of a general lottery with additional qualification that each of the outcome in the set Ω has monetary consequence.

To illustrate, consider a probability space for the experiment of casting a fair dice, (Ω, ϑ, μ). Here, $\Omega = \{s_1, ..., s_6\}$. Now consider a set $R = \{x_1, ..., x_6\}$, where x_i the monetary consequence related with the state s_i, for $i = 1, .., 6$. Specifically, we can let $\{x_1, ..., x_6\} = \{0, 10, 100, 50, 20, 200\}$. We can read this scenario as follows: if the dice land with side s_i on the top, you win x_i amount of money. Since dice is fair, we have $\mu(s_i) = \frac{1}{6}$, for $i = 1, .., 6$.

We can formally define this money lottery in terms of the probability space (Ω, ϑ, μ) and a random variable a, where a is function with following properties: $a : \Omega \mapsto R$, $a(s_i) = x_i$. We interpret $x \in R$ as wealth.

For the above experiment of casting a fair dice, the random variable a can also be described as the vector $(0, 10, 100, 50, 20, 200)$. In this context, what is the probability of winning at most 50? Note that you win an amount less than or equal to 50, if the dice lands up with any of the followings sides; s_1, s_2, s_4, s_5. Therefore, probability of winning at most 50 is probability of the event $\{s_1, s_2, s_4, s_5\}$. This probability is given by $\mu[s_1, s_2, s_4, s_5] = 4/6$. Let

ζ be set of subsets of X, and a^{-1} be a function such that:

$$a^{-1} : \zeta \mapsto \vartheta.$$

Note that a^{-1} is NOT inverse function of a defined above. Recall, elements of ζ are subsets of X. So, if we take a subset of X, a^{-1} function gives its pre-image which belongs to ϑ. For example, $a^{-1}(0, 10, 50, 20) = \{s_1, s_2, s_4, s_5\}$. Now consider the following function $F(.) : R \mapsto [0, 1]$ such that: $F(x) = \mu \circ a^{-1}(-\infty, x]$, where μ the above defined probability measure on Ω, and a^{-1} is the above defined function.

$a^{-1}(-\infty, x]$ is the set of outcomes whose monetary consequences are less than or equal to x. For example, in the above context $a^{-1}(-\infty, 50] = \{s_1, s_2, s_4, s_5\}$. Therefore, $F(50) =$

$\mu[s_1, s_2, s_4, s_5] = \frac{4}{6}$. Similarly, you can easily check that $F(100) = \mu[s_1, s_2, s_3, s_4, s_5] = \frac{5}{6}$. It is obvious that F is the distribution function associated with this money lottery. It turns out that, we can study the above money lottery with the help of the random variable a along with the distribution function $F(.)$.

For the above probability space, (Ω, ϑ, μ), consider another function $\hat{a} : \Omega \mapsto R$, $\hat{a}(s_i) = x_i$. Let the set of monetary consequences be $\{x_1, ..., x_6\} = \{20, 10, 100, 50, 40, 45\}$. For $\hat{a}$, let the corresponding distribution function be $\hat{F}(.) : R \mapsto [0, 1]$, e.g., let $\hat{F}(50) = \mu[s_1, s_2, s_4, s_5, s_6] = \frac{5}{6}$, $F(100) = \mu[s_1, s_2, s_3, s_4, s_5, s_6] = 1$.

Again, we can represent this random variable $\hat{a}$ as the vector $(0, 10, 100, 50, 20, 200)$. Moreover, we can study this second lottery with the help of this random variable along with the distribution function $\hat{F}(.)$.

When Ω is finite, we can denote the act by the vector $(x_1, ..., x_S)$, where S is the number of states in Ω and x_s is the monetary payoff in state s. $F(x) = \sum_{\{s:a(s)\leq x\}} p_s$, $p_s = \mu(s)$.

Note that for a given probability space, different random variables will generate different distribution functions, i.e., different lotteries. Also, different probability spaces will generate different lotteries. Different probability spaces will generate different money lotteries and different distribution functions.

To sum up, we can represent and analyze a money lottery simply by studying the associated distribution function $F(.) : R \mapsto [0, 1]$. Let $\mathbb{L}$ be the set of all money lotteries, i.e., the set of all probability distribution functions defined over say $[x_0, \infty)$, where $x_0 \in R$.

$$\mathbb{L} = \{F(.)|F(.) : R \mapsto [0, 1]\}.$$

Take any lottery represented by the distribution function $F(.)$. For this lottery, the probability that the realized monetary payoff is less than or equal to x is given by $F(x)$. Note that distribution functions preserve the linearity of lotteries. The final distribution of the compound lottery $(F_1(x), ..., F_K(x); \alpha_1, ..., \alpha_K)$ is the weighted average of the individual distributions, i.e., $F(x) = \alpha_1 F_1(x) + ... + \alpha_K F_K(x)$.

As before, we assume that the decision maker has a rational preference relation $\succeq$ defined over $\mathbb{L}$. As an implication of the Expected Utility Theorem, it follows that when $\succeq$ satisfies the Axioms of continuity and independence, there exist a function $u : [x_0, \infty) \mapsto R$ such that

$$U(F) = \int u(x) dF(x).$$

Remark: Notice that so far we have not imposed any conditions on u. However, here for simplicity and to avoid some paradoxes, we will assume that u is continuous, increasing and bounded above. (*St. Petersburg-Menger paradox).* When u is continuous and bounded above, we can be sure of the existence of the above integral.[3] Moreover, there exists a function, $f(x)$, such that:

$$U(F) = \int u(x) dF(x) = \int u(x) f(x) dx.$$

[3]It called Riemann-Stieltjes integral.

Since,

$$U(\gamma F(x) + G(x)) = \int u(x) d[\gamma F(x) + G(x)] =$$

$$\gamma \int u(x) dF(x) + \int u(x) dG(x) = \gamma U(F(x)) + U(G(x)), i.e.,$$

$U(.)$ is linear in $F(.)$. We will call the function u as Bernoulli utility function.

Money Lottery: General description*

This subsection is optional. For a given probability space, a general lottery can be defined as an *act* or a gamble or a random variable a, where the random variable a is a function that maps Ω into C, i.e., $a : \Omega \mapsto \mathbf{C}$, where $\mathbf{C}$ is the set of consequences. Formally, for a given probability space, (Ω, ϑ, μ),

Let $\mathbf{A}$ be the set of all acts or gambles.

Let ζ be the Borelian algebra on R. In this set-up it can be shown that each act induces a probability measure p on (R, ζ), such that

$(\forall B \in \zeta)[p(B) = \mu \circ a^{-1}(B)]$.

In other words, each act induces a lottery describable by the distribution function $F(.) : R \mapsto [0, 1]$ such that for all $x \in R$, $F(x) = \mu \circ a^{-1}(-\infty, x]$.

Let's assume that x is continuous. When $\mu \circ a^{-1}$ is continuous with respect to the Lebesgues measure λ, from Radon-Nikodym theorem it follows that $\mu \circ a^{-1}$, i.e., $F(x)$ has a density function w.r.t. λ such that $F(x) = \int_{-\infty}^{x} f(t)dt$ for all x.

3 Risk and Premia

We begin by defining what are called degenerate lotteries. A lottery represented by $F(.)$ is called Degenerate Lottery if for $x' \in R$, we have have $(\forall x' < x)[F(x') = 0]$ and $(\forall x') \geq x[F(x') = 1)]$. In that case, the lottery gives amount x' with probability 1. You can verify that a lottery is degenerate if the following holds: $\mu \circ a^{-1}([x_0, x) = 1$ for some $x \in R$. A degenerate lottery is risk free lottery.

Here are some examples: Let $a(s_i) = x_i$. Now, the lottery $(1, 0)$ over $\Omega = \{s_1, s_2\}$ is a degenerate lottery - it produces outcome s_1 with probability 1. The lottery $(1, 0, 0)$ over $\hat{\Omega} = \{s_1, s_2, s_3\}$ is also a degenerate lottery. The money lottery $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ over $\hat{\Omega} = \{s_1, s_2, s_3\}$ such that $a(s_i) = 100$ is also a degenerate money lottery - the gives amount 100 with probability 1. A lottery that does not produce an outcome with probability 1 is called non-degenerate.

Take a lottery $F(.) \in \mathbb{L}$. The expected monetary value of (return from) this lottery is $\sum p_i x_i$, if x takes only finite values. When if x is a continuous variable the expected monetary value of the lottery is $\int x dF(x)$. Let, $F_d(.)$ be a (degenerate) lottery that yields $\int x dF(x)$ with probability 1. We call $F_d(.)$ the degenerate lottery associated with lottery $F(.)$.

Risk Aversion: Consider a non-degenerate lottery $F'(.)$, let $F'_d(.)$ be the associated degenerate lottery. A decision maker with preference relation $\succeq$ is risk averse for the lottery $F'(.)$ if $F'_d(.) \succeq F'(.)$.

Definition 4 A decision maker with preference relation $\succeq$ is risk averse if $(\forall F(.) \in \mathbb{L})[F_d(.) \succeq F(.)]$, *i.e.,*

$$(\forall F(.) \in \mathbb{L})[u(\int x dF(x)) \geq \int u(x) dF(x)].$$

where $F(.)$ is non-degenerate. He is called strictly risk averse if $(\forall F(.) \in \mathbb{L})[F_d(.) \succ F(.)]$.

Definition 5 Strict Risk Aversion: A decision maker with $\succeq$ is strictly risk averse if for every non-degenerate lottery $\forall F(.)$, *we have* $F_d(.) \succ F(.)$, *i.e.,*

$$u(\int x dF(x)) > \int u(x) dF(x).$$

By definition, if u exhibits (strict) risk aversion then u is (strictly) concave. (Jensen's inequality). Also, you should note that attitude towards risk is a feature of the underlying preference relation. However, when EUT holds that is the preference relation can be represented by utility function then we can study the attitude towards risk simply by examining the properties of relevant vN-M function u.

Definition 6 Risk Neutrality: A decision maker with $\succeq$ is risk-neutral if $(\forall F(.) \in \mathbb{L})[F_d(.) \sim F(.)]$, *i.e., if*

$$(\forall F(.) \in \mathbb{L})[u(\int x dF(x)) = \int u(x) dF(x)].$$

Next we define *Certainty Equivalent* of a lottery. For given u and the initial wealth level $\bar{x}$, the certainty equivalent of the lottery $F(\tilde{z})$, $c(F, u, \bar{x})$, is given by

$$u(\bar{x} + c(F, u, \bar{x})) = \int u(\bar{x} + \tilde{z}) dF(\tilde{z}). \tag{1}$$

It is easy to show that $(\forall F(.) \in \mathbb{L})[c(F, u, \bar{x}) \leq \int \tilde{z} dF(\tilde{z})]$ iff u exhibits risk aversion, i.e., iff u is concave. To see this note that $c(F, u, \bar{x}) \leq \int \tilde{z} dF(\tilde{z}) \Leftrightarrow [\bar{x} + c(F, u, \bar{x}) \leq \int (\bar{x} + \tilde{z}) dF(\tilde{z})] \Leftrightarrow [u(\bar{x} + c(F, u, \bar{x})) \leq u(\bar{x} + \int (\tilde{z}) dF(\tilde{z}))]$. Since $u(\bar{x} + c(F, u, \bar{x})) = \int u(\bar{x} + (\tilde{z})) dF(\tilde{z})$, we get $\int u(\bar{x} + \tilde{z}) dF(\tilde{z}) \leq u(\int (\bar{x} + \tilde{z}) dF(\tilde{z})$, i.e., u is concave. Further, note that $\int \tilde{z} dF(\tilde{z}) \leq 0 \Rightarrow c(F, u, \bar{x}) \leq 0$.

Proposition 4 The following statements are equivalent:
u is concave;
u exhibits risk-aversion;
$(\forall F(.) \in \mathbb{L})[c(F, u, \bar{x}) \leq \int \tilde{z} dF(\tilde{z})]$

For given wealth level $\bar{x}$, let's add risk $\tilde{z}$ with distribution $F(\tilde{z})$. The *Risk Premium*, $\rho(\bar{x}, \tilde{z})$, is given by

$$\int u(\bar{x} + \tilde{z}) dF(\tilde{z}) = u\left(\bar{x} + \int \tilde{z} dF(\tilde{z}) - \rho(\bar{x}, \tilde{z})\right). \tag{2}$$

That is, at the wealth level $\bar{x}$, the decision maker is indifferent b/w bearing the risk $\tilde{z}$ and having a sure amount of $\int \tilde{z} dF(z) - \rho(\bar{x}, \tilde{z})$. From (1) and (2),

$$c(F, u, \bar{x}) = \int \tilde{z} dF(\tilde{z}) - \rho(\bar{x}, \tilde{z}), i.e.,\ \rho(\bar{x}, \tilde{z}) = \int \tilde{z} dF(\tilde{z}) - c(F, u, \bar{x}). \tag{3}$$

Note that when u exhibits risk-aversion, i.e., $(\forall F(.) \in \mathbb{L})[c(F, u, \bar{x}) \leq \int \tilde{z} dF(\tilde{z})]$, $\rho(\bar{x}, \tilde{z}) \geq 0$.

Also, note that if a decision maker who is at wealth level $\bar{x}$ and has risk $\tilde{z}$, s/he will be willing to sell it at a price of $c(F, u, \bar{x})$. Therefore, $c(F, u, \bar{x}) = c_a(F, u, \bar{x})$ is the (asking) price of the risk $\tilde{z}$, from the perspective of the decision maker who owns the risk. On the other hand, if s/he does not own the risk $\tilde{z}$ but is thinking of buying it, at the wealth level $\bar{x}$ the maximum amount s/he will be willing to pay to buy the risk $\tilde{z}$ is given by

$$u(\bar{x}) = \int u(\bar{x} + \tilde{z} - c_b(F, u, \bar{x})) dF(\tilde{z}).$$

Insurance Premium: For given wealth level $\bar{x}$, the insurance premium, $c_I(F, u, \bar{x})$ is the amount that makes the decision maker indifferent b/w accepting the risk $\tilde{z}$ and a payment of $c_I(F, u, \bar{x})$. let's add risk $\tilde{z}$ with distribution $F(\tilde{z})$. $c_I(F, u, \bar{x})$ is given by

$$u(\bar{x} - c_I(F, u, \bar{x})) = \int u(\bar{x} + \tilde{z}) dF(\tilde{z}). \tag{4}$$

From (1)- (4), we get the following relations

$$c_I(F, u, \bar{x}) = -c(F, u, \bar{x}) = \rho(\bar{x}, \tilde{z}) - \int \tilde{z} dF(\tilde{z}). \tag{5}$$

Clearly, when the risk is actuarially fair, i.e., when $\int \tilde{z} dF(\tilde{z}) = 0$, $c_I(F, u, \bar{x}) = -c(F, u, \bar{x}) = \rho(\bar{x}, \tilde{z})$. Since, for a risk-averse person $\rho(\bar{x}, \tilde{z}) \geq 0$, the decision maker will pay a non-negative amount to get rid of the risk. For additional properties of risk premium, see the Appendix.

4 Risk Aversion: Measurements and comparisons

4.1 Coefficient of Absolute Risk Aversion

As one would expect, risk aversion is related to the curvature of u, i.e., the magnitude of u'' but cannot be fully captured by it. Why?. Suppose, $v(.) = \beta u(.) + \gamma$, where $\beta > 0$, then $u(.)$ and $v(.)$ represent the same preference relations, so a satisfactory measure of risk aversion should give us the same answer for these two functions. However, whenever $\beta \neq 1$, $v''(.) \neq u''(.)$ therefore we get different answers.
Coefficient of Absolute Risk Aversion CARA: Arrow-Pratt coefficient of absolute risk aversion at wealth level $\bar{x} \in R$ is given by

$$r_A(\bar{x}, u) = -\frac{u''(\bar{x})}{u'(\bar{x})}.$$

Obviously, r_A is defined only when $u'(\bar{x}) \neq 0$; $u''(\bar{x}) = 0$ implies risk neutrality. However, you can see that for any $v(\bar{x}) = \beta u(\bar{x}) + \gamma$, where $\beta > 0$, we have $r_A(\bar{x}, v) = r_A(\bar{x}, u)$. That is, unlike u'', $r_A(\bar{x}, u)$ is invariant to linear transformation of the utility function. Moreover, from $r_A(\bar{x}, u)$ we can recover u up to two constants of integration. But any Bernoulli utility function is unique only up to two constants; the origin and scale. This means that we can recover the preference relation fully. For example, let $r_A(\bar{x}, u) = -\frac{u''(\bar{x})}{u'(\bar{x})} = t$,

$t > 0$. Integrating, $r_A(\bar{x}, u)$ gives us $u(\bar{x}) = -\beta e^{-tx} + \gamma$ for some $\beta > 0$. But, $-\beta e^{-tx} + \gamma$ represents the same preference relation, regardless of $\beta > 0$ and γ. As a special case, we get $u(x) = -e^{-tx}$. In general, we can write

$$u(x) = \int e^{-\int r(x)dx} dx.$$

If $u(x)$ and $\bar{u}(x)$ represent the same preference relation, we write $u(x) \sim \bar{u}(x)$. We know that $\rho(\bar{x}, \tilde{z}) = \frac{1}{2}\sigma_{\tilde{z}}^2 r_A(\bar{x}, u)$, *i.e.*,

$$r_A(\bar{x}, u) = 2\frac{\rho(\bar{x}, \tilde{z})}{\sigma_{\tilde{z}}^2}. \tag{6}$$

That is, $r_A(\bar{x}, u)$ is proportional to $\rho(\bar{x}, \tilde{z})$ and, therefore, is a measure of the local risk aversion. Similarly, $-r_A(\bar{x}, u)$ is a measure of the local propensity to gamble.

Note that while $r_A(x, u)$ is a measure of small risk aversion, $\rho(x, \tilde{z})$ is a better measure when the risk is not small.

4.2 Comparing Risk Aversion Across Individuals

Consider $u_i = -e^{-c_i x}$, for individual $i = 1, 2$. Remember this function exhibits CARA preference. For this functional form, it is easy to see that $\forall x[r_A(x, u_2) \geq r_A(x, u_1)] \Leftrightarrow c_2 \geq c_1$. In general, we have the following result.

Proposition 5 Take two *BUFs* $u_1(.)$ and $u_2(.)$. The individual with $u_2(.)$ is more risk averse than the individual with $u_1(.)$ if any of the following holds:
(i) $\forall x[r_A(x, u_2) \geq r_A(x, u_1)]$;
(ii) There exists an increasing and concave function $\psi(.)$ such that
$\forall x[u_2(x) = \psi(u_1(x))]$;
(iii) $u_2 \circ u_1^{-1}(.)$ is concave;
(iv) $(\forall x)(\forall F(.))[c(F, u_2, x) \leq c(F, u_1, x)]$;
(v) $(\forall x)(\forall z)[\pi(x, z, u_2) \geq \pi(x, z, u_1)]$;
(vi) $\forall F(.)[\int u_2(x + \tilde{z})dF(x) \geq u_2(\bar{x}) \Rightarrow \int u_1(x + \tilde{z})dF(x) \geq u_1(\bar{x})]$.

For proof see Appendix.

4.3 Comparing Risk Aversion across Wealth Levels

A decision maker whose preference is represented by u is said to exhibit decreasing absolute risk aversion with wealth if $\frac{\partial r_A(x,u)}{\partial x} \leq 0$. Consider a BUF, u, and two different wealth levels say x_1 and x_2, $x_1 > x_2$. For any risk $\tilde{z}$ with distribution $F(\tilde{z})$, we can study the decision maker's choice behavior by considering functions $u_1(\tilde{z}) = u(x_1 + \tilde{z})$ and $u_2(\tilde{z}) = u(x_2 + \tilde{z})$. It is intuitive that if u exhibits decreasing absolute risk aversion than $r_A(\tilde{z}, u_2) \geq r_A(\tilde{z}, u_1)$.

Proposition 6 A BUF, u, exhibits decreasing absolute risk aversion iff any of the following holds:
(i) $x_1 > x_2$ implies that $u_2(\tilde{z})$ is more concave than $u_1(\tilde{z})$, i.e., $u_2(\tilde{z})$ is a concave transformation of $u_1(\tilde{z})$;
(ii) $x - c(F, u, x)$ is a decreasing function of x;

(iii) $\pi(x, z, u)$ is a decreasing function of x;
(iv) $\int u(x_2 + \tilde{z})dF(\tilde{z}) \geq u(x_2) \Rightarrow \int u(x_1 + \tilde{z})dF(\tilde{z}) \geq u(x_1)$.

For proof see Appendix.

4.4 Coefficient of Relative Risk Aversion

Suppose we consider values of the risk $\tilde{z}$ relative to the wealth level. Then, we can say that the risk $\tilde{z}$ is proportional, in that an arbitrary realization of the risk can be described as $z\bar{x}$, where $z \in [0, 1]$ say. For a given BUF, u, at wealth level $\bar{x}$, the Coefficient of Relative Risk Aversion is

$$r_R(u, \bar{x}) = -\bar{x}\frac{u''(\bar{x})}{u'(\bar{x})}.$$

Clearly, $r_R(u, \bar{x}) = \bar{x}r_A(u, \bar{x})$. For example, the function $u(x) = \log x$ exhibits constant relative risk aversion. You can verify that for $u(x) = \log x$, we have $r_R(u, x) = 1$. More generally, for $u(x) \sim \log x$, $r_R(u, x) = 1$. Also, for $u(x) \sim x^{1-c}$, where $c < 1$, $r_R(u, x) = c$, and for $u(x) \sim -x^{(1+c)}$, where $c > 1$, $r_R(u, x) = -c$.

The relative risk premium $\hat{\rho}(\bar{x}, \tilde{z})$ is given by

$$\int u(\bar{x}(1 + \tilde{z}))dF(\tilde{z}) = u(\bar{x}(1 - \hat{\rho}(\bar{x}, \tilde{z})).$$

Remark: A decreasing relative risk aversion implies a decreasing absolute risk aversion, but not vice-versa, i.e., for a BUF, u, $\frac{\partial r_R(u,x)}{\partial x} \leq 0 \Rightarrow \frac{\partial r_A(u,x)}{\partial x} \leq 0$. Let $u(x) = -e^{-q^{-1}(x+\beta)^q}$. It can be checked that for this function, when $\beta \geq 0$ and $q < 1$ and $q \neq 0$, $r_A(u, x)$ is a decreasing function of x for $x > 0$. However, $r_R(u, x) = x(x + b^{-1}[(x + b)^q + 1 - q])$ is strictly increasing if $0 < q < 1$; strictly decreasing if $q < 0$ and $\beta = 0$ and neither if $q < 0$ and $\beta > 0$.

5 Appendix*

Reading of results listed here is strictly optional.

Risk Premium: Additional Properties

Replace $\bar{x}$ with $\bar{x} + \mu$ and $\tilde{z}$ with $\tilde{z} - \mu$ in (2) to get

$$\int (u(\bar{x} + \mu + \tilde{z} - \mu))dF(\tilde{z}) = u\left(\bar{x} + \mu + \int (\tilde{z} - \mu)dF(\tilde{z}) - \rho(\bar{x} + \mu, \tilde{z} - \mu)\right).$$

Now, comparing it (2) we can see that $\rho(\bar{x}, \tilde{z}) = \rho(\bar{x} + \mu, \tilde{z} - \mu)$. That is, we can analysis any given risk $\tilde{z}$ by studying the risk $\tilde{z} - \mu$. WLOG, we can assume that $\int \tilde{z}dF(\tilde{z}) = \mu = 0$, i.e., we can assume that the risk is actuarially neutral or fair. In the following, we will make this assumption. Therefore, the above equality can be written as

$$\int (u(\bar{x} + \tilde{z}))dF(\tilde{z}) = u\left(\bar{x} + \int \tilde{z}dF(\tilde{z}) - \rho(\bar{x}, \tilde{z})\right) = u(\bar{x} - \rho(\bar{x}, \tilde{z})). \tag{7}$$

That is, the decision maker is indifferent b/w bearing the risk $\tilde{z}$ and having a sure amount of $\bar{x} - \rho(\bar{x}, \tilde{z})$. When $\tilde{z}$ is small, for any value z of $\tilde{z}$ (from Taylor Series), $u(\bar{x} + z) \approx u(\bar{x}) + zu'(\bar{x}) + \frac{z^2}{2}u''(\bar{x}), i.e., \int u(\bar{x} + \tilde{z})dF(\tilde{z}) \approx u(\bar{x}) + \frac{\sigma^2_{\tilde{z}}}{2}u''(\bar{x})$. Also, assuming $\tilde{z}$ and therefore $\rho(\bar{x}, \tilde{z})$ to be very small, $u(\bar{x} - \rho(\bar{x}, \tilde{z})) \approx u(\bar{x}) - \rho(\bar{x}, \tilde{z})u'(\bar{x})$. Therefore, when $\tilde{z}$ is very small

$$\rho(\bar{x}, \tilde{z}) = -\frac{1}{2}\sigma^2_{\tilde{z}}\frac{u''(\bar{x})}{u'(\bar{x})}. \tag{8}$$

When $\mu \neq 0$, it is easy to see that we get

$$\rho(\bar{x}, \tilde{z}) = -\frac{1}{2}\sigma^2_{\tilde{z}}\frac{u''(\bar{x} + \mu)}{u'(\bar{x} + \mu)}.$$

Probability Premium: Given a BUF u, for the given wealth level $\bar{x}$ and positive number z the Probability Premium, $\pi(\bar{x}, z, u)$, is given by:

$$u(\bar{x}) = (\frac{1}{2} + \pi(\bar{x}, z, u))u(\bar{x} + z) + (\frac{1}{2} - \pi(\bar{x}, z, u))u(\bar{x} - z). \tag{9}$$

That is, $\pi(\bar{x}, z, u)$ is the excess in wining probability that makes the decision maker indifferent between a sure amount of $\bar{x}$, on one hand, and lottery of two outcomes $(x+z)$ and $(x-z)$, on the other hand. (9) can be written as $(\frac{1}{2} + \pi(\bar{x}, z, u))u(\bar{x} + z) + (\frac{1}{2} - \pi(\bar{x}, z, u))u(\bar{x} - z)$ $= (\frac{1}{2} + \pi(\bar{x}, z, u))u(\bar{x}) + (\frac{1}{2} - \pi(\bar{x}, z, u))u(\bar{x})$,i.e., $(\frac{1}{2} + \pi(\bar{x}, z, u))[u(\bar{x} + z) - u(\bar{x})] = (\frac{1}{2} - \pi(\bar{x}, z, u))[u(\bar{x}) - u(\bar{x} - z)]$. Now, $\pi(\bar{x}, z, u) \geq 0 \Rightarrow [u(\bar{x} + z) - u(\bar{x}) \leq u(\bar{x}) - u(\bar{x} - z)]$, i.e., $\frac{u(\bar{x}+z)+u(\bar{x}-z)}{2} \leq u(\bar{x})$, i.e., u is concave at $\bar{x}$. More formally, for all x and $z > 0$, if $\pi(x, z, u) \geq 0$, then u is concave.

Proofs

Proof of Proposition 5: $(i) \Rightarrow (ii)$: Since $u_1(.)$ and $u_2(.)$ are ordinally identical, i.e., both are increasing, there exists an increasing function $\psi(.)$ such that $\forall x[u_2(x) = \psi(u_1(x))]$, and $\psi' > 0$. Differentiating $u_2(x) = \psi(u_1(x))$, $u_2'(x) = \psi'(u_1(x))u_1'(x)$ and

$$u_2''(x) = \psi'(u_1(x))u_1''(x) + \psi''(u_1(x))(u_1'(x))^2, i.e.,$$

$$r_A(x, u_2) = r_A(x, u_1) - \frac{\psi''(u_1(x))}{\psi'(u_1(x))}u_1'(x), i.e.,$$

$$r_A(x, u_2) \geq r_A(x, u_1) \Leftrightarrow \psi''(.) \leq 0.$$

$(i) \Rightarrow (iii)$:

$$[r_A(x, u_2) \geq r_A(x, u_1)] \Leftrightarrow [\frac{u_2''(x)}{u_2'(x)} - \frac{u_1''(x)}{u_1'(x)} \leq 0] \Leftrightarrow [\frac{d}{dx}\log\frac{u_2'(x)}{u_1'(x)} \leq 0]. \tag{10}$$

Now,

$$\frac{d}{dx}u_2 \circ u_1^{-1}(x) = \frac{u_2'[u_1^{-1}(x)]}{u_1'[u_1^{-1})(x)]}.^4$$

[4] Let $u_1^{-1}(x) = t(x)$, i.e., $x = u_1(t(x))$. So $\frac{\partial}{\partial x}(u_1(t(x))) = 1$, i.e.,$u_1'(t(x))\frac{d}{dx}t = 1$, i.e., $u_1'(t(x))\frac{d}{dx}u_1^{-1}(x) = 1$, i.e., $\frac{d}{dx}u_1^{-1}(x) = \frac{1}{u_1'(u_1^{-1}(x))}$.

Let $u_1^{-1}(x) = t(x)$, i.e., $x = u_1(t(x))$. Since $u_1(.)$, is increasing, $t(.)$ is increasing as well. Therefore, $u_2 \circ u_1^{-1}(x)$ is concave iff $\frac{d}{dx} u_2 \circ u_1^{-1}(x)$, *i.e.*, $\frac{u_2'(u_1^{-1})(x)}{u_1'(u_1^{-1})(x)}$, i.e., $\frac{u_2'(t(x))}{u_1'(t(x))}$ is decreasing. But, $\frac{u_2'(t(x))}{u_1'(t(x))}$ is decreasing iff $\log \frac{u_2'(t(x))}{u_1'(t(x))}$ is decreasing, i.e., $\frac{d}{dx}(\frac{u_2'(t(x))}{u_1'(t(x))}) \leq 0 \Leftrightarrow \frac{d}{dx} \log \frac{u_2'(t(x))}{u_1'(t(x))} \leq 0$, which is true in view of (10).

(iv) $\Rightarrow$ *(vi)*: Take any x and $F(.)$. Let $u_1(c(F, u_1, x)) = \int u_1(x + \tilde{z}) dF(\tilde{z})$ and $u_2(c(F, u_2, x)) = \int u_2(x + \tilde{z}) dF(\tilde{z})$. Take any $\bar{x}$ such that $\int u_2(x + \tilde{z}) dF(\tilde{z}) \geq u_2(\bar{x})$. Clearly, $u_2(c(F, u_2, \bar{x})) \geq u_2(\bar{x}) \Rightarrow c(F, u_2, \bar{x}) \geq \bar{x}$.

Now $c(F, u_1) \geq c(F, u_2) \Rightarrow c(F, u_1) \geq \bar{x}$, i.e., $u_1(c(F, u_1, x)) \geq u_1(\bar{x})$, i.e., $\int u_1(x + \tilde{z}) dF(\tilde{z}) \geq u_1(\bar{x})$.

(iii) $\Rightarrow$ *(iv)*: From equation (**??**), $u_i(\bar{x} - \rho_i(\bar{x}, \tilde{z})) = \int u_i(\bar{x} + \tilde{z}) dF(\tilde{z})$, $i = 1, 2$, i.e., $\bar{x} - \rho_i(\bar{x}, \tilde{z}) = (u_i)^{-1}(\int u_i(\bar{x} + \tilde{z}) dF(\tilde{z}))$,

$$\rho_i(\bar{x}, \tilde{z}) = \bar{x} - (u_i)^{-1}(\int u_i(\bar{x} + \tilde{z}) dF(\tilde{z})),$$

$i = 1, 2$. Therefore,

$$\rho_2(\bar{x}, \tilde{z}) - \rho_1(\bar{x}, \tilde{z}) = (u_1)^{-1}(\int u_1(\bar{x} + \tilde{z}) dF(\tilde{z})) - (u_2)^{-1}(\int u_2(\bar{x} + \tilde{z}) dF(\tilde{z})). \tag{11}$$

Let, $\tilde{t} = u_1(\bar{x} + \tilde{z}), i.e., \bar{x} + \tilde{z} = u_1^{-1}(\tilde{t})$, then the above equation can be written as

$$\rho_1(\bar{x}, \tilde{z}) - \rho_2(\bar{x}, \tilde{z}) = (u_1)^{-1}(\int \tilde{t} dF(\tilde{z})) - (u_2)^{-1} \int u_2((u_1)^{-1}(\tilde{t})) dF(\tilde{z}). \tag{12}$$

Now, consider the RHS of (12). Since, $u_2 \circ (u_1)^{-1}$ is concave, from Jensen's inequality $\int u_2 \circ (u_1)^{-1}(\tilde{t}) dF(\tilde{z}) \leq u_2 \circ (u_1)^{-1} \int (\tilde{t}) dF(\tilde{z})$, i.e.,

$$(u_1)^{-1}(\int \tilde{t} dF(\tilde{z})) - (u_2)^{-1} \int u_2((u_1)^{-1}(\tilde{t})) dF(\tilde{z}) \geq$$

$$(u_1)^{-1}(\int \tilde{t} dF(\tilde{z})) - (u_2)^{-1} u_2 \circ (u_1)^{-1} \int (\tilde{t}) dF(\tilde{z}) = (u_1)^{-1}(\int \tilde{t} dF(\tilde{z})) - (u_1)^{-1} \int (\tilde{t}) dF(\tilde{z}) = 0.$$

That is

$$\rho_2(\bar{x}, \tilde{z}) \geq \rho_1(\bar{x}, \tilde{z}), i.e., \int \tilde{z} dF(\tilde{z}) - \rho_2(\bar{x}, \tilde{z}) \leq \int \tilde{z} dF(\tilde{z}) - \rho_1(\bar{x}, \tilde{z}), i.e.,$$

$c(F, u_2, x) \leq c(F, u_1, x)$.

Proof of Proposition 6: The above claim follows immediately from the arguments given in the proof of Proposition 5.

(i) is straightforward. Note that from Proposition 5, when $u_2(\tilde{z})$ is more concave than $u_1(\tilde{z})$ than $\rho_2(x, \tilde{z}) \geq \rho_1(x, \tilde{z})$, i.e., $\rho(x, \tilde{z})$ is decreasing function of x.