202: Dynamic Macroeconomics

Neoclassical Growth with Optimizing Agents (Ramsey-Cass-Koopmans Model)

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Critique of Solow Model and Subsequent Extensions:

- Recall the two basic critisisms of the Solow Model:
    - Even though it is supposed to be a growth model - it cannot really explain long run growth:
    - The steady state in the Solow model may be dynamically inefficient.
- The basic Solow growth model has subsequently been extended to counter some of these critisisms.
- We have already looked at one such extension: **Solow Model with Exogenous Technological Progress**
- In today's class we shall look at the other extension: **Neoclassical Growth Model with Optimizing Agents**

Neoclassical Growth with Optimizing Agents:

- Let us now extend the Solow model to allow for optimizing agents.
- There are two frameworks which allow for optimizing consumption/savings behaviour by households:
  1. The Ramsey-Cass-Koopmans Inifinite Horizon Framework (henceforth R-C-K);
  2. The Samuelson-Diamond Overlapping Generations Framework (henceforth OLG).
- The basic difference between the two is that in the R-C-K model agents optimize over infinite horizon; while in the OLG model, agents optimize over a finite time horizon (usually 2 periods).
- As we shall see later, this apparently innocuous difference in terms of time horizon spells out very different growth trajectories for the two models.

Neoclassical Growth with Optimizing Agents: The R-C-K Model

- We start with the R-C-K model. This model is still Neoclassical - beacuse it retains **all** the assumptions of the Neoclassical production function (including the diminishing returns property and the Inada conditions.)
- In fact the production side story is exactly identical to Solow.
- As before, the economy starts with a given stock of capital (K_t) and a given level of population (N_t) at time t. (We are ignoring technological progress for now).
- Since the production side story is identical to Solow, we know that the firm-specific production functions can be aggregated to generate an aggregate production function: $Y_t = F(K_t, N_t)$.
- And at every point of time the market clearing wage rate and the rental rate of capital are given by:

$$w_t = F_N(K_t, N_t); \quad r_t = F_K(K_t, N_t).$$

The R-C-K Model: The Household Side Story

- There are H identical households indexed by h.
- Each household consists of a single **infinitely lived** member to begin with (at $t = 0$). However population within a household increases over time at a constant rate n. (And each newly born member is infinitely lived too!) This implies that total population also increases at the rate n.
- At any point of time t, the total capital stock and the total labour force in the economy are equally distributed across all the households, which are supplied inelastically to the market at the market wage rate w_t and the market rental rate r_t.
- Thus total earning of a household at time t: $w_t N_t^h + r_t K_t^h$.
- Corresponding **per member** earning: $y_t^h = w_t + r_t k_t^h$,

  where k_t^h is the **per member capital stock** in household h, which is also the **per capita capital stock** (or the capital-labour ratio, k_t) in the economy.

The Household Side Story (Contd.):

- In every time period, the instantaneous utility of the household depends on its **per member** consumption:

$$u_t = u\left(c_t^h\right); \; u' > 0; \; u'' < 0; \; \lim_{c^h \to 0} u'(c^h) = \infty; \; \lim_{c^h \to \infty} u'(c^h) = 0.$$

- The household at time 0 chooses its entire consumption profile $\left\{c_t^h\right\}_{t=0}^{\infty}$ so as to maximise the discounted sum of its life-time utility:

$$U_0^h = \int_{t=0}^{\infty} u\left(c_t^h\right) \exp^{-\rho t} dt; \;\; \rho > 0,$$

  subject to **the household's budget constraint** in every time period.

- Notice that identical households implied that **per member** consumption (c_t^h) of any household is also equal to the per capita consumption (c_t) in the economy at time t.

Interpretation of the Discount Rate:

- Notice that the objective function of the household is an integral defined over *infinite horizon*, where future utilities are discounted at a constant rate ρ. The discount rate (ρ) may have three possible interpretations.
- It is easier to understand these interpretations if we write down the discrete time counterpart of the above objective function:

$$U_0^h \simeq u\left(c_0^h\right) + \frac{u\left(c_1^h\right)}{1+\rho} + \frac{u\left(c_2^h\right)}{(1+\rho)^2} + \ldots\ldots\ldots$$

1. One interpretation of the above inifinte horizon utility function is that agents are immortal (live for ever), but they have an innate (psychological) tendency to prefer current consumption over future consumption; hence they discount utilities from consumption that happen in future dates. In this case ρ is interpreted as the **"pure" rate of time preference** of an agent.

Interpretation of the Discount Rate (Contd.):

- Notice that the curvature of the utility function itself to some extent captures the preference of an agent over consumptions at two dates: t and $t+1$. But this measure is 'impure' in the sense that it depends on the precise amounts of c_t^h and c_{t+1}^h.
  - If I already have too much of c_t^h and too little of c_{t+1}^h, then I might prefer an extra unit of future consumption more than an extra unit of today's consumption; and it would be the other way round if my c_t^h is too low compared to c_{t+1}^h.
  - This happens simply because the utility function is concave in c, which induces this kind on 'consumption smoothing'.
- A 'pure' rate of time preference measures the agent's preference for current consumption over future consumption even when the actual consumption at the two time periods are exactly equal (i.e., $c_t^h = c_{t+1}^h$). This way it neutralizes the effect of concavity of the utility function and looks at the pure psychological preference for today vis-a-vis tomorrow - which is independent of consumption smoothing.

Interpretation of the Discount Rate (Contd.):

- In a two period set up where the total lifetime utility is defined by $U(c_t, c_{t+1})$, the 'pure' rate time preference is defined as:

$$\left. \frac{\frac{\partial U}{\partial c_t} - \frac{\partial U}{\partial c_{t+1}}}{\frac{\partial U}{\partial c_{t+1}}} \right|_{c_t = c_{t+1} = \bar{c}}$$

- In other words, it measures the **rate of change** in the marginal valuation on an extra unit of consumption available today vis-a-vis available tomorrow - **along a constant consumption path**.
- It is easy to see that in our additive utility specification where

$$U(c_t, c_{t+1}) = u(c_t) + \frac{u(c_{t+1})}{1+\rho}$$

the above definition coincides with the discount rate ρ.

Interpretation of the Discount Rate (Contd.):

2. Another interpretation of the discount rate ρ follows if we read the infinite horizon utility function of the household as the sum of utilities of successive generations of agents who themselves are finitely lived, but who care for their future generations.

- To understand the idea better, suppose each member of the household lives exactly for one period. But in every successive period $(1+n)$ proprtion of new members are born (also with a life-time of exactly one period), each of whom are an exact replica of the previous set of agents (i.e, have identical tastes and preferences).
- Each agent cares for the utility of her child, who in turn cares for the utility of her child and so on....In other words, the agents are altruistic towards their children. But the altruism is '*imperfect*' in the sense that they care a little less for their children than they do for themselves.

Interpretation of the Discount Rate (Contd.):

- By this definition then, utlity of an agent belonging to genertaion t :

$$U_t = u(c_t) + \frac{1}{1+\rho} U_{t+1}.$$

- If we now expand the successive values of U_t, then we shall get back the utility function of an agent belonging to generation 0 as:

$$U_0 = u(c_t) + \frac{1}{1+\rho} u(c_{t+1}) + \frac{1}{(1+\rho)^2} u(c_{t+2}) + \ldots\ldots.$$

- In other words, we shall get back the infinite horizon utility function as had been defined earlier, except that the term ρ now measures the '**degree of parental altruism**'.
- The lower is ρ, the higher is the parental altruism.
- When $\rho = 0$, there is '*perfect' altruism* (i.e., parents care as much for their children as they care for themselves).

Interpretation of the Discount Rate (Contd.):

3. A third interpretation of the discount rate ρ follows if we allow each agent to potentially live forever, but introduce a constant (age-independent) mortality risk at every time period.

- Suppose an agent lives for sure in the first period of his life (when he is born); but at every subsequent period he faces a constant probablility of dealth, denoted by p.
- If the agent is alive in any time period t, then he can enjoy utility from consumption at that point of time, given by $u(c_t)$. But if he dies then he gets zero utility.
- Thus beginning at time 0, the *expected* life-time utility of the agent will be given by:

$$U_0 = u(c_t) + pu(c_{t+1}) + p^2 u(c_{t+2}) + ...$$

- Without any loss of generality, replace p by $\frac{1}{1+\rho}$ and we shall get back the infinite horizon utility function as had been defined earlier.

The R-C-K Model: Centralized Version (Optimal Growth)

- There are two version of the R-C-K model:
  - A centralized version - which analyses the problem from the perspective of a social planner.
  - A decentralized version - which analyses the problem from the perspective of a perfectly competitive market economy where 'atomistic' households and firms take optimal decisions in their respective individual spheres.
- The centralized version was developed by Ramsey (way back in 1928) and is oftem referred to as the 'optimal growth' problem.
- It is assumed that there exists an **omniscient**, **omnipotent**, **benevolent** social planner who wants to maximise the citizens' welfare.
- Since all households are identical, the objective function of the social planner is identical to that of the households:

$$U_0 = \int_{t=0}^{\infty} u\left(c_t\right) \exp^{-\rho t} dt. \tag{1}$$

The R-C-K Model: Centralized Version (Contd.)

- The social planner maximises (1) subject to **the planner's budget constraint** in every period.
- Notice that in a centrally planned economy there are no markets (hence no market wage rate or market rental rate), and there is no private ownership of assets (capital) and no personalized income.
- The social planner employs the existing capital stock in the economy (either collectively owned or owned by the government) and the existing labour force to produce the final output -using the aggregate production technology.
- After production it distributes a part of the total output among its citizens for consumption puoposes and invests the rest.
- Thus the budget constraint faced by the planner in period t is nothing but the aggregate resource constraint:

$$C_t + I_t = Y_t = F(K_t, N_t).$$

The R-C-K Model: Centralized Version (Contd.)

- Investment augments next period's capital stock: $\dfrac{dK}{dt} = I_t$.
- Thus the budget constraint faced by the planner in period t is given by:

$$C_t + \frac{dK}{dt} = F(K_t, N_t) - \delta K_t.$$

- Writing in per capita terms:

$$c_t + \frac{dk}{dt} = f(k_t) - \delta k_t - n k_t.$$

- Thus the dynamic optimization problem of the social planner is:

$$\int_{t=0}^{\infty} u(c_t) \exp^{-\rho t} dt \tag{I}$$

subject to

$$\frac{dk}{dt} = f(k_t) - (\delta + n) k_t - c_t; \;\; k_t \geqq 0 \text{ for all } t; \;\; k_0 \text{ given}.$$

A Digression: Dynamic Optimization in Continuous Time (Optimal Control)

- Consider the following optimization problem which is defined over a finite time horizon from 0 to T:

$$W = \int_{t=0}^{T} F\left(u_t, x_t, t\right) dt \tag{2}$$

subject to

$$\text{(i) } \frac{dx}{dt} = g(u_t, x_t, t); \;\; u_t \in U; \;\; x_0 \text{ given.}$$

- Here u_t is called the *control variable*; x_t is called the *state variable*; F represents the instantaneous payoff function, or the *felicity* function.
- (i) specifies the evolution of the state variable as a function of the state and control variables.
- It is called the equation of motion or the *state transition equation*.

Optimal Control (Contd.):

- The objective function here is an integral, and our task is to find out a time path of the time dependent variable u from the corresponding choice set U, (i.e., to choose a $u \in U$ for each point of time t starting from 0 to T) such that the value of this integral is maximized.
- But our choice is not unconstrained. (Had it been so, a simple point-by-point static optimization exercise would have given us the required solution path).
- Note that the F function depends not only on u but also on another time dependent variable x. And our choice of u at each point of time affects the next period's value of x through the given differential equation.
- Thus our choice of u affects the objective function directly, as well as indirectly through x.

Optimal Control (Contd.):

- u_t is called the *control variable* because we choosing its value directly.
- Once the value of u_t is chosen in any time period t, the value of the state variable evolves automatically through the state transistion equation.
- Notice that when we are considering the problem at time 0, the initial value of the state variable is given to us, but not that of the control variable.
- The **initial value of the control variable will also be optimally chosen**, along with all its subsequent values.

Optimal Control: Pontryagin's Maximum Principle

- Let u_t^* be a solution path to the problem specified in (2), and let x_t^* be the associated path for the state variable, where u_t^* is a piece-wise continuous function of t and x_t^* is a strictly continuous but piece-wise differentiable in t. Then there exist a strictly continuous and piece-wise differentiable variable λ_t, and a function H defined as:

$$H(u, x, \lambda, t) \equiv F(u_t, x_t, t) + \lambda_t g(u_t, x_t, t),$$

such that

1. H is maximized with respect to u at u_t^* for all $t \in [0, T]$;
2. $\left.\frac{\partial H}{\partial x}\right|_{(u_t^*, x_t^*, \lambda, t)} = -\frac{d\lambda}{dt}$;
3. $\left.\frac{\partial H}{\partial \lambda}\right|_{(u_t^*, x_t^*, \lambda, t)} = \frac{dx}{dt}$;
4. $\lambda_T = 0$

Optimal Control (Contd.):

- The function H is called the **Hamiltonian Function** associated with the given dynamic optimizations problem.
- The newly introduced time dependent variable λ_t is called the *co-state variable* associated with the state variable x_t.
- The co-state variable λ_t measures the change in the value of the objective function W associated with an infinitesimal change in the state variable x at time t (which is the same a change in the $\frac{dx}{dt}$ function or the constraint function.
- If there were an exogenous tiny increment to the state variable at time t, and if the problem were modified optimally thereafter, then the increment in the total value of the objective would be λ_t. Thus it is *the marginal valuation of an incremental change in the state variable at time t*
- λ_t is therefore often referred to as the *shadow price of the state variable at time t*.

- **Pontryagin's Maximum Principle** gives us four *first order necessary conditions* for the optimization problem defined in (2).
- These necessary conditions are also *sufficient* if additionally the following conditions (due to Mangasarian) hold:
  - the functions F and f are concave in (u, x);
  - $\lambda_t \geqq 0$ for all t whenever f is nonlinear in either u or x.
- The first three F.O.N.C's are defined in terms of the Hamiltonian function.
- Note that if the Hamiltonian function is non-linear in u, then (1) can be replaced by the condition

$$\left.\frac{\partial H}{\partial u}\right|_{(u_t^*, x_t^*, \lambda, t)} = 0,$$

  provided the second order check is verified.
- The last condition of the Maximum Principle, which specifies a terminal condition for λ_t, is called the **Transversality Condition**.

Optimal Control (Contd.):

- Sometimes depending on the specification of the problem the transversality condition may change.
- In the above problem we are given an initial condition about the state state variable, but nothing has be specified about the terminal value of the state. This type of problems are called problems with **a free terminal state**, and the relevant transversality condition for this set of problems are given by (4).
- Alternatively you may have an optimization problem where not only the initial state value, but the terminal value of the state is also given: $x_T = \bar{x}$ (given).
- This is a problem with **a fixed terminal state**. In this case the first three F.O.N.C.s will again be given by (1) – (3). Only condition (4) will be replaced by a new transversality condition now, given by

$$x_T = \bar{x}.$$

Optimal Control (Contd.):

- Yet another type of problem specifies a terminal condition on the state variable in the form of an inequality. These are problems with **a truncated vertical terminal line**. : $x_T \geqq \bar{x}$ (given).
- In this case once again the first three F.O.N.C.s will again be given by (1) – (3). But condition (4) will be replaced by a new transversality condition now, given by the following Complementray Slackness condition:

$$\lambda_T \geqq 0;\ x_T \geqq \bar{x};\ \lambda_T \left(x_T - \bar{x}\right) = 0.$$

The R-C-K Model Revisited:

- Let us now go back to the centralized version of the R-C-K model.
- Recall that the social planner's problem is given by:

$$\int_{t=0}^{\infty} u(c_t) \exp^{-\rho t} dt \tag{I}$$

subject to

$$\frac{dk}{dt} = f(k_t) - (\delta + n)k_t - c_t; \; k_t \geqq 0 \text{ for all } t; \; k_0 \text{ given.}$$

- Notice that the choice set for the control variable is: $c_t \in \mathbb{R}^+$.
- Also the terminal condition on the state variable can be written as: $\lim_{t\to\infty} k_t \geqq 0$.
- Corresponding Hamiltonian Function:

$$H_t = u(c_t) \exp^{-\rho t} + \lambda_t [f(k_t) - (\delta + n)k_t - c_t]$$

The R-C-K Model: Centralized Version (Contd.)

- The Corresponding FONCs (which are also sufficient in this case):

$$H \text{ is maximized with respect to } c_t \quad \Rightarrow \quad \frac{\partial H}{\partial c_t} = 0 \text{ for all } t; \qquad \text{(i)}$$

$$\left(\text{verify:} \frac{\partial^2 H}{\partial c_t^2} < 0\right)$$

$$\frac{\partial H}{\partial k_t} = -\frac{d\lambda}{dt}; \qquad \text{(ii)}$$

$$\frac{\partial H}{\partial \lambda_t} = \frac{dk}{dt}; \qquad \text{(iii)}$$

$$\text{TVC: } \lim_{t \to \infty} \lambda_t k_t = 0. \qquad \text{(iv)}$$

The R-C-K Model: Centralized Version (Contd.)

- Sometimes instead of the Hamiltonian Function, we use the *Current-value* Hamiltonian Function, defined as:

$$\begin{aligned} \hat{H}_t &= H_t \exp^{\rho t} \\ &= u(c_t) + \mu_t \left[f(k_t) - (\delta + n) k_t - c_t \right], \end{aligned}$$

where $\mu_t = \lambda_t \exp^{\rho t}$ is called the *Current-value co-state variable*.

- FONCs in terms of the Current-value Hamiltonian:

$$\hat{H} \text{ is maximized with respect to } c_t \Rightarrow \frac{\partial \hat{H}}{\partial c_t} = 0 \text{ for all } t; \qquad \text{(i)}$$

$$\frac{\partial \hat{H}}{\partial k_t} = -\frac{d\mu}{dt} + \mu\rho; \qquad \text{(ii)}$$

$$\frac{\partial \hat{H}}{\partial \mu_t} = \frac{dk}{dt}; \qquad \text{(iii)}$$

$$\text{TVC: } \lim_{t \to \infty} \mu_t \exp^{-\rho t} k_t = 0. \qquad \text{(iv)}$$

Interpretation of the Optimality Conditions:

- FONC (i):

$$\frac{\partial \hat{H}}{\partial c_t} = 0 \Rightarrow u'(c_t) = \mu_t \text{ for all } t$$

implies that the marginal utility from consumption at every point of time must be equal to the shadow price of capital (i.e., the incremental utility associated with a unit increase in capital stock).

- FONC (ii):

$$\frac{\partial \hat{H}}{\partial k_t} = -\frac{d\mu}{dt} + \rho\mu_t \Rightarrow \left[f'(k_t) - \delta - n\right] + \frac{1}{\mu_t}\frac{d\mu}{dt} = \rho$$

implies that the 'net' rate of return (inclusive of capital gains/losses) on savings must be equal to the minimum compensation required to induce people to forego a unit of current consumption for the sake of tomorrow (i.e., the agents' subjective rate of time preference).

Interpretation of the Optimality Conditions (Contd.):

- FONC (iii):

$$\frac{\partial \hat{H}}{\partial \mu_t} = \frac{dk}{dt} \Rightarrow \frac{dk}{dt} = f(k_t) - (\delta + n)k_t - c_t$$

  denotes the per capita budget constraint of the social planner.
- Finally, the Transversality Condition:

$$\lim_{t \to \infty} \mu_t \exp^{-\rho t} k_t = 0$$

  implies that **at the terminal time**
  - if the shadow price of capital is positive, no capital stock should be left unused (unconsumed) and the economy must end up with zero capital stock ($\mu_T > 0 \Rightarrow k_T = 0$);
  - on the other hand, if some capital stock is indeed left unused then it must be the case that the corresponding shadow price is zero (i.e., consuming further generates no utility value) ($k_T > 0 \Rightarrow \mu_T = 0$).
  - Needless to say in this infinite horizon problem, the above conditions hold in a limiting sense.

Interpretation of the Current Value Hamiltonian Function:

- The *Current-value* Hamiltonian Function:

$$\hat{H}_t = u(c_t) + \mu_t [f(k_t) - (\delta + n)k_t - c_t]$$

measures the utility valuation of the per capita GDP at any point of time t.

- Note that the per capita output at any time period $f(k_t)$ can be used for two purposes: to be enjoyed as consumption (c_t) and to augment the capital stock $\left(\frac{dk}{dt}\right)$.
- The part that is consumed generates direct utility given by $u(c_t)$.
- That part that is used for investment generates potential future consumption and associated with an utility valuation of μ_t.
- Thus the Current Value Hamiltonian measures the direct as well as the indirect utility associated with the per capita output at any time period t.
- The Hamitonian (or the Present-value Hamiltonian), H_t, measures the present discounted utility value of the per capita output at time t.

R-C-K Model (Centralized Version): Characterization of the Optimal Path

- To summarise, the optimal trajectories of c_t , k_t and μ_t must satisfy the following set of equations at every point of time t:

$$\begin{aligned} u'(c_t) &= \mu_t; && \text{(i)} \\ \frac{1}{\mu_t}\frac{d\mu}{dt} &= \rho - \left[f'(k_t) - \delta - n\right]; && \text{(ii)} \\ \frac{dk}{dt} &= f(k_t) - (\delta + n)k_t - c_t; && \text{(iii)} \\ \lim_{t\to\infty} \mu_t \exp^{-\rho t} k_t &= 0. && \text{(iv)} \end{aligned}$$

- Notice that even though we have three time-dependent variables (c_t , k_t and μ_t), c_t and μ_t are always tied to each other by virtue of equation (i); hence their dynamic paths are also inter-dependent.
- Thus we can eliminate one of them to get a system of differential equations either in (c_t and k_t) or in (k_t and μ_t).

Characterization of the Optimal Path (Contd.):

- Here we shall eliminate μ_t and work with c_t.
(**Verify that you reach the same conclusions when you eliminate c_t and work with μ_t instead**).
- Log-differentiating (i), and using (ii):

$$\begin{aligned} \frac{u''(c_t)}{u'(c_t)}\frac{dc}{dt} &= \frac{1}{\mu_t}\frac{d\mu}{dt} = \rho - \left[f'(k_t) - \delta - n\right] \\ \Rightarrow \frac{dc}{dt} &= \frac{c_t}{\left(\frac{-c_t\, u''(c_t)}{u'(c_t)}\right)}\left[f'(k_t) - \delta - n - \rho\right]. \qquad \text{(v)} \end{aligned}$$

- We also know:

$$\frac{dk}{dt} = f(k_t) - (\delta + n)k_t - c_t. \qquad \text{(iii)}$$

- Equations (iii) & (v) represent a 2×2 system of differential equations which **along with the Transversality Condition** characterize the optimal path of the economy under the centralized R-C-K model.

Intertemporal Elasticity of Substitution:

- Notice that in equation (v), there is a term:

$$\left(\frac{-c_t \;\; u''\left(c_t\right)}{u'\left(c_t\right)} \right) \equiv \sigma(c_t).$$

- This term has multiple interpretations.

1. The most obvious interpretation is that it is **the elasticity of marginal utility with respect to consumption**.
2. In choices under uncertainty, $\sigma(c_t)$ coincides with **the Arrow-Pratt measure of relative risk aversion**.
3. The $\sigma(c_t)$ terms is also **the inverse of the elasticity of substitution between current and future consumption.**

Intertemporal Elasticity of Substitution (Contd.):

- Note that elasticity of substitution between consumption at date t and consumption at date $t + \Delta t$ is defined as

$$\epsilon = -\frac{d\left(\frac{c_t}{c_{t+\Delta t}}\right) / \left(\frac{c_t}{c_{t+\Delta t}}\right)}{d\left(\frac{u'(c_t)}{u'(c_{t+\Delta t})}\right) / \left(\frac{u'(c_t)}{u'(c_{t+\Delta t})}\right)}.$$

- It can be shown that as $\Delta t \longrightarrow 0$, $\epsilon \longrightarrow \frac{1}{\sigma}$. (**Verify this.**)
- It is sometimes convenient to work with utility functions where $\sigma(c_t)$ is a constant.
  - Examples:
    - Log utility function: $u(c_t) = \log c_t$
    - CRRA utility function: $u(c_t) = \dfrac{(c_t)^{1-\theta}}{1-\theta}; \theta \neq 1.$
- For the time being, however, we shall work with a general utility function where $\sigma(c_t)$ need not be a constant.

Characterization of the Optimal Path (Contd.):

- The 2×2 non-linear and autonomous system of equations for the centralized economy are given by:

$$\frac{dc}{dt} = \frac{c_t}{\sigma(c_t)}\left[f'(k_t) - \delta - n - \rho\right]; \qquad \text{(v)}$$

and

$$\frac{dk}{dt} = f(k_t) - (\delta + n)k_t - c_t. \qquad \text{(iii)}$$

- Both (iii) and (v) are non-linear differential equations; so we have to use phase diagram technique to qualitatively characterize the optimal path.

R-C-K Model (Centralized Version): Steady State(s)

- First let us identify the possible staedy state(s) of the dynamic system.
- The steady state is now defined as pair of values (k, c) such that neither values change over time.
- In other words, the steady states are defined by the following two equation:

$$\begin{aligned} \frac{c_t}{\sigma(c_t)}\left[f'(k_t) - \delta - n - \rho\right] &= 0; \\ f(k_t) - (\delta + n)k_t - c_t &= 0. \end{aligned}$$

- Notice that $\sigma(c_t) > 0$. Hence from the above equations we can identify three possbile steady states of the system:

$$\begin{aligned} \text{Trivial steady state}: &\quad c = 0; k = 0; \\ \text{Semi-trivial steady state}: &\quad c = 0; k = \bar{k} \text{ such that } f(\bar{k}) = \delta + n; \\ \text{Non-trivial steady state}: &\quad c = c^* > 0; k = k^* > 0 \text{ such that} \\ f'(k^*) &= \delta + n + \rho;\ c^* = f(k^*) - (\delta + n)k^*. \end{aligned}$$

R-C-K Model (Centralized Version): Constrcution of the Phase Diagram

- From equation (v):

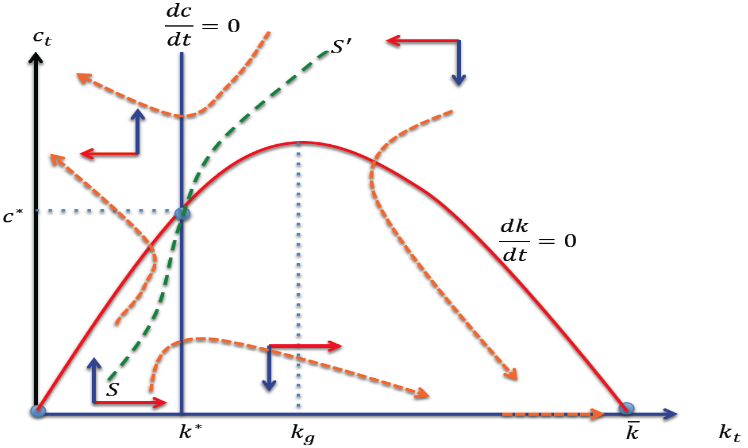
$$\begin{aligned} \frac{dc}{dt} &\gtreqless 0 \text{ according as} \\ \text{either } c_t &= 0 \text{ or } f'(k_t) \gtreqless \delta + n + \rho. \end{aligned}$$

- On the other hand, from equation (iii):

$$\begin{aligned} \frac{dk}{dt} &\gtreqless 0 \text{ according as} \\ c_t &\lesseqgtr f(k_t) - (\delta + n)k_t. \end{aligned}$$

- Now we can trace the level curves $\frac{dc}{dt} = 0$ and $\frac{dk}{dt} = 0$ in the (k_t, c_t) plane and draw the coresponding directional arrows to get the corresponding phase diagram.
(**How should the Phase Diagram look?**)

R-C-K Model (Centralized Version): Phase Diagram



R-C-K Model (Centralized Version): Characterization of the Optimal Path (Contd.)

- Notice that any pair of directional arrows in the phase diagram satisfy (by construction) the dynamic equations (iii) and (v) and therefore satisfy the first three FONCs of the given dynamic optimization problem .
- Also note that k_0 is given, but c_0 is not. In fact **our choice of** c_0 would generate multiple possible time paths of c_t and k_t - all satisfying the first three FONCs.
- One can classify these multiple trajectories in three broad categories:
  - **Category I**: Trajectories that move towards the horizontal axis over time and eventually approach the point $(\bar{k}, 0)$ as $t \rightarrow \infty$;
  - **Category II**: Trajectories that approach the vertical axis over time;
  - **Category III**: A **unique** trajectory that represents the stable arm of the saddle point (k^*, c^*) and approaches the non-trivial steady state point (k^*, c^*) over time.
- Which one of these is **the optimal trajectory**?
- Here the transversility condition comes to our rescue.

R-C-K Model (Centralized Version): Characterization of the Optimal Path (Contd.)

- Recall that the TVC is part of the necessary (and sufficinet) conditions for optimality.
- So among all these trajectories, the one which satisfies the TVC will indeed be the optimal path. (What if there are multiple such trajectories?)
- As it turns out, *only the unique trajectory belonging to* **Category III** (*represented by the line SS' in the diagram*) *satisfies all the four FONCs including the transversility condition*.
- We now provide heuristic arguments as to why trajectories belonging to the other two categories cannot be optimal.

Proof that Trajectories of Type I cannot be Optimal:

- Recall that the Transversality condition for the central planner's problem is specified as:

$$\text{TVC: } \lim_{t\to\infty} \mu_t \exp^{-\rho t} k_t = 0. \tag{iv}$$

- Also noting that along the optimal path, $\mu_t = u'(c_t)$, we can write the TVC as:

$$\text{TVC: } \lim_{t\to\infty} u'(c_t) \exp^{-\rho t} k_t = 0. \tag{iv$\prime$}$$

- Now let us check whether trajectories belonging to Category I satisfy condition (iv$\prime$).
- Notice that along a trajectory of type I, $\lim_{t\to\infty} k_t = \bar{k} > 0$.
- Hence condition (iv$'$) will be satisfied along these trajectories if and only if

$$\text{TVC: } \lim_{t\to\infty} u'(c_t) \exp^{-\rho t} = 0. \tag{iv$''$}$$

Proof that Trajectories of Type I cannot be Optimal: (Contd.)

- Now along a trajectory of type I, $\lim_{t\to\infty} c_t = 0$ which implies that $\lim_{t\to\infty} u'(c_t) \to \infty$.
- At the same time, $\lim_{t\to\infty} \exp^{-\rho t} = 0$.
- Thus it is not immediately clear whether condition (iv$'$) will be satisfied or not.
- It depends on whether in the neighbourhood of $(\bar{k}, 0)$, $u'(c_t)$ is increasing at a faster/slower rate than the rate of fall of $\exp^{-\rho t}$.
- The exponential term $\exp^{-\rho t}$ is of course decreasing at a constant rate ρ.
- On the other hand, the $u'(c_t)$ term is increasing at the rate $[(n + \delta - f'(k)) + \rho]$.
- In the neighbourhood of $(\bar{k}, 0)$, $[(n + \delta - f'(k)) + \rho] > \rho$. (**Why?**)
- Thus $u'(c_t)$ is increasing at a **faster** rate than the rate of fall of $\exp^{-\rho t}$.

Proof that Trajectories of Type I cannot be Optimal: (Contd.)

- Therefore along any trajectory of type I,

$$\lim_{t\to\infty} u'(c_t)\exp^{-\rho t} \longrightarrow \infty,$$

  which violates the TVC (iv$''$).

- **Hence trajectories of type I cannot be optimal.**

Proof that Trajectories of Type II cannot be Optimal:

- The trajectories of type II approach the vertical axis over time.
- If they approach the vertical axis **asympototically** (never actually hitting it at any finite point of time) then indeed the TVC (iv) will be satisfied and such trajectories would be optimal.
- However we now argue that this is not possible. Indeed **every trajectory belonging to category II actually hit the vertical axis within a finite point of time**.
- We prove this by contradiction.
- Suppose, if possible, that trajectories of type II approach but do not hit the vertical axis within finite time.
- In other words, suppose, if possible, that trajectories of type II are asymptotic either to the vertical axis or some line parallel to the vertical axis.
- If that is true then starting form any given finite initial value, k_0,

$$\lim_{t \to \infty} k_t = M \text{ (where } M \text{ is a non-negative constant} < k_0).$$

Proof that Trajectories of Type II cannot be Optimal: (Contd.)

- Now from FONC (iii), we know that,

$$\frac{dk}{dt} = f(k_t) - (\delta + n)k_t - c_t.$$

- Therefore,

$$k_t = k_0 + \int_0^t [f(k_\tau) - (\delta + n)k_\tau - c_\tau] \, d\tau.$$

- Hence

$$\lim_{t \to \infty} k_t = k_0 + \int_0^\infty [f(k_\tau) - (\delta + n)k_\tau - c_\tau] \, d\tau = M.$$

Proof that Trajectories of Type II cannot be Optimal: (Contd.)

- Rearranging terms:

$$\int_0^\infty \left[c_\tau - \left\{f(k_\tau) - (\delta + n)k_\tau\right\}\right] d\tau = k_0 - M \equiv N \text{ (a finite constant).}$$

- We now argue that along any trajectory of type II, the integral defined by the LHS above will diverge away to $+\infty$ and therefore can never converge to finite constant N.
  (**Prove this yourself. Hint:** A *necessary* condition for an infinite integral $I \equiv \int_0^\infty a_\tau d\tau$ to converge is that $\frac{da_\tau}{d\tau} < 0$. Define $a_\tau \equiv \left[c_\tau - \left\{f(k_\tau) - (\delta + n)k_\tau\right\}\right]$ here and show that along any trajectory of type II this necessary condition is violated.)
- Hence there is a contradiction.

Proof that Trajectories of Type II cannot be Optimal: (Contd.)

- We have just proved that **any trajectory belonging to category II cannot be asymptotic to the vertical axis; it must hit the vertical axis within a finite period of time.**
- Now take any such trajectory. Can it still be optimal?
- The answer is "No".
- The reason is as follows:
  - Suppose the trajectory hits the vertical axis precisely at time T.
  - then exactly at time T, k_t reaches zero;
  - Consequently, c_t falls from a finite value to zero (since a positive value of consumption cannot be sustained with zero capital stock);
  - This implies that precisely at time T, μ_t jumps from a finite value to infinity.
  - Such a discrete jump of μ_t violates FONC (ii) - which presupposes continuity of μ_t.
- **Hence trajectories of type II cannot be optimal.**

R-C-K Model (Centralized Version): Identification of the Optimal Trajectory

- We have now seen that trejectories belonging to either category I or category II cannot be optimal because they violate one of the FONCs (i)-(iv).
- That leaves the unique trajectory beloging to category III, which is the stable arm of the saddle point (k^*, c^*) and represented by the line SS' in the diagram.
- Along this trajectory, as $t \longrightarrow \infty$, c_t and k_t approach c^* and k^* respectively.
- It is easy to verify that this trajectory satisfies all the four FONCs, **including the Transversality Condition**.
- Hence this is the unique optimal trajectory for the social planner's problem.
- Thus, given k_0, it is optimal for the social planner to choose the corresponding c_0 that lies on trajectory III and then let the economy evolve according to the two dynamic equations (iii) & (v).

R-C-K Model (Centralized Version): Growth Implications

- In the centralized version of the R-C-K model (without technical progress), once again the economy goes to a steady state in the long run, where per capita income becomes constant.
- So there is no long run growth of per capita income; aggreagte income in the long run grows at the constant rate n.
- Thus the growth conclusions of the centralized R-C-K model are exactly identical to that of the Solow model (without technical progress).
- **Excercise:** Introduce exgoneous technical progress in this R-C-K model as we did in Solow; show that even then the growth conclusions would be the same - *except that you now need more restrictions on the utility function so that the TVC holds.*
- **What happens during transition? Does conditional convergence hold? (Verify using a log utility function and a Cobb-Douglas production function.)**

R-C-K Model (Centralized Version): Implications for Dynamic Efficiency

- Although the growth implications of the centralized R-C-K model are exactly identical to that of Solow, there is a big difference.
- The steady state of the R-C-K model is always *dynamically efficient*. (**Why?)**
- In fact with a positive rate of time preference (so that $\rho > 0$), the steady state point $(k^* : f'(k^*) = n + \delta + \rho)$ represents the "best point' in the sense that it maximises the welfare of the household.
- This point is called the '*modified golden rule*' which is different from the '*golden rule*' capital stock defined earlier $(k_g : f'(k_g) = n + \delta)$
- Indeed the 'golden rule' point is no longer the 'best point' for the agents (unless they have zero rate of time preference.) Even if I start with an initial capital stock $k_0 = k_g$, it is **not optimal** for me to stay at (k_g, c_g) forever. I would like to move to the corresponding point of the saddle path SS' and evetually approach (k^*, c^*) because that, by construction, maximizes my welfare (life-time utility).

R-C-K Model: the De-centralized Version

- So far we have analysed the centralized version of the R-C-K model, where a benevolent social planner makes all the production and savings/investment decisions on behalf of the households.
- We now turn to the corresponding problem for a decentralized market economy, where all these economic decisions are undertaken by 'atomistic' firms and households.
- Perfect competition prevails - which means that while optimizing, the firms and households take all the market prices (in the commodity market as well as in the factor markets) as given.

R-C-K Model (De-centralized Version): Production Side Story

- Recall that the production side story in the R-C-K model is identical to that of Solow.
- Thus the economy starts with a given stock of capital (K_t) and a given level of population (N_t) at time t. (We are ignoring technological progress for now).
- From previous analysis, we also know that all firms have access to an identical production technology - **which satisfies all standard neoclassical properties**.
- The firm-specific production functions can be aggregated to generate an aggregate production function such that :

$$Y_t = F(K_t, N_t).$$

- At every point of time the market clearing wage rate and the rental rate of capital are given by:

$$w_t = F_N(K_t, N_t); \quad r_t = F_K(K_t, N_t).$$

R-C-K Model (De-centralized Version): Household Side Story

- The household at time 0 chooses its entire consumption profile $\left\{c_t^h\right\}_{t=0}^{\infty}$ so as to maximise the discounted sum of its life-time utility:

$$U_0^h = \int_{t=0}^{\infty} u\left(c_t^h\right) \exp^{-\rho t} dt; \;\; \rho > 0,$$

subject to **the household's budget constraint** in every time period. $\left(u\left(c_t^h\right)\right.$ of course satisfies all the standard properties specified earlier).

- If we do not allow intra-household borrowing, then household h's budget constraint (in aggregate terms) would be given by:

$$C_t^h + I_t^h = w_t N_t^h + r_t K_t^h, \text{ where } \frac{dK_t^h}{dt} = I_t^h - \delta K_t^h.$$

- In per member terms, household h's budget constraint becomes:

$$\frac{dk_t^h}{dt} = w_t + r_t k_t^h - (n+\delta) k_t^h - c_t^h.$$

R-C-K Model (De-centralized Version): Household Side Story (Contd.)

- Thus in the absence of intra-household borrowing the optimization problem of the representative household h is given by:

$$\underset{\{c_t^h\}_{t=0}^{\infty}}{Max.}\ U_0^h = \int_{t=0}^{\infty} u\left(c_t^h\right) \exp^{-\rho t} dt; \ \ \rho > 0,$$

  subject to

$$\frac{dk_t^h}{dt} = w_t + r_t k_t^h - (n+\delta)k_t^h - c_t^h; \ \ k_t^h \geqq 0 \text{ for all } t \geqq 0; \ \ k_0^h \text{ given},$$

  where c_t^h is the control variable and k_t^h is the state variable.
- Notice that in order to solve this problem the households would have to have some expectation about the entire time paths of w_t and r_t.
- We shall assume that **households' have perfect foresight**. So they can correctly **guess** all the future values of the market wage rate and rental rate.

De-centralized R-C-K Model: Household Side Story (Contd.)

- Note that the household's problem is different from the social planner problem in an important way: the households can borrow from one another.
- Allowing for intra-household borrowings means that a household's consumption at any point of time t need not be limited by its current income and current capital stock.
- The household can consume beyond its current income at any point of time - by borrowing from others.
- Allowing for intra-household borrowings also means that a household now has two forms of assets that it can invest its savings into:
  1. physical capital (K_t^h);
  2. financial capital, i.e., lending to other households $(L_t^h \equiv -B_t^h)$.

De-centralized R-C-K Model: Household Side Story (Contd.)

- Let the interest rate on financial assets be denoted by $\hat{r}_t$.
- We already know that the (net) interest rate on investment in physical capital is given by $(r_t - \delta)$.
- Arbitrage in the asset market ensures that in equlibrium two interest rates are the same :

$$\hat{r}_t = r_t - \delta.$$

- Hence we can define the **total asset holding** by the household as $A_t^h \equiv K_t^h + L_t^h$, where $L_{t,}^h < 0$ if the household is a net borrower.
- Thus the aggregate budget constraint of the household now becomes:

$$C_t^h + \frac{dA_t^h}{dt} = w_t N_t^h + \hat{r}_t A_t^h.$$

- Writing in per member terms:

$$\frac{da_t^h}{dt} = w_t + (\hat{r}_t - n)a_t^h - c_t^h.$$

Household Side Story: Ponzi Game

- But allowing for intra-household borrowing brings in the possibility of **Ponzi game**.
- Consider the following plan by a household:
  - Suppose in period 0, the household borrows a huge amount $\bar{B}$ - which would allow him to maintain a very high level of consumption at all subsequent points of time. Thus
$$B_0^h = \bar{B}.$$
  - In the next period (period 1), he pays back his period 0 debt with interest by borrowing again (presumably from a different lender). Thus his period 1 borrowing would be:
$$B_1^h = (1 + \hat{r}_0) B_0.$$
  - In period 2 he again pays back his period 1 debt with interest by borrowing afresh:
$$B_2^h = (1 + \hat{r}_1) B_1^h = (1 + \hat{r}_0)(1 + \hat{r}_1) B_0^h.$$

Household Side Story: Ponzi Game (Contd.)

- Proceeding this way,

$$B^h_{t+1} = (1+\hat{r}_t)B^h_t = (1+\hat{r}_0)(1+\hat{r}_1)...(1+\hat{r}_t)B^h_0.$$

- In other words, the household's debt grows at the rate $\hat{r}_t$.
- Notice that by playing this game, the household effectively never pays back its initial loan $\bar{B}$; it is simply rolling it over period after period.
- In the process it is able to maintain an arbitrarily high level of consumption (over and above it's current income).
- This kind of financing scheme is called **Ponzi finance**.
- If a household is allowed to play such a Ponzi game, then the household's budget constraint becomes irrelevant. There is effectively no budget constraint for the household any more; it can maintain any arbitrarily high consumption path by playing a Ponzi game.
- To rule this out, we impose an additional constraint on the household's optimization problem - called the **No-Ponzi Game Condition**.

Household Side Story: No-Ponzi Game Condition

- Recall that the when the the Household is playing the Ponzi game, it's debt grows at the rate $\hat{r}_t$:

$$\frac{1}{B_t^h}\frac{dB_t^h}{dt} = \hat{r}_t.$$

- Hence the household's debt in per capita (per member) terms $\left(b_t^h \equiv \frac{B_t^h}{N_t^h}\right)$ grows at the rate $\hat{r}_t - n$.
- One Version of **No-Ponzi Game (NPG) Condition**, which rules out this kind of behaviour is given below:

$$\textbf{NPG Condition}: \lim_{t\to\infty} a_t^h \exp^{-\int_0^t (\hat{r}_v - n)dv} \geqq 0.$$

- This Non-Ponzi Game condition states that as $t \to \infty$, the present discounted value of the household's asset (when the discount rate is the population-adjusted interest rate) must be non-negative.

Implication of the No-Ponzi Game Condition:

- Notice that one can always rule out Ponzi behaviour by banning intra-household borrowing altogether.
- But that is a stronger restriction than required.
- In fact the above NPG condition is very flexible in the sense that it allows households to borrow perpetually, but at the same time rules out Ponzi behaviour.
- How?
- Let us first see how the NPG condition rules out Ponzi behaviour:
- Recall that when the household is playing a Ponzi game, its per member borrowing is increasing at the rate $\hat{r}_t - n$, i.e.,

$$\frac{1}{b_t^h}\frac{db_t^h}{dt} = \hat{r}_t - n.$$

- Solving:

$$b_t^h = b_0^h \exp^{\int_0^t (\hat{r}_v - n) dv}$$

Implication of the No-Ponzi Game Condition: (Contd.)

- Since the household is playing a Ponzi Game, $b_0^h > 0$. Also it must have already consumed all its initial capital stock and its net asset stock for any $t > 0$ must be given by: $a_t^h = -b_t^h$.
  - Thus,

$$\begin{aligned} a_t^h &= -b_0^h \exp^{\int_0^t (\hat{r}_v - n) dv} \\ \Rightarrow \quad a_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv} &= -b_0^h \end{aligned}$$

- Taking the limiting value:

$$\lim_{t \to \infty} a_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv} = -b_0^h < 0,$$

which violates the given NPG condition.

- In other words, playing a Ponzi game necessarily violates the NPG condition.
- Does it rule out borrowing altogether? The answer is "no".

Implication of the No-Ponzi Game Condition: (Contd.)

- Suppose the household follows a consumption path such that $b_0^h > 0$ and also any subequent time period beyond the initial point (i.e., for any $t > 0$) $a_t^h = -b_t^h$. In other words, suppose the household is once again borrowing perpetually.
- But now it's borrowing grows at a rate $g_t < \hat{r}_t - n$. (This implies that the household is paying at least part of the interest payment in every period from its own pocket.)
- Once again,

$$
\begin{aligned}
b_t^h &= b_0^h \exp^{\int_0^t g_v dv} \\
\Rightarrow \quad & a_t^h = -b_t^h = -b_0^h \exp^{\int_0^t g_v dv} \\
\Rightarrow \quad & a_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv} = -b_0^h \exp^{-\int_0^t [(\hat{r}_v - n) - g_v] dv}
\end{aligned}
$$

Implication of the No-Ponzi Game Condition: (Contd.)

- If g_t is low enough (in relation to $(\hat{r}_t - n)$) then, as $t \longrightarrow \infty$, the intergral $\int_0^t [(\hat{r}_v - n) - g_v]\, dv$ would diverge to infinity and hence the $\exp^{-\int_0^t [(\hat{r}_v - n) - g_v] dv}$ term will converge to zero.
- Notice that in this case the NPG condition will be satisfied, despite the fact that the household is perpetually borrowing.
- In other words, the NPG condition does not rule out borrowing (even perpetual borrowing); it just requires the household to start paying back its debt from its own pocket from some point of time onwards.
- The implication of the NPG condition becomes clearer if we combine it with the household's flow budget constraint.

Economic Implication of the NPG/TVC:

- Recall that the household's flow budgent constraint (when intra-household borrowing is allowed) is given by:

$$\begin{aligned}
\frac{da_t^h}{dt} &= w_t + \hat{r}_t a_t^h - n a_t^h - c_t^h \\
&\Rightarrow \frac{da_t^h}{dt} - \left(\hat{r}_t - n\right) a_t^h = w_t - c_t^h \\
&\Rightarrow \exp^{-\int_0^t (\hat{r}_v - n) dv} \left[\frac{da_t^h}{dt} - \left(\hat{r}_t - n\right) a_t^h\right] = \left(w_t - c_t^h\right) \exp^{-\int_0^t (\hat{r}_v - n) dv} \\
&\Rightarrow \frac{d\hat{a}_t^h}{dt} = w_t \exp^{-\int_0^t (\hat{r}_v - n) dv} - c_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv}
\end{aligned}$$

where $\hat{a}_t^h \equiv a_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv}$.

Economic Implication of the NPG/TVC (Contd.):

- Moving dt to the other side and intergrating both sides (from 0 to ∞):

$$\int_0^\infty d\hat{a}_t^h = \int_0^\infty w_t \exp^{-\int_0^t (\hat{r}_v - n)dv} dt - \int_0^\infty c_t^h \exp^{-\int_0^t (\hat{r}_v - n)dv} dt$$

$$\Rightarrow \lim_{t\to\infty} \hat{a}_t^h - \hat{a}_0^h = \int_0^\infty w_t \exp^{-\int_0^t (\hat{r}_v - n)dv} dt - \int_0^\infty c_t^h \exp^{-\int_0^t (\hat{r}_v - n)dv}$$

- Now recall that $\hat{a}_t^h \equiv a_t^h \exp^{-\int_0^t (\hat{r}_v - n)dv}$.
- Hence, $\hat{a}_0^h = a_0^h$, and $\lim_{t\to\infty} \hat{a}_t^h = \lim_{t\to\infty} a_t^h \exp^{-\int_0^t (\hat{r}_v - n)dv} \geqq 0$ (by NPG).

Economic Implication of the NPG/TVC (Contd.):

- Thus,

$$\lim_{t\to\infty} \hat{a}_t^h = \hat{a}_0^h + \int_0^\infty w_t \exp^{-\int_0^t (\hat{r}_v - n)dv} dt - \int_0^\infty c_t^h \exp^{-\int_0^t (\hat{r}_v - n)dv} dt \geqq 0.$$

- By rearranging terms and substituting for $\hat{a}_0^h$,

$$\int_0^\infty c_t^h \exp^{-\int_0^t (\hat{r}_v - n)dv} dt \leqq \int_0^\infty w_t \exp^{-\int_0^t (\hat{r}_v - n)dv} dt + a_0^h.$$

- In other words, the NPG condition implies that no matter what the consumption path (and the consequent borrowing pattern) is for the household, eventually the present value of the consumption stream must be limited by the sum of its non-human and human wealth (namely the discounted value of its labour earnings).

Household's Optimization Problem:

- Imposing the No-Ponzi Game condition, the household' optimization problem becomes:

$$\underset{\left\{c_t^h\right\}_{t=0}^{\infty}}{Max.}\ U_0^h = \int_{t=0}^{\infty} u\left(c_t^h\right) \exp^{-\rho t} dt; \ \ \rho > 0,$$

subject to

$$\text{(i)}\ \frac{da_t^h}{dt} = w_t + \hat{r}_t a_t^h - n a_t^h - c_t^h; \ \ a_0^h \text{ given.}$$

$$\text{(ii)}\ \textbf{NPG Condition} \ : \ \lim_{t \to \infty} a_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv} \geqq 0.$$

- As before, we can write down the FONCs (which are also sufficient) in terms of the corresponding Hamiltonian/Current-value Hamiltonian function.

Household's Problem: FONCs in terms of Hamiltonian

- The Hamiltonian Function:

$$H_t = u\left(c_t^h\right)\exp^{-\rho t} + \lambda_t\left[w_t + \hat{r}_t a_t^h - n a_t^h - c_t^h\right]$$

- Corresponding FONCs:

$$H \text{ is maximixed with respect to } c_t^h \quad \Rightarrow \quad \frac{\partial H}{\partial c_t^h} = 0 \ \text{ for all } t$$

$$\text{i.e., } u'\left(c_t^h\right)\exp^{-\rho t} \quad = \quad \lambda_t \qquad \text{(i)}$$

$$\frac{\partial H}{\partial a_t^h} \quad = \quad -\frac{d\lambda}{dt}$$

$$\text{i.e., } -\frac{d\lambda}{dt} \quad = \quad \lambda_t\left[\hat{r}_t - n\right] \qquad \text{(ii)}$$

Household's Problem: FONCs in terms of Hamiltonian (Contd.)

$$\frac{\partial H}{\partial \lambda_t} = \frac{da_t^h}{dt}$$

$$\text{i.e., } \frac{da_t^h}{dt} = w_t + \hat{r}_t a_t^h - n a_t^h - c_t^h \qquad \text{(iii)}$$

- $$\text{TVC: } \lim_{t \to \infty} \lambda_t a_t^h = 0. \qquad \text{(iv)}$$
- In addition we have the NPG Condition:

$$\lim_{t \to \infty} a_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv} \geqq 0. \qquad \text{(v)}$$

(Too many boundary conditions?)

Household's Problem: FONCs in terms of Current-value Hamiltonian

- *Current-value* Hamiltonian Function:
$$\hat{H}_t = H_t \exp^{\rho t} = u\left(c_t^h\right) + \mu_t\left[w_t + \hat{r}_t a_t^h - n a_t^h - c_t^h\right],$$
where $\mu_t = \lambda_t \exp^{\rho t}$.
- FONCs in terms of the Current-value Hamiltonian:
$$\hat{H} \text{ is maximized with respect to } c_t \quad \Rightarrow \quad \frac{\partial \hat{H}}{\partial c_t^h} = 0 \quad \text{for all } t$$
$$\text{i.e., } u'\left(c_t^h\right) = \mu_t \qquad \text{(i)}$$

$$\frac{\partial \hat{H}}{\partial a_t^h} = -\frac{d\mu}{dt} + \mu\rho$$
$$-\frac{d\mu}{dt} = \mu_t\left[\hat{r}_t - n - \rho\right] \qquad \text{(ii)}$$

Household's Problem: FONCs in terms of Current-value Hamiltonian (Contd.)

- $$\begin{aligned} \frac{\partial \hat{H}}{\partial \mu_t} &= \frac{da_t^h}{dt} \\ \text{i.e., } \frac{da_t^h}{dt} &= w_t + \hat{r}_t a_t^h - n a_t^h - c_t^h \qquad \text{(iii)} \end{aligned}$$

$$\text{TVC: } \lim_{t \to \infty} \mu_t \exp^{-\rho t} a_t^h = 0. \qquad \text{(iv)}$$

- And the NPG Condition:

$$\lim_{t \to \infty} a_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv} \geqq 0. \qquad \text{(v)}$$

Household's Problem: Optimal Solutions

- From FONCs (i)-(iii) of the household's optimization problem we get the following dynamic equations:

$$\frac{dc_t^h}{dt} = \frac{c_t^h}{\sigma(c_t^h)}\left[\hat{r}_t - n - \rho\right] \quad (1)$$

$$\frac{da_t^h}{dt} = w_t + \hat{r}_t a_t^h - n a_t^h - c_t^h \quad (2)$$

- Equations (1) and (2) represents a $2X2$ system of difference equations which implicitly defines the 'optimal' trajectories of household h.
- However we now have two dynamic equations, but three boundary conditons: the initial condition a_0^h, the TVC and the NPG condition.
- But note that along the optimal trajectory, when the TVC holds, then the NPG condition is also satisfied at the margin. Hence we can combine the two together to get the following boundary condition:

$$\lim_{t\to\infty} a_t^h \exp^{-\int_0^t (\hat{r}_v - n)dv} = 0. \quad \text{(vi)}$$

De-centralized R-C-K Model: Solution Paths for the Aggregate Economy

- Given the optimal solutions to the household's problem, we can now aggregate over all households to get the corresponding time paths for the economy-wide averages.
- When households are all identical, then of course aggregation becomes trivial:

$$\begin{aligned} c_t^1 &= c_t^2 = c_t^3 = \ldots.. = c_t^H = c_t \text{ (average consumption);} \\ a_t^1 &= a_t^2 = a_t^3 = \ldots.. = a_t^H = a_t \text{ (average asset holding).} \end{aligned}$$

- Also, notice that average asset holding in the economy:

$$a_t = \frac{\sum A_t^h}{N_t} = \frac{\sum K_t^h + \sum L_t^h}{N_t} = \frac{\sum K_t^h}{N_t} = k_t,$$

since aggregate borrowing (or lending) across all households must be zero.

- So when the households are identical, we can directly replace a_t^h and c_t^h in equations (1) and (2) by k_t and c_t respectively.

De-centralized R-C-K Model: Solution Paths for the Aggregate Economy (Contd.)

- Thus,

$$\frac{dc_t}{dt} = \frac{c_t}{\sigma(c_t)}\left[\hat{r}_t - n - \rho\right] \quad (1')$$

$$\frac{dk_t}{dt} = w_t + \hat{r}_t k_t - nk_t - c_t \quad (2')$$

- Finally noting that in a competitive market economy: $w_t = f(k_t) - k_t f'(k_t)$ and $\hat{r}_t = r_t - \delta = f'(k_t) - \delta$, we get dynamics of average consumption and per capita capita stock for the aggregate economy as:

$$\frac{dc_t}{dt} = \frac{c_t}{\sigma(c_t)}\left[f'(k_t) - \delta - n - \rho\right] \quad (3)$$

$$\frac{dk_t}{dt} = f(k_t) - (n+\delta)k_t - c_t \quad (4)$$

De-centralized R-C-K Model: Solution Paths for the Aggregate Economy (Contd.)

- Compare equations (3) and (4) with the dynamic equations derived for the social planner earlier. Observe that **they are exactly identical**!
- This implies that under the R-C-K model, **the optimal trajectories for the decentralized market economy and the centralized planning economy would be identical**.
- Since we have already proved that the steady state for the social planner's problem would be dynamically efficient, so would be the steady state for the market economy.

R-C-K Model: Equivalence between Centralized and De-centralized Economy

- We have just seen that in the R-C-K model with identical households, the solution paths of the social planner and that of the market economy will coincide.
- But in deriving this strong equivalence result, we have assumed the households are identical in every respect.
- In fact with identical household allowing for intra-household borrowing and the consequent NPG condition become superfluous because one side of the lending/borrowing market will be always missing!
- A more interesting questio is: will this strong equivalence result hold even when households are heterogenous?
- The answer is "yes", provided the utility function satisfies certain additional properties.

Decentralized R-C-K Model with Heterogenous Households:

- Let us now introduce heterogenous households in the de-centralized R-C-K model.
- In particular let us assume that households have identical preferences but **they differ in terms of initial asset holdings.**
- Let the H households have initial asset holdings denoted by a_0^1, a_0^2, a_0^3 ....and a_0^H respectively such that

$$a_0^1 \neq a_0^2 \neq a_0^3 \neq a_0^4 .... \neq a_0^H.$$

- We already know that along the optimal path, per member consumption and asset stock of any household h will follow the dynamic equations given below:

$$\frac{dc_t^h}{dt} = \frac{c_t^h}{\sigma(c_t^h)}\left[r_t - \delta - n - \rho\right] \quad (1)$$

$$\frac{da_t^h}{dt} = w_t + (r_t - \delta - n)\, a_t^h - c_t^h \quad (2)$$

Decentralized R-C-K Model with Heterogenous Households (Contd.):

- Now aggregating over all households (which are no longer identical), we can write the average asset holding in the economy as:

$$\begin{aligned} a_t &= k_t = \frac{\sum A_t^h}{N_t} = \frac{A_t^1}{N_t} + \frac{A_t^2}{N_t} + \frac{A_t^3}{N_t} + ... + \frac{A_t^H}{N_t} \\ &= \left(\frac{A_t^1}{N_t^1}\frac{N_t^1}{N_t}\right) + \left(\frac{A_t^2}{N_t^2}\frac{N_t^2}{N_t}\right) + \left(\frac{A_t^3}{N_t^3}\frac{N_t^3}{N_t}\right) + ... + \left(\frac{A_t^H}{N_t^H}\frac{N_t^H}{N_t}\right). \end{aligned}$$

- Noting that population is equally divided across all households, we get:

$$a_t = k_t = \frac{1}{H}\left(a_t^1 + a_t^2 + a_t^3 + ... + a_t^H\right). \tag{5}$$

- Likewise, aggregating over all households, we can write the average consumption in the economy as:

$$c_t = \frac{\sum C_t^h}{N_t} = \frac{1}{H}\left(c_t^1 + c_t^2 + c_t^3 + ... + c_t^H\right). \tag{6}$$

Decentralized R-C-K Model with Heterogenous Households (Contd.):

- Differentiating (5) with respect to t:

$$\frac{dk_t}{dt} = \frac{da_t}{dt} = \frac{1}{H}\left(\frac{da_t^1}{dt} + \frac{da_t^2}{dt} + \frac{da_t^3}{dt} + ... + \frac{da_t^H}{dt}\right).$$

Then using (2):

$$\begin{aligned}
\frac{dk_t}{dt} &= \frac{da_t}{dt} = \frac{1}{H}\sum \frac{da_t^h}{dt} \\
&= \frac{1}{H}\sum\left[w_t + (r_t - \delta - n)\, a_t^h - c_t^h\right] \\
&= w_t + (r_t - \delta - n)\,\frac{1}{H}\sum a_t^h - \frac{1}{H}\sum c_t^h \\
&= w_t + (r_t - \delta - n)\, a_t - c_t
\end{aligned}$$

Decentralized R-C-K Model with Heterogenous Households (Contd.):

- Once again, recognising that in this competitive market economy: $w_t = f(k_t) - k_t f'(k_t)$ and $\hat{r}_t = r_t - \delta = f'(k_t) - \delta$, we get dynamics of average capital stock for the aggregate economy with heterogenous households as:
$$\frac{dk_t}{dt} = f(k_t) - (\delta + n) k_t - c_t.$$
- However aggregating for the average consumption is not that easy.
- Differentiating (7) with respect to t:
$$\frac{dc_t}{dt} = \frac{1}{H}\left(\frac{dc_t^1}{dt} + \frac{dc_t^2}{dt} + \frac{dc_t^3}{dt} + ... + \frac{dc_t^H}{dt}\right).$$
- Then using (1):
$$\frac{dc_t}{dt} = \frac{1}{H}\sum\left(\frac{dc_t^h}{dt}\right) = \frac{1}{H}\sum\left(\frac{c_t^h}{\sigma(c_t^h)}\left[r_t - \delta - n - \rho\right]\right).$$

Decentralized R-C-K Model with Heterogenous Households (Contd.):

- Notice however that if $\sigma(c_t^h)$ is not a constant, then consumption of different households will grow at different rates and therefore aggregation for the entire economy becomes an issue.
- The dynamics that hold for a household may not hold for the economy-wide average.
- This aggregation problem can however be avoided **if the utility function is of CRRA variety**, so that σ is a constant.
- In this case, along the optimal path, the rate of growth of consumption for all households would be the same:

$$\frac{1}{c_t^h}\frac{dc_t^h}{dt} = \frac{1}{\sigma}\left[r_t - \delta - n - \rho\right]$$

and therefore so would be the rate of growth of average consumption:

$$\frac{1}{c_t}\frac{dc_t}{dt} = \frac{1}{\sigma}\left[r_t - \delta - n - \rho\right]$$

Decentralized R-C-K Model with Heterogenous Households (Contd.):

- Once again substituting for r_t:

$$\frac{1}{c_t}\frac{dc_t}{dt} = \frac{1}{\sigma}\left[f\left(k_t\right) - \delta - n - \rho\right]$$

- Thus the equivalence between the centralized and decentralized solution prevails despite households being heterogenous in terms of initial asset holding (provided of course their utility is of CRRA variety)..

Decentralized R-C-K Model with Heterogenous Households (Contd.):

- Finally, when households have different initial asset holding, we have just seen that their *rate of growth of consumption* would be the same.
- But how about the *level of consumption*?
- Here the inital asset holding makes difference.
- In fact, the initially rich households will always maintain a higher level of consumption than the initially poor households and the initial level difference will perpetuate in the long run.
- To see this, note that for any household h :

$$\frac{dc_t^h}{dt} = \frac{c_t^h}{\sigma}\left[r_t - \delta - n - \rho\right]$$

$$\Rightarrow \quad c_t^h = c_0^h \exp^{\int_0^t \frac{(r_v - \delta - n - \rho)}{\sigma} dv}.$$

Decentralized R-C-K Model with Heterogenous Households (Contd.):

- Now, we have see earlier (from the household's budget constraint and the NPG condition) that:

$$\int_0^\infty c_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv} dt = \int_0^\infty w_t \exp^{-\int_0^t (\hat{r}_v - n) dv} dt + a_0^h$$

$$\Rightarrow \int_0^\infty c_t^h \exp^{-\int_0^t (\hat{r}_v - n) dv} = \hat{W}_0 + a_0^h$$

- Plugging the solution for c_t^h in the RHS above:

$$\int_0^\infty c_0^h \exp^{\int_0^t \left[\frac{(r_v - \delta - n - \rho)}{\sigma} - (\hat{r}_v - n)\right] dv} = \hat{W}_0 + a_0^h$$

Decentralized R-C-K Model with Heterogenous Households (Contd.):

- Simplifying:

$$c_0^h = \frac{\hat{W}_0 + a_0^h}{R_0}, \text{ where } R_0 \equiv \int_0^\infty \exp^{\int_0^t \left[\frac{(r_v - \delta - n - \rho)}{\sigma} - (\hat{r}_v - n) \right] dv}.$$

- Notice that $\hat{W}_0$ and R_0 are the same for all households, but a_0^h are not.
- Thus a rich household will enjoy a higher level initial consumption than a relatively poor households.
- But the rate of growth of consumption for all households is the same.
- This implies that the initial consumption difference between the rich and the poor will persist in the long run.

Equivalence between a Planned Economy & the Market Economy under R-C-K Model: Implications

- The equivalence of outcomes between the socially planned economy and the competitive market economy is a very strong result.
- It implies that the actions of 'atomistic' agents acting in their individual spheres result in an outcome which is exactly identical to that of the omniscient, omnipotent social planner (or governement).
- Thus there is no logical scope for government intervention here - either in terms of influencing the long run growth rate (which it cannot affect anyway) or in terms of improving efficiency (which is superfluous, because now the market economy is already efficient)!
- Note however that **this equivalence result depends crucially on the assumption that households have perfect foresight/rational expectations.**
- Without this assumption, the equivalence result breaks down; the optimal trajectory chosen by the market economy will no longer be the socially optimal one.

R-C-K Model: References

- **Reference for the R-C-K Model (Centralized & De-centralized versions):**
    - Barro & Sala-i-Martin, Economic Growth (2nd Edition), Chapter 2
- **Reference for Dynamic Optimization Technique in Continuous Time** (for students who are interested in diggging deeper in terms of technique)**:**
    - A.C. Chiang: Elements of Dynamic Optimization, Chapters 7,8 & 9
    - M. Kamien & N. Schwartz: Dynamic Optimization, Part II, Sections 1-9.
      (The second book is more rigorous, but also more terse. Consult at your own risk!)