Multi-Tasks: General Model and Applications

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1 Basics

So far the production technology used by us allowed the Agent to perform only one task; and there was only one output y. In real world, employees at work perform multi-tasks. For example, workers produce output using firm assets as well as Maintain assets. Managers/CEO supervise existing workers/employees, train existing workers/employees, and hire new workers/employees. Salespersons promote sale with existing customers as well make new customers. Teachers are expected to teach, do research, and perform some administrative works.

The real-world output is also multi-dimensional. Workers output consists of quantity/units of output and the residual value of assets. Managers/CEO decisions affect the current profits as well as the value of stocks/shares of company. Teachers efforts affect the teaching quality and quantity and also Research output. In this module we address the following questions:

1. Why are many incentive schemes 'low powered'? Specifically, why wages that do not depend on some measure of output?
2. Why at times some verifiable signals of effort are left out of the contract?
3. Should tasks be performed through employment contract or be purchased using (market) contracts?

2 Model

The Agent performs multiple tasks. Each task requires specific effort on the part of the Agent.[1] The vector of efforts put in by the Agent is represented

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[†]This write up has not been proof-read. Please watch out for typos, and if possible bring them to my notice.

[1]Model is based on Holmstrom and Milgrom (1991, J Law Eco and Organizations)

by $e = (e_1, ..., e_n)$, where $e \in \mathcal{E} = \Re^n_+$. The (money) cost of effort function $\psi(e) = \psi(e_1, ..., e_n)$ is such that $\frac{\partial \psi(e)}{\partial e_i} > 0$, for all $i = 1, 2, ..., n$. Moreover, $\psi(e)$ strictly convex. As a result of efforts, an output vector $q = (q_1, ..., q_n)$ is produced; i.e., $q \in \Re^n_+$ and $q : \Re^n_+ \mapsto \Re^n_+$. It is standard to assume that $q = q(e, \epsilon)$, $E(\epsilon) = 0$. Besides, as a result of efforts, a vector of contractible signals, x, is also produced. In general, we let $x = (x_1, ..., x_k) \in \Re^k$. Specifically, $x : \Re^n_+ \mapsto \Re^k$ such that

$$x = \mu(e) + \epsilon.$$

Assume $\mu : \Re^n_+ \mapsto \Re^k$ is concave, and that the vector ϵ is multi-normal, i.e., $\epsilon \sim N(0, \Sigma)$, where 0 is k-vector of zeros, and Σ is $k \times k$ variance-covariance matrix. The signal can be the output n-vector $q \in \Re^n$ itself, or some other signal. In the former case, $x = q \in \Re^n_+$. Different outputs/signals can have different 'measurability', as discussed below. Assume that principal offers a linear contract:

$$w(x) = t + s^T x = t + \sum_{i=1}^{k} s_i x_i,$$

where $s_i \geq 0$, $i = 1, ..., k$. Payoffs of the parties are as defined below: Principal is risk-neutral with expected payoff $V = V(q, w)$ or simply $V = V(e, w)$. Agent is risk-averse with payoff function $u(w, e) = E\left(-e^{-r(w-\psi(e))}\right)$, $r > 0$, where $r = -\frac{u''(.)}{u'(.)} > 0$, i.e., the agent holds CARA preference.

3 A Simple Version

Suppose the Agent performs only two tasks; $i = 1, 2$. As a result, two signals/outputs are produced: $q = x = \mu(e) + \epsilon = e + \epsilon$, where $q, e, \epsilon \in \Re^2$. Specifically, $q_i(e_i, \epsilon_i) = e_i + \epsilon_i$, where

$$\begin{aligned} q_1(e_1, \epsilon_1) &= e_1 + \epsilon_1 \\ q_2(e_2, \epsilon_2) &= e_2 + \epsilon_2, \end{aligned}$$

$\epsilon = (\epsilon_1, \epsilon_2) \sim N(0, \Sigma)$, where Σ is 2×2 variance-covariance matrix; $\Sigma = \begin{pmatrix} \sigma_1^2 & R \\ R & \sigma_2^2, \end{pmatrix}$ and $R \in \Re$. As before, let the contract be linear: $w(x) = t + s_1 q_1 + s_2 q_2$, where $s_i \geq 0$. Also, let Principal's payoff function be: $V(q_1, q_2, w) = E(q_1 + q_2 - w) = e_1 + e_2 - E(w)$. Let $\psi(e) = \frac{1}{2} c_1 e_1^2 + \frac{1}{2} c_2 e_2^2 + \delta e_1 e_2$. Clearly, $\frac{\partial \psi(e_1, e_2)}{\partial e_1} = c_1 e_1 + \delta e_2$ and $\frac{\partial \psi(e_1, e_2)}{\partial e_2} = c_2 e_2 + \delta e_1$. Note the following:

$$\begin{cases} \delta = 0 & \text{tasks are independent;} \\ \delta > 0 & \text{tasks are technological substitutes;} \\ \delta < 0 & \text{tasks are technological complements.} \end{cases}$$

Tasks are perfect substitutes if $\delta = \sqrt{c_1 c_2}$; imperfect substitutes if $0 < \delta < \sqrt{c_1 c_2}$. Moreover, $E(w(x)) = E(t + s_1(e_1 + \epsilon_1) + s_2(e_2 + \epsilon_2)) = t + s_1 e_1 + s_2 e_2$. And, $Var(t + s_1(e_1 + \epsilon_1) + s_2(e_2 + \epsilon_2)) = s_1^2 \sigma_1^2 + s_2^2 \sigma_2^2 + 2R s_1 s_2$.

3.1 First Best:

The first best is solution to

$$\max_{e_i,t,s_i} E\left(\sum q_i - w\right)$$

s.t. $-e^{-r[w-\psi(e_1,e_2)]} = -e^{-r\bar{w}}$, i.e., $w-\psi(e_1,e_2)=\bar{w}$, i.e., $w=\bar{w}+\psi(e_1,e_2)$, where $\bar{w}$ denotes the *Certainty equivalent* of the reservation (outside) wage. Therefore, the first best is solution to $\max_{e_1,e_2} E(q_1+q_2-\bar{w}-\psi(e_1,e_2)), i.e.,$ $\max_{e_1,e_2} E(e_1+\epsilon_1+e_2+\epsilon_2-\bar{w}-\psi(e_1,e_2)), i.e.,$

$$\max_{e_1,e_2}\left\{e_1+e_2-\left[\frac{1}{2}c_1e_1^2+\frac{1}{2}c_2e_2^2+\delta e_1e_2\right]\right\}$$

Therefore, the first best efforts, e_1^* and e_2^*, solve the following FOCs:

$$\begin{aligned} e_1 &: \quad c_1e_1+\delta e_2-1=0 \quad (1)\\ e_2 &: \quad c_2e_2+\delta e_1-1=0. \quad (2)\end{aligned}$$

3.2 Second Best:

Under the second-best, e is not contractible but q is. Note that in this subsection $x=q$ by assumption. As before, for given $w(s,t)$ offered by the principal, the Agent solves $\max_{e_1,e_2}\{\hat{w}(e_1,e_2)\}$, where

$$\underbrace{\hat{w}(e_1,e_2)}_{certainty-equivalent\ wage} = \underbrace{E[w(e_1,e_2)]}_{expected\ wage}-\underbrace{\psi(e_1,e_2)}_{effort\ cost}-\underbrace{\frac{r}{2}Var[w(e_1,e_2)]}_{risk-premium}, i.e.,$$

So the Agent solves:

$$\max_{e_1,e_2}\left\{\underbrace{t+s_1e_1+s_2e_2}_{expected\ wage}-\underbrace{\left[\frac{1}{2}c_1e_1^2+\frac{1}{2}c_2e_2^2+\delta e_1e_2\right]}_{effort\ cost}-\frac{r}{2}\left[\underbrace{s_1^2\sigma_1^2+s_2^2\sigma_2^2+2Rs_1s_2}_{risk-premium}\right]\right\}$$

So, given $w(.)$ opted by the Principal, the Agent will choose e_1 and e_2 that satisfy the following FOCs:

$$\begin{aligned} s_1 &- c_1e_1-\delta e_2=0 \quad (3)\\ s_2 &- c_2e_2-\delta e_1=0. \quad (4)\end{aligned}$$

That is, the effort vector chosen by the Agent is solution to

$$s-\nabla\psi(e)=0,$$

where $s = (s_1, s_2)^T$ and $\nabla\psi(e) = (\psi_1(e), \psi(e)_2)^T$ is the gradient vector of $\psi(e)$. The IR is given by $u(\hat{w}(e_1, e_2)) \geq u(\bar{w}), i.e., \quad \hat{w}(e_1, e_2) \geq \bar{w}, i.e.,$

$$t + s_1e_1 + s_2e_2 - \left[\frac{1}{2}c_1e_1^2 + \frac{1}{2}c_2e_2^2 + \delta e_1e_2\right] - \frac{r}{2}\left[s_1^2\sigma_1^2 + s_2^2\sigma_2^2 + 2Rs_1s_2\right] \geq \bar{w} \quad (5)$$

So, the Principal will solve: $\max_{e_1,e_2,t,s_1,s_2} E[q_1 + q_2 - w(q_1, q_2)] = E[q_1 + q_2 - (t + s_1q_1 + s_2q_2)], i.e.,$

$$\max_{e_1,e_2,t,s_1,s_2} E[e_1 + (1 - s_1)\epsilon_1 + e_2 + (1 - s_2)\epsilon_2 - (t + s_1e_1 + s_2e_2)]$$

s.t. (3) – (5) hold. Clearly, (5) will bind. Therefore, the Principal's problem can be written as:

$$\max_{e_1,e_2,s_1,s_2} \left\{e_1 + e_2 - \left[\frac{1}{2}c_1e_1^2 + \frac{1}{2}c_2e_2^2 + \delta e_1e_2\right] - \frac{r}{2}\left[s_1^2\sigma_1^2 + s_2^2\sigma_2^2 + 2Rs_1s_2\right]\right\}$$

s.t. (3) and (4) hold. More generally, the Principal's programme is:

$$\max_{e} \left\{V(e) - \psi(e) - \frac{r}{2}s^T\Sigma s\right\}$$

s.t. $e = \arg\max\{s^T\mu(e) - \psi(e)\}$

3.2.1 $R = 0$:

Using (3) and (4), the FOC w.r.t. e_1 is

$$1 - [c_1e_1 + \delta e_2] - r[c_1s_1\sigma_1^2 + s_2\sigma_2^2\delta] = 0.$$

From (3), we know $s_1 = c_1e_1 + \delta e_2$ and $s_2 = c_2e_2 + \delta e_1$. Therefore, we can re-write the FOC:

$$s_1 = \frac{1 - r\sigma_2^2\delta s_2}{1 + r\sigma_1^2c_1} \quad (6)$$

By symmetry FOC w.r.t. e_2 gives

$$s_2 = \frac{1 - r\sigma_1^2\delta s_1}{1 + r\sigma_2^2c_2}, i.e., \quad (7)$$

in view of (6), we get $s_2 = \frac{1 - r\sigma_1^2\delta\frac{1 - r\sigma_2^2\delta s_2}{1 + r\sigma_1^2c_1}}{1 + r\sigma_2^2c_2}, i.e.,$

$$s_2^{SB} = \frac{1 + r\sigma_1^2(c_1 - \delta)}{(1 + r\sigma_1^2c_1)(1 + r\sigma_2^2c_2) - \delta^2\sigma_1^2\sigma_2^2r^2} \quad (8)$$

Similarly,

$$s_1^{SB} = \frac{1 + r\sigma_2^2(c_2 - \delta)}{(1 + r\sigma_1^2c_1)(1 + r\sigma_2^2c_2) - \delta^2\sigma_1^2\sigma_2^2r^2} \quad (9)$$

From (8) and (9), it can be checked that $\frac{\partial s_i}{\partial \sigma_i} < 0$ and $\frac{\partial s_i}{\partial \sigma_j} < 0$. That is, if the measurability (preciseness) of the first task comes down the Principal reduces incentives on both the tasks. Similarly, if the measurability (preciseness) of the Second task comes down. Moreover, $\sigma_2^2 \Rightarrow \infty$ implies

$$\begin{aligned} s_2 &\Rightarrow 0 \\ s_1 &\Rightarrow \frac{r(c_2 - \delta)}{(1 + r\sigma_1^2 c_1) r c_2 - \delta^2 \sigma_1^2 r^2} \end{aligned}$$

That is, Principal induces Agent to specialize on the first task. When $\delta = 0$, from (8) and (9) we get

$$s_i = \frac{1}{1 + r\sigma_i^2 c_i} = \frac{1}{1 + r\sigma_i^2 \psi_{ii}}$$

Remark: From (3) and (4) note: if $\delta = 0$, $\frac{de_1}{ds_1} = \frac{1}{c_1} = \frac{1}{\psi_{11}} > 0$ and $\frac{de_2}{ds_2} = \frac{1}{c_2} = \frac{1}{\psi_{22}} > 0$. Also, from (6) and (7), if $\delta > 0$, $s_1(\delta) < s_1(0)$ & $s_2(\delta) < s_2(0)$; and if $\delta < 0$, $s_1(\delta) > s_1(0)$ & $s_2(\delta) > s_2(0)$. Therefore, 'power' of the incentives is inversely proportional to δ.

3.2.2 $R \neq 0$:

For simplicity assume $\delta = 0$, $\sigma_1^2 = \sigma_2^2 = \sigma^2$, $c_1 = c_2 = c = 1$. Now, ICs are

$$s_i = e_i = \frac{\psi(e|\delta = 0)}{\partial e_i}, \quad i = 1, ..., n.$$

So, Principal solves:

$$\max_{e_1, e_2} \left\{ e_1 + e_2 - \left[\frac{1}{2} e_1^2 + \frac{1}{2} e_2^2 + \delta e_1 e_2 \right] - \frac{r}{2} \left[e_1^2 \sigma_1^2 + e_2^2 \sigma_2^2 + 2R e_1 e_2 \right] \right\}$$

FOCs are

$$\begin{aligned} 1 - rRe_2 - e_1 - r\sigma^2 e_1 &= 0 \\ 1 - rRe_1 - e_2 - r\sigma^2 e_2 &= 0 \end{aligned}$$

So,

$$e_1 = e_2 = e^{SB} = \frac{1}{1 + r\sigma^2 + rR}$$

Clearly, $\frac{\partial e_i^{SB}}{\partial R} = \frac{\partial s_i^{SB}}{\partial R} < 0$. That is, if the error terms are positively correlated, i.e., $R > 0$, compared to the case when $R = 0$, the Principal will reduce the power of the incentive. In contrast, if $R < 0$, the Principal will increase the power of the incentive.

Exercise 1 *Provide an intuitive explanation for the result* $\frac{\partial e_i^{SB}}{\partial R} = \frac{\partial s_i^{SB}}{\partial R} < 0$.

4 General Model

Recall, $x : \Re^n_+ \mapsto \Re^k_+$ such that: $x = \mu(e) + \epsilon$, where $\mu : \Re^n_+ \mapsto \Re^k$ is concave, $\epsilon \sim N(0, \Sigma)$ and $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_{kk} \end{pmatrix}$, where $\sigma_{jj} = \sigma_j^2$, for all $j = 1, 2..., k$.
In the FB, the Principal solves:

$$\max_{e \in \Re^n_+} \{V(e) - \psi(e)\}.$$

Assume that the above programme is concave. In the SB, suppose contract is linear: $w = s^T \mu(e) + t$. Now, the certainty equivalent wage for Agent is:

$$\hat{w}(s^T, t, e) \;=\; E\left(-e^{-r(w(s^T,t)-\psi(e))}\right) = \underbrace{s^T \mu(e) + t}_{expecedwage} - \underbrace{\psi(e)}_{effortcost} - \underbrace{\frac{r}{2} s^T \Sigma s}_{riskpremium}.$$

As before, for given $s \in \Re^k$ opted by the Principal, the Agent will choose $e \in \Re^n_+$ to maximize $\hat{w}(s^T, t, e)$. Assume $\mu(e) = e$. So, the FOCs for the Agent's programme are given by:

$$\begin{aligned} (\forall i = 1, ..., n)[s_i - \psi_i(e) &= 0], i.e., \\ s(e) - \nabla \psi(e) &= 0. \end{aligned} \tag{10}$$

(10) further gives us $\nabla s(e) = [\psi_{ij}]$, where $[\psi_{ij}]$ is the $n \times n$ matrix of second derivatives of $\psi(e)$. That is, $[\psi_{ij}] = \begin{pmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{nn} \end{pmatrix}$. Use of the inverse function theorem gives us $\nabla e(s) = [\psi_{ij}]^{-1}$. Now, the Principal programme is:

$$\max_{e \in \Re^n_+} \left\{V(e) - s^T \mu(e) - t\right\}$$

s.t.

$$\begin{aligned} IC: &\quad e = \arg\max \left\{s^T \mu(e) - \psi(e)\right\} \\ IR: &\quad s^T \mu(e) + t - \psi(e) - \frac{r}{2} s^T \Sigma s \geq 0. \end{aligned}$$

IR will bind. So, P's programme can be written as can be written as

$$\max_{e \in \Re^n_+} \left\{V(e) - \psi(e) - \frac{r}{2} s^T(e) \Sigma s(e)\right\}$$

s.t. IC, i.e., $e = \arg\max\{s^T\mu(e) - \psi(e)\}$. In view the fact that $s(e) = \nabla\psi(e)$, i.e., $s_i = \psi_i(e)$, the FOCs for P's programme w.r.t. e are given by:

$$(\forall i = 1, .., n)\left[\frac{\partial V(e)}{\partial e_i} = s_i(e) + r\sum_{k=1}^{n}\sum_{j=1}^{n} s_j(e)\sigma_{jk}\psi_{ki}(e)\right]. \tag{11}$$

In vector form, the FOCs are given by:

$$\nabla V(e) = [I + r[\psi_{ij}]\Sigma]s(e) \tag{12}$$

where I is the $n \times n$ identity matrix. (12) gives us

$$s(e) = [I + r[\psi_{ij}]\Sigma]^{-1}\nabla V(e) \tag{13}$$

Note that when Σ and $[\psi_{ij}]$ are both diagonal, from (11), we get

$$(\forall i = 1, ..., n)\left[s_i = \frac{\frac{\partial V(e)}{\partial e_i}}{1 + r\sigma_i^2\psi_{ii}}\right]$$

If we assume that $\psi^T(e)\Sigma\psi(e)$ is convex, the unique solution is identified by the FOCs.

5 Applications

5.1 Dependent Tasks: Conclusions

When tasks are interdependent and the worker is risk averse: The owner will

- use incentive contract for the measurable tasks.
- however, will use low-powered incentive contracts
- due to multi-tasking, the incentive pay encourages substitution among tasks
- desirability of high-power incentive contracts for measurable tasks reduces as the measurably of some other tasks reduces
- low-powered incentive contracts
    - reduce the undesirable substitution among tasks, where the employee focuses only on the tasks that are awarded and ignore other tasks
    - reduce the undesirable consequences of measurable talks.

The measurability of tasks is an important determinant of integration of tasks

- an employee is allowed to engage in 'outside' activities only if the 'inside' tasks are measurable.
- when 'inside' tasks are measurable, the worker can be induced to work for firm by using incentives on the output produced.
- when 'inside' tasks are NOT measurable, the worker can be induced to work for firm by ruling out the possibility of working for someone else.
- That is, when 'inside' tasks are NOT measurable, the worker will be employed as and employee of the firm rather than working independently.
- So, non-measurability of outputs increases the 'size' of the firm, (in terms of number of employees).

5.2 Low incentives within firms

Nobel prize winning economist Williamson has argued that firms use low-powered employment contracts, rather than the high powered contracts predicted by the theory of moral hazard.[2] Here is a formal explanation. Assume that the Agent can use some assets to perform two tasks; $t = 1, 2$. As a result, two outputs are produced; q_1 and q_2. Besides, two signals are generated: $x_i(t_i, \epsilon_i) = \mu(t_i) + \epsilon_i = t_i + \epsilon_i$, $i = 1, 2$, $\epsilon \sim N(0, \Sigma)$, where Σ is variance-covariance matrix; $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$. q_2 is enjoyed by the owner of the assets and cannot be contracted away. So if the Principal is the owner of assets then his payoff is $V(q_1, q_2, w) = v_1(q_1) + v_2(q_2) - w$. In contrast, if A is the owner of assets, the P's payoff is $V(q_1, q_2, w) = v_1(q_1) - w$, where $v_i' > 0$, etc. Let, $\psi(t)$ be the cost of effort function. Assume, only q_1 is measurable and contract is linear. So, $w = s_1 q_1 + t$. Let

$$\begin{aligned} \pi^1 &= \max_{t_1}\{v_1(t_1) - \psi(t_1)\} \\ \pi^2 &= \max_{t_2}\{v_2(t_2) - \psi(t_2)\} \\ \pi^{12} &= \max_{t_1, t_2}\{v_1(t_1) + v_2(t_2) - \psi(t)\} \end{aligned}$$

Proposition 1 *Suppose, P own the assets, $\psi(.) = \psi(t_1 + t_2)$ and the Agent's choice meets $t_1 + t_2 = \bar{t}$. Under this conditions, if $\pi^{12} \geq \max\{\pi^1, \pi^2\}$, then $s_1 = 0$.*

Note $s_1 > 0 \Rightarrow t_2 = 0$ and t_1 will solve $t_1 = \bar{t}$. Moreover, the P's profit will be $v_1(\bar{t}) - \psi(\bar{t}) - \frac{r}{2} s_1^2 \sigma_1^2$. However,

$$v_1(\bar{t}) - \psi(\bar{t}) - \frac{r}{2} s_1^2 \sigma_1^2 \quad < \quad \pi^1 \leq \pi^{12}.$$

[2] See for example Williamson (1985).

In contrast, if $s_1 = 0$, the Agent will be indifferent among various possible combinations of t_1 and t_2, so will choose the one that gives π^{12}.

Alternatively, suppose P own the assets, $\psi(.) = \psi(t_1 + t_2)$ and the Agent's choice meets the following condition; for some $\bar{t} > 0$, $\psi'(\bar{t}) = 0$, and $\psi''(t) > 0$. Under this conditions, the Agent will choose $t_1 + t_2 = \bar{t}$ and does not mind the exact proportion. For such a context, the following holds:

Proposition 2 *If $V'() > 0$ and $V(t_1, 0) = 0$, i.e., the second task is essential, then the optimum $s_1 = 0$.*

Next, suppose we do not insist on condition $t_1 + t_2 = \bar{t}$; suppose the Agent can choose $t_i \in \Re_+$. Again $s_1 > 0 \Rightarrow t_2 = 0$ and t_1 will solve $\psi'(t_1) = s_1$. Moreover, the P's profit will be $v_1(t_1(s_1)) - \psi(t_1(s_1)) - \frac{r}{2} s_1^2 \sigma_1^2$.

Exercise 2 *Suppose the benefit of t_2 is enjoyed by the Agent (alternatively, suppose the Agent owns the assets). Find out the conditions under which optimum $s_1 > 0$.*