# Multi-tasks: General Model

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# Multiple Tasks I

So far, we modeled production wherein

- Agent performed only one task;
- There was only one output $q$.

In real world,

- employees at work perform multi-tasks
- produce several outputs

For example,

- Workers
    - Produce output (using firm's assets)
    - Maintain assets

# Multiple Tasks II

- Managers/CEO
  - Supervise existing workers/employees
  - Train existing workers/employees
  - Hire new workers/employees
- Salespersons
  - Promote sale with existing customers
  - Make new customers
  - Launch sale of new products
- Teachers
  - Teach
  - Research
  - Serve on administrative committees

# Multiple Tasks III

The output is also multi-dimensional

- Workers output
  - Quantity/units of output
  - Residual value of assets
- Managers/CEO
  - Current profits
  - Value of stocks/shares of company
- Teachers
  - Teaching quality and quantity
  - Research output

# Model I

Holmstrom and Milgrom (1991, J Law Eco and Organizations)

- Multiple tasks; $e$ is multi-dimensional, i.e., $e = (e_1, ..., e_n) \in \mathcal{E} = \mathfrak{R}_+^n$
- The (money) cost of effort function: $\psi(e) = \psi(e_1, ..., e_n)$ is strictly convex.
- As a result of efforts, an output vector $q$ is produced; it standard to assume that $q = q(e, \epsilon')$, $E(\epsilon') = 0$ and

$$q = (q_1, ..., q_n) \in \mathfrak{R}_+^n, \; i.e., \; q : \mathfrak{R}_+^n \mapsto \mathfrak{R}_+^n.$$

- As a result of efforts, a vector of contractible signals $x$ is also produced; i.e., $x = q \in \mathfrak{R}_+^n$. In general, let $x = (x_1, ..., x_k) \in \mathfrak{R}_+^k$ such that

$$x = \mu(e) + \epsilon$$

  $\mu : \mathfrak{R}_+^n \mapsto \mathfrak{R}^k$ is concave, and $\epsilon \sim N(0, \Sigma)$, where 0 is k-vector of zeros, and $\Sigma$ is variance-covariance matrix.
- Different outputs/signals have different 'measurability'

# Model II

Payoffs:

- Contract: $w(x) = t + s^T x = t + \sum_{i=1}^{k} s_i x_i$, where $s_i \geq 0$
- Principal is risk-neutral with expected payoff $V = V(q, w)$, i.e., $V = V(e, w)$
- Agent is risk-avers: $u(w, e) = -e^{-r(w-\psi(e))}$, $r > 0$, where
- $r = -\frac{u''}{u'} > 0$, i.e., CARA, and

# Model III

A simple version:

- Two tasks; $i = 1, 2$
- Two signals/outputs: $q_i(e_i, \epsilon_i) = e_i + \epsilon_i$, $i = 1, 2$. Specifically, $q_i(e_i, \epsilon_i) = e_i + \epsilon_i$, where

$$\begin{aligned} q_1(e_1, \epsilon_1) &= e_1 + \epsilon_1 \\ q_2(e_2, \epsilon_2) &= e_2 + \epsilon_2, \end{aligned}$$

  $\epsilon = (\epsilon_1, \epsilon_2) \sim N(0, \Sigma)$, where $\Sigma$

- $\epsilon \sim N(0, \Sigma)$, where $\Sigma$ is variance-covariance matrix;

$$\Sigma = \begin{pmatrix} \sigma_1^2 & R \\ R & \sigma_2^2 \end{pmatrix}$$

- Principal's payoff: $V(q_1, q_2, w) = E(q_1 + q_2 - w) = e_1 + e_2 - E(w)$

# Model IV

- $\psi(e) = \frac{1}{2}c_1 e_1^2 + \frac{1}{2}c_2 e_2^2 + \delta e_1 e_2$
- $\psi_1(.) = \frac{\partial \psi(e_1,e_2)}{\partial e_1} = c_1 e_1 + \delta e_2$ and $\psi_2(.) = \frac{\partial \psi(e_1,e_2)}{\partial e_2} = c_2 e_2 + \delta e_1$. So
- $\begin{cases} \delta = 0 & \text{tasks are independent;} \\ \delta > 0 & \text{tasks are technological substitutes;} \\ \delta < 0 & \text{tasks are technological complements.} \end{cases}$
- Tasks are perfect substitutes if $\delta = \sqrt{c_1 c_2}$; imperfect substitutes if $0 < \delta < \sqrt{c_1 c_2}$
- Contract: $w(x) = t + s_1 q_1 + s_2 q_2$, where $s_i \geq 0$. Note
- $$\begin{aligned} E(w(x)) &= E(t + s_1(e_1 + \epsilon_1) + s_2(e_2 + \epsilon_2)) \\ &= t + s_1 e_1 + s_2 e_2. \end{aligned}$$
- $Var(t + s_1(e_1 + \epsilon_1) + s_2(e_2 + \epsilon_2)) = s_1^2\sigma_1^2 + s_2^2\sigma_2^2 + 2Rs_1 s_2$
- $\bar{w}$ = Certainty equivalent of the reservation (outside) wage

# First Best

The first best is solution to

$$\max_{e_i,t,s_i} E(\sum q_i - w)$$

s.t. $-e^{-r[w-\psi(e_1,e_2)]} = -e^{-r\bar{w}}$, i.e., $w - \psi(e_1, e_2) = \bar{w}$, i.e.,

$$w = \bar{w} + \psi(e_1, e_2).$$

Therefore, the first best is solution to

$$\max_{e_1,e_2} E(e_1 + \epsilon_1 + e_2 + \epsilon_2 - \bar{w} - \psi(e_1, e_2)), i.e.,$$

$$\max_{e_1,e_2}\{e_1 + e_2 - [\frac{1}{2}c_1 e_1^2 + \frac{1}{2}c_2 e_2^2 + \delta e_1 e_2]\}$$

Therefore, the first best efforts, $e_1^*$ and $e_2^*$, solve the following foc

$$\psi_1(e) = c_1 e_1 + \delta e_2 = 1 \quad (0.1)$$

$$\psi_2(e) = c_2 e_2 + \delta e_1 = 1. \quad (0.2)$$

# Second Best I

$e$ is not contractible but $q$ is. As before, the agent solves

$$\max_{e_1, e_2}\{\hat{w}(e_1, e_2)\},$$

where

$$\underbrace{\hat{w}(e_1, e_2)}_{certainty-equivalent\ \ wage} = \underbrace{E[w(e_1, e_2)]}_{expected\ \ wage} - \underbrace{\psi(e_1, e_2)}_{effort\ \ cost} - \underbrace{\frac{r}{2} Var[w(e_1, e_2)]}_{risk-premium}, i.e.,$$

$$\begin{aligned}
\underbrace{\hat{w}(e_1, e_2)}_{certainty-equivalent\ \ wage} &= \underbrace{t + s_1 e_1 + s_2 e_2}_{expected\ \ wage} \\
&- \underbrace{[\frac{1}{2} c_1 e_1^2 + \frac{1}{2} c_2 e_2^2 + \delta e_1 e_2]}_{effort\ \ cost} \\
&- \frac{r}{2}\underbrace{[s_1^2 \sigma_1^2 + s_2^2 \sigma_2^2 + 2R s_1 s_2]}_{risk-premium}
\end{aligned}$$

# Second Best II

The foc w.r.t. $e_1$ and $e_2$ are

$$s_1 = c_1 e_1 + \delta e_2 \tag{0.3}$$
$$s_2 = c_2 e_2 + \delta e_1 \tag{0.4}$$

That is,

$$s(e) = \nabla \psi(e).$$

IR is given by

$$u(\hat{w}(e_1, e_2)) \geq u(\bar{w}), i.e., \quad \hat{w}(e_1, e_2) \geq \bar{w}, i.e.,$$

$$t + s_1 e_1 + s_2 e_2 - [\frac{1}{2} c_1 e_1^2 + \frac{1}{2} c_2 e_2^2 + \delta e_1 e_2] - \frac{r}{2}[s_1^2 \sigma_1^2 + s_2^2 \sigma_2^2 + 2R s_1 s_2] \geq \bar{w} \tag{0.5}$$

The principal solves $\max_{e_1, e_2, t, s_1, s_2} E[q_1 + q_2 - w(q_1, q_2)]$, i.e.,

$$\max_{e_1, e_2, t, s_1, s_2} E[q_1 + q_2 - (t + s_1 q_1 + s_2 q_2)], i.e.,$$

# Second Best III

$$\max_{e_1,e_2,t,s_1,s_2} E[e_1 + (1-s_1)\epsilon_1 + e_2 + (1-s_2)\epsilon_2 - (t + s_1 e_1 + s_2 e_2)]$$

s.t. (0.3) – (0.5) hold. Clearly, (0.5) will bind. Therefore, the Principal's problem can be written as

$$\max_{e_1,e_2,s_1,s_2} \{e_1 + e_2 - [\frac{1}{2}c_1 e_1^2 + \frac{1}{2}c_2 e_2^2 + \delta e_1 e_2] - \frac{r}{2}[s_1^2\sigma_1^2 + s_2^2\sigma_2^2 + 2Rs_1 s_2]\}$$

s.t. (0.3) and (0.4) hold.

Note that the Principal programme can be written as

$$\max_{e}\{V(e) - \psi(e) - \frac{r}{2}s^T\Sigma s\}$$

s.t. $e = \arg\max\{s^T\mu(e) - \psi(e)\}$

# Second Best IV

**Special Case 1: R=0**

Using (0.3) and (0.4), the foc w.r.t. $e_1$ is

$$1-\underbrace{[c_1e_1+\delta e_2]}_{=s_1 \ \textit{from}\ (0.3)}-r[c_1s_1\sigma_1^2+s_2\sigma_2^2\delta]=0, i.e.,$$

$$s_1=\frac{1-r\sigma_2^2\delta s_2}{1+r\sigma_1^2c_1} \tag{0.6}$$

By symmetry foc w.r.t. $e_2$ gives

$$s_2=\frac{1-r\sigma_1^2\delta s_1}{1+r\sigma_2^2c_2}, i.e., \tag{0.7}$$

in view of (0.6)

$$s_2=\frac{1-r\sigma_1^2\delta\frac{1-r\sigma_2^2\delta s_2}{1+r\sigma_1^2c_1}}{1+r\sigma_2^2c_2}, i.e.,$$

# Second Best V

$$s_2^{SB} = \frac{1 + r\sigma_1^2(c_1 - \delta)}{(1 + r\sigma_1^2 c_1)(1 + r\sigma_2^2 c_2) - \delta^2\sigma_1^2\sigma_2^2 r^2} \quad (0.8)$$

Similarly,

$$s_1^{SB} = \frac{1 + r\sigma_2^2(c_2 - \delta)}{(1 + r\sigma_1^2 c_1)(1 + r\sigma_2^2 c_2) - \delta^2\sigma_1^2\sigma_2^2 r^2} \quad (0.9)$$

From (0.8) and (0.9), it can be checked that $\frac{\partial s_i}{\partial \sigma_i} < 0$ and $\frac{\partial s_i}{\partial \sigma_j} < 0$.

Moreover, $\sigma_2^2 \Rightarrow \infty$ implies

$$\begin{aligned} s_2 &\Rightarrow 0 \\ s_1 &\Rightarrow \frac{r(c_2 - \delta)}{(1 + r\sigma_1^2 c_1)rc_2 - \delta^2\sigma_1^2 r^2} \end{aligned}$$

# Second Best VI

**Subcase :** $\delta = 0$: In this subcase, from (0.8) and (0.9)

$$s_i = \frac{1}{1 + r\sigma_i^2 c_i} = \frac{1}{1 + r\sigma_i^2 \psi_{ii}}$$

### Remark

- From (0.3) and (0.4) note: if $\delta = 0$, $\frac{de_1}{ds_1} = \frac{1}{c_1} = \frac{1}{\psi_{11}} > 0$ and $\frac{de_2}{ds_2} = \frac{1}{c_2} = \frac{1}{\psi_{22}} > 0$.
- From (0.6) and (0.7), if $\delta > 0$,

  $$s_1(\delta) < s_1(0) \ \& \ s_2(\delta) < s_2(0);$$

  and if $\delta < 0$,

  $$s_1(\delta) > s_1(0) \ \& \ s_2(\delta) > s_2(0).$$

- Therefore, ‘power’ of the incentives is inversely proportional to $\delta$.

# Second Best VII

**Special Case 2:** $R \neq 0$**:**

For simplicity assume $\delta = 0$, $\sigma_1^2 = \sigma_2^2 = \sigma^2$, $c_1 = c_2 = c = 1$:
Now, ICs are

$$s_i = e_i = \psi_i(e|\delta = 0).$$

So, Principal solves

$$\max_{e_1, e_2}\{e_1 + e_2 - [\frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \delta e_1 e_2] - \frac{r}{2}[s_1^2\sigma_1^2 + s_2^2\sigma_2^2 + 2Re_1 e_2]\}$$

foc are

$$\begin{aligned} 1 - rRe_2 - e_1 - r\sigma^2 e_1 &= 0 \\ 1 - rRe_1 - e_2 - r\sigma^2 e_2 &= 0 \end{aligned}$$

# Second Best VIII

So,

$$e_1 = e_2 = e^{SB} = \frac{1}{1 + r\sigma^2 + rR}$$

Clearly,

$$\frac{\partial e_i^{SB}}{\partial R} = \frac{\partial s_i^{SB}}{\partial R} < 0.$$

That is,

- If $R > 0$, compared to the case when $R = 0$, the principal will reduce the power of the incentive.
- If $R < 0$, the principal will increase the power of the incentive.

# General Model I

In the FB, the Principal solves

$$\max_e \{ V(e) - \psi(e) \}$$

In the SB, let $w = s^T \mu(e) + t$. Now, the certainty equivalent wage for agent is

$$CE = s^T \mu(e) + t - \psi(e) - \frac{r}{2} s^T \Sigma s$$

assume $\mu(e) = e$. So, the focs for the Agent's programme are given by:

$$\begin{aligned} (\forall i = 1, ..., n)[s_i &= \psi_i(e)], i.e., \\ s(e) &= \nabla \psi(e). \end{aligned} \tag{0.10}$$

(0.10) further gives us $\nabla s(e) = [\psi_{ij}]$. The inverse function theorem gives us

$$\nabla e(s) = [\psi_{ij}]^{-1}.$$

# General Model II

The Principal programme is:

$$\max_{e} \left\{ V(e) - s^T \mu(e) - t \right\}$$

s.t.

$$\begin{aligned} IC: &\quad e = \arg\max\{s^T \mu(e) - \psi(e)\} \\ IR: &\quad s^T \mu(e) + t - \psi(e) - \frac{r}{2} s^T \Sigma s \geq 0. \end{aligned}$$

IR will bind. Now, P's programme can be written as can be written as

$$\max_{e}\{V(e) - \psi(e) - \frac{r}{2} s^T(e) \Sigma s(e)\}$$

s.t. $e = \arg\max\{s^T \mu(e) - \psi(e)\}$

# General Model III

The P's programme is

$$\max_{e}\{V(e) - \psi(e) - \frac{r}{2}s^T(e)\Sigma s(e)\}$$

In view the fact that $s(e) = \nabla\psi(e)$, i.e., $s_i = \psi_i(e)$, the foc's for P's programme w.r.t. $e$ are given by

$$\nabla V(e) = [I + r[\psi_{ij}]\Sigma]s$$

which gives us

$$s(e) = [I + r[\psi_{ij}]\Sigma]^{-1}\nabla V(e)$$

# Dependent Tasks: Conclusions I

When tasks are interdependent and the worker is risk averse: The owner will

- use incentive contract for the measurable tasks.
- however, will use low-powered incentive contracts
- due to multi-tasking, the incentive pay encourages substitution among tasks
- desirability of high-power incentive contracts for measurable tasks reduces as the measurably of some other tasks reduces

The measurability of tasks is an important determinant of integration of tasks

- an employee is allowed to engage in ‘outside’ activities only if the ‘inside’ tasks are measurable.
- when ‘inside’ tasks are NOT measurable, the worker will be employed as and employee of the firm rather than working independently.
- So, non-measurability of outputs increases the ‘size’ of the firm, (in terms of number of employees).

# Application: Low incentives within firms I

Assume

- Two tasks; $i = 1, 2$
- Two signals/outputs: $q_i(t_i, \epsilon_i) = t_i + \epsilon_i$, where $i = 1, 2$.
- $\epsilon \sim N(0, \Sigma)$, where $\Sigma$ is variance-covariance matrix;

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

- $q_2$ is enjoyed by the owner of the assets and cannot be contracted away
- Principal's payoff: If P is the owner of assets then $V(q_1, q_2, w) = v_1(t_1) + v_2(t_2) - w$; If A is the owner of assets; $V(q_1, q_2, w) = v_1(t_1) - w$; where $v_i' > 0$, etc
- $\psi = \psi(\bar{t}) = \psi(t_1 + t_2)$, where $\bar{t} = t_1 + t_2$
- only $q_1$ is measurable.

- So, $w = s_1 q_1 + t$

  Let

$$\pi^1 = \max_{t_1}\{v_1(t_1) - \psi(t_1)\} \tag{0.11}$$

$$\pi^2 = \max_{t_2}\{v_2(t_2) - \psi(t_2)\} \tag{0.12}$$

$$\pi^{12} = \max_{t_1, t_2}\{v_1(t_1) + v_2(t_2) - \psi(\bar{t})\} \tag{0.13}$$

# Application: Low incentives within firms III

**Proposition**

*Suppose, P owns the assets, the agent's choice has to meet* $t_1 + t_2 = \bar{t}$ *and* $\pi^{12} \geq \max\{\pi^1, \pi^2\}$*, then* $s_1 = 0$*.*

Note $s_1 > 0 \Rightarrow t_2 = 0$ and $t_1$ will solve $t_1 = \bar{t}$. Moreover, the P's profit will be $v_1(\bar{t}) - \psi(\bar{t}) - \frac{r}{2}s_1^2\sigma_1^2$. But,

$$v_1(\bar{t}) - \psi(\bar{t}) - \frac{r}{2}s_1^2\sigma_1^2 \quad < \quad \pi^1 \leq \pi^{12}.$$