Relative Performance Evaluation

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Relative Performance Evaluations are widely used when individual performances can be observed:

- At School: In grading, ranking etc.
- At Work: In promotions, hiring and firing, etc.
- In Sports: In declaring winner, runners-up etc.

Question

- *Is relative performance evaluation efficient?*
- *Should the wage/reward be based only on the absolute value of the output, or also on the relative ranking of performances?*

Consider:

- One Principal and Two agents and Two outputs
Model II

- Two agents produce two (possibly different) individually observable outputs.
- Principal is Risk-neutral but agents are Risk-averse with CARA preferences.

The production technology: \( Q = q_1 + q_2 \), where

\[
q_1(e_1, \epsilon_1, \epsilon_2) = e_1 + \epsilon_1 + \alpha \epsilon_2
\]

\[
q_2(e_2, \epsilon_1, \epsilon_2) = e_2 + \epsilon_2 + \alpha \epsilon_1
\]

where \( \epsilon_1 \) and \( \epsilon_2 \) are iid with \( \epsilon_i \sim N(0, \sigma^2) \).

Three cases:

- \( \alpha = 0 \): Technologically independent outputs
- \( \alpha > 0 \): Positively correlated outputs
- \( \alpha < 0 \): Negatively correlated outputs
Model III

- Principal is risk-neutral. \( V(q_1, q_2, w) = E[q_1 + q_2 - w_1 - w_2] \)

- Agents are risk-averse. \( u_i(w_i, e) = -e^{-r_i(w_i - \psi_i(e))}, r_i > 0 \), where

- \( r_i = -\frac{u_i''}{u_i'} > 0 \), i.e., CARA, and

- \( \psi_i(e) = \frac{1}{2} c_i e^2 \) is the (money) cost of effort \( e \) by agent \( i \).

- \( e \) is not contractible but \( q_i \)'s are.

For simplicity assume:

- \( r_1 = r_2 = r \), and \( c_1 = c_2 = c \), as a result, \( \psi_1(.) = \psi_2(.) = \psi(.) = \frac{1}{2} ce^2 \)

- Linear Contracts:

\[
\begin{align*}
w_1(q_1, q_2) &= t_1 + s_1 q_1 + \tilde{s}_1 q_2 \\
w_2(q_1, q_2) &= t_2 + s_2 q_2 + \tilde{s}_2 q_1
\end{align*}
\]
\( \tilde{s}_1 = 0 \) and \( \tilde{s}_2 = 0 \) will imply no relative performance evaluation.

When is it optimum to have \( \tilde{s}_1 \neq 0 \) and \( \tilde{s}_2 \neq 0 \)?
Model V

**Second Best:** The principal will solve

$$\max \ E(\sum q_i - \sum w_i)$$

However, since the agents are assumed to be identical, for each agent the principal solves

$$\max \ E(q_i - w_i)$$

say

$$\max \ E(q_1 - w_1)$$

s.t.

$$E(u_1(w_1, e_1)) = E(-e^{-r(w_1 - \psi(e_1))}) \geq -e^{-r(\bar{w})} = u(\bar{w}) \quad (IR)$$

$$e_1 = \arg \max_{e} E(-e^{-r(w_1 - \psi(e))}) \quad (IC)$$
Model VI

e_1 is the effort chosen by first agent. Let’s define

\[-e^{-r\hat{w}_1(e)} = E(-e^{-r(w_1 - \psi_1(e)))}\]

\[\hat{w}_1(e) = \underbrace{\text{certainty-equivalent wage}}_{\text{expected wage}} - \underbrace{\text{effort cost}}_{\text{risk-premium}}\]

Note that:

\[w_1(q_1, q_2) = t_1 + s_1 q_1 + \tilde{s}_1 q_2\]
\[= t_1 + s_1 (e_1 + \epsilon_1 + \alpha \epsilon_2) + \tilde{s}_1 (e_2 + \epsilon_2 + \alpha \epsilon_1)\]
\[= t_1 + s_1 e_1 + \tilde{s}_1 e_2 + s_1 (\epsilon_1 + \alpha \epsilon_2) + \tilde{s}_1 (\epsilon_2 + \alpha \epsilon_1)\]

Therefore,
\[ Var[w_1(q_1, q_2)] = Var[s_1(\epsilon_1 + \alpha \epsilon_2) + \tilde{s}_1(\epsilon_2 + \alpha \epsilon_1)], \text{ i.e.,} \]
\[ = Var[(s_1 + \alpha \tilde{s}_1)\epsilon_1 + (\tilde{s}_1 + \alpha s_1)\epsilon_2], \text{ i.e.,} \]
\[ = \sigma^2[(s_1 + \alpha \tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2]. \]

The two agents will choose efforts independently in a N.E.

For given \( e_2 \) opted by the second agent, the certainty equivalent payoff of the first agent is a function of his effort level \( e_1 \) and is given by

\[ \hat{w}_1(e) = E(w_1(q_1, q_2)) - \frac{1}{2} ce^2 - \frac{r\sigma^2}{2}[(s_1 + \alpha \tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2], \text{ i.e.,} \]

in view of \( w_1(q_1, q_2) = t_1 + s_1 q_1 + \tilde{s}_1 q_2; \ q_1(e_1, \epsilon_1, \epsilon_2) = e_1 + \epsilon_1 + \alpha \epsilon_2, \) and \( q_2(e_2, \epsilon_1, \epsilon_2) = e_2 + \epsilon_2 + \alpha \epsilon_1, \) we have

\[ E(w_1(q_1, q_2)) = t_1 + s_1 e + \tilde{s}_1 e_2. \]
Model VIII

Therefore,

\[ \hat{w}_1(e) = t_1 + s_1 e + \tilde{s}_1 e_2 - \frac{1}{2} ce_1^2 - \frac{r \sigma^2}{2} [ (s_1 + \alpha \tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2 ] \]  

(1)

So, given \( e_2 \), the agent 1 will solve

\[ \max_{e} \{ t_1 + s_1 e + \tilde{s}_1 e_2 - \frac{1}{2} ce_1^2 - \frac{r \sigma^2}{2} [ (s_1 + \alpha \tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2 ] \} \]  

(2)

That is, \( e_1^{SB} \) solves the following foc

\[ e_1^{SB} = \frac{s_1}{c} \]  

(3)

Now from (1) and (3), we get

\[ \hat{w}_1(e_1^{SB}) = t_1 + \frac{s_1^2}{2c} + \frac{\tilde{s}_1 s_2}{c} - \frac{r \sigma^2}{2} [ (s_1 + \alpha \tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2 ] \]  

(4)
Now, in view of $e_{1}^{SB} = \frac{s_1}{c}$, the P’s problem can be written as

$$\max_{t_1, \tilde{s}_1, s_1} \left\{ \frac{s_1}{c} - (t_1 + \frac{s_1^2}{c} + \frac{\tilde{s}_1 s_2}{c}) \right\}$$

s.t.

$$\hat{w}_1 = t_1 + \frac{s_1^2}{2c} + \frac{\tilde{s}_1 s_2}{c} - \frac{r \sigma^2}{2} \left[ (s_1 + \alpha \tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2 \right] = \bar{w}_1$$  \hspace{1cm} (5)

Using the value of $t_1$ from (5) and ignoring $\bar{w}_1$, the P’s problem can be rewritten as

$$\max_{\tilde{s}_1, s_1} \left\{ \frac{s_1}{c} - \frac{s_1^2}{2c} - \frac{r \sigma^2}{2} \left[ (s_1 + \alpha \tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2 \right] \right\}$$

**Remark:** Note: For given $s_1$, optimizing the above w.r.t. $\tilde{s}_1$ is equivalent to solving

$$\min_{\tilde{s}_1} \left\{ \frac{r \sigma^2}{2} \left[ (s_1 + \alpha \tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2 \right] \right\}$$
So, for given $s_1$, $\tilde{s}_1^{SB}$ solves the following the following foc

$$\tilde{s}_1^{SB} = -\left(\frac{2\alpha}{1 + \alpha^2}\right)s_1$$  \hspace{1cm} (6)

In view of (6), the P’s problem reduces to

$$\max_{s_1} \left\{ \frac{s_1}{c} - \frac{s_1^2}{2c} - \frac{r\sigma^2}{2} \frac{s_1^2 (1 - \alpha^2)^2}{(1 + \alpha^2)} \right\}$$

So, the $s_1^{SB}$ solves the following foc

$$s_1^{SB} = \frac{1 + \alpha^2}{1 + \alpha^2 + r c \sigma^2 (1 - \alpha^2)^2}$$  \hspace{1cm} (7)
Remark

- From (6) note that $\alpha = 0 \Rightarrow \tilde{s}_{SB}^{1} = 0$ and $\alpha = 0 \Rightarrow s_{SB}^{1} = \frac{1}{1 + r_c \sigma^2}$. That is, if the outputs are technologically independent than relative performance evaluation is not optimum. Why?

- $\alpha > 0 \Rightarrow \tilde{s}_{SB}^{1} < 0$, i.e., an agent is penalized[rewarded] when the other individual’s performance is higher[lower].

- However, $\alpha < 0 \Rightarrow \tilde{s}_{SB}^{1} > 0$. In this case, an agent is compensated[penalized] when the other agent’s performance is higher [lower].
Remark

- From (6), \((\alpha \to 1) \Rightarrow [\tilde{s}_{1}^{SB} = -s_{1}^{SB}]\) and from (7), \((\alpha \to 1) \Rightarrow [s_{1}^{SB} = 1].\)

- When \(\alpha = 1\), there is a common shock affects the two performances. In this case, the relative performance evaluation allows filtering out of the common shock.

- Therefore, the FB can be implemented even with risk-averse agents.

Question

Suppose \(\alpha = 1\). Can the agents collude and choose \(e = 0\) each?