Relative Performance Evaluation

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March 11, 2015

Model I

Relative Performance Evaluations are widely used when individual performances can be observed:

- At School: In grading, ranking etc.
- At Work: In promotions, hiring and firing, etc.
- In Sports: In declaring winner, runners-up etc.

Question

- *Is relative performance evaluation efficient?*
- *Should the wage/reward be based only on the absolute value of the output, or also on the relative ranking of performances?*

Consider:

- One Principal and Two agents and Two outputs

Model II

- Two agents produce Two (possibly different) individually observable outputs
- Principal is Risk-neutral but agents are Risk-averse with CARA preferences

The production technology: $Q = q_1 + q_2$, where

$$\begin{aligned} q_1(e_1, \epsilon_1, \epsilon_2) &= e_1 + \epsilon_1 + \alpha\epsilon_2 \\ q_2(e_2, \epsilon_1, \epsilon_2) &= e_2 + \epsilon_2 + \alpha\epsilon_1 \end{aligned}$$

where ϵ_1 and ϵ_2 are *iid* with $\epsilon_i \sim N(0, \sigma^2)$.
Three cases:

- $\alpha = 0$: Technologically independent outputs
- $\alpha > 0$: Positively correlated outputs
- $\alpha < 0$: Negatively correlated outputs

Model III

- Principal is risk-neutral. $V(q_1, q_2, w) = E[q_1 + q_2 - w_1 - w_2]$
- Agents are risk-averse. $u_i(w_i, e) = -e^{-r_i(w_i - \psi_i(e))}$, $r_i > 0$, where
- $r_i = -\frac{u_i''}{u_i'} > 0$, i.e., CARA, and
- $\psi_i(e) = \frac{1}{2}c_i e^2$ is the (money) cost of effort e by agent i.
- e is not contractible but q_is are.

For simplicity assume:

- $r_1 = r_2 = r$, and $c_1 = c_2 = c$, as a result, $\psi_1(.) = \psi_2(.) = \psi(.) = \frac{1}{2}ce^2$
- Linear Contracts:

$$
\begin{aligned}
w_1(q_1, q_2) &= t_1 + s_1 q_1 + \tilde{s}_1 q_2 \\
w_2(q_1, q_2) &= t_2 + s_2 q_2 + \tilde{s}_2 q_1
\end{aligned}
$$

Model IV

- $\tilde{s}_1 = 0$ and $\tilde{s}_2 = 0$ will imply no relative performance evaluation.
- When is it optimum to have $\tilde{s}_1 \neq 0$ and $\tilde{s}_2 \neq 0$?

Model V

Second Best: The principal will solve

$$\max_{s_i, \tilde{s}_i, t_i} E\left(\sum q_i - \sum w_i\right)$$

However, since the agents are assumed to be identical, for each agent the principal solves

$$\max_{s_i, \tilde{s}_i, t_i} E(q_i - w_i)$$

say

$$\max_{s_1, \tilde{s}_1, t_1} E(q_1 - w_1)$$

s.t.

$$\begin{aligned} E(u_1(w_1, e_1)) = E(-e^{-r(w_1 - \psi(e_1))}) &\geq -e^{-r(\bar{w})} = u(\bar{w}) && (IR) \\ e_1 &= \arg\max_{e} E(-e^{-r(w_1 - \psi(e))}) && (IC) \end{aligned}$$

Model VI

e_1 is the effort chosen by first agent. Let's define

$$-e^{-r\hat{w}_1(e)} = E(-e^{-r(w_1-\psi_1(e))})$$

$$\underbrace{\hat{w}_1(e)}_{certainty-equivalent\ \ wage} = \underbrace{\cdot}_{expected\ \ wage} - \underbrace{\cdot}_{effort\ \ cost} - \underbrace{\cdot}_{risk-premium}$$

Note that:

$$\begin{aligned} w_1(q_1, q_2) &= t_1 + s_1 q_1 + \tilde{s}_1 q_2 \\ &= t_1 + s_1(e_1 + \epsilon_1 + \alpha\epsilon_2) + \tilde{s}_1(e_2 + \epsilon_2 + \alpha\epsilon_1) \\ &= t_1 + s_1 e_1 + \tilde{s}_1 e_2 + s_1(\epsilon_1 + \alpha\epsilon_2) + \tilde{s}_1(\epsilon_2 + \alpha\epsilon_1) \end{aligned}$$

Therefore,

Model VII

$$
\begin{aligned}
Var[w_1(q_1, q_2)] &= Var[s_1(\epsilon_1 + \alpha\epsilon_2) + \tilde{s}_1(\epsilon_2 + \alpha\epsilon_1)], i.e., \\
&= Var[(s_1 + \alpha\tilde{s}_1)\epsilon_1 + (\tilde{s}_1 + \alpha s_1)\epsilon_2], i.e., \\
&= \sigma^2[(s_1 + \alpha\tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2].
\end{aligned}
$$

The two agents will choose efforts independently in a N.E.

For given e_2 opted by the second agent, the certainty equivalent payoff of the first agent is a function of his effort level e_1 and is given by

$$\hat{w}_1(e) = E(w_1(q_1, q_2)) - \frac{1}{2}ce^2 - \frac{r\sigma^2}{2}[(s_1 + \alpha\tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2], i.e.,$$

in view of $w_1(q_1, q_2) = t_1 + s_1 q_1 + \tilde{s}_1 q_2;\; q_1(e_1, \epsilon_1, \epsilon_2) = e_1 + \epsilon_1 + \alpha\epsilon_2$, and $q_2(e_2, \epsilon_1, \epsilon_2) = e_2 + \epsilon_2 + \alpha\epsilon_1$, we have

$$E(w_1(q_1, q_2)) = t_1 + s_1 e + \tilde{s}_1 e_2.$$

Model VIII

Therefore,

$$\hat{w}_1(e) = t_1 + s_1 e + \tilde{s}_1 e_2 - \frac{1}{2} c e_1^2 - \frac{r\sigma^2}{2}[(s_1 + \alpha\tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2] \quad (1)$$

So, given e_2, the agent 1 will solve

$$\max_e \{t_1 + s_1 e + \tilde{s}_1 e_2 - \frac{1}{2} c e_1^2 - \frac{r\sigma^2}{2}[(s_1 + \alpha\tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2]\} \quad (2)$$

That is, e_1^{SB} solves the following foc

$$e_1^{SB} = \frac{s_1}{c} \quad (3)$$

Now from (1) and (3), we get

$$\hat{w}_1(e_1^{SB}) = t_1 + \frac{s_1^2}{2c} + \frac{\tilde{s}_1 s_2}{c} - \frac{r\sigma^2}{2}[(s_1 + \alpha\tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2] \quad (4)$$

Model IX

Now, in view of $e_1^{SB} = \frac{s_1}{c}$, the P's problem can be written as

$$\max_{t_1, \tilde{s}_1, s_1} \{\frac{s_1}{c} - (t_1 + \frac{s_1^2}{c} + \frac{\tilde{s}_1 s_2}{c})\}$$

s.t.

$$\hat{w}_1 = t_1 + \frac{s_1^2}{2c} + \frac{\tilde{s}_1 s_2}{c} - \frac{r\sigma^2}{2}[(s_1 + \alpha\tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2] = \bar{w}_1 \tag{5}$$

Using the value of t_1 from (5) and ignoring $\bar{w}_1$, the P's problem can be rewritten as

$$\max_{\tilde{s}_1, s_1}\{\frac{s_1}{c} - \frac{s_1^2}{2c} - \frac{r\sigma^2}{2}[(s_1 + \alpha\tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2]\}$$

Remark: Note: For given s_1, optimizing the above w.r.t. $\tilde{s}_1$ is equivalent to solving

$$\min_{\tilde{s}_1}\{\frac{r\sigma^2}{2}[(s_1 + \alpha\tilde{s}_1)^2 + (\tilde{s}_1 + \alpha s_1)^2]\}$$

Model X

So, for given s_1, $\tilde{s}_1^{SB}$ solves the following the following foc

$$\tilde{s}_1^{SB} = -\left(\frac{2\alpha}{1+\alpha^2}\right)s_1 \tag{6}$$

In view of (6), the P's problem reduces to

$$\max_{s_1}\left\{\frac{s_1}{c} - \frac{s_1^2}{2c} - \frac{r\sigma^2}{2}s_1^2\frac{(1-\alpha^2)^2}{(1+\alpha^2)}\right\}$$

So, the s_1^{SB} solves the following foc

$$s_1^{SB} = \frac{1+\alpha^2}{1+\alpha^2+rc\sigma^2(1-\alpha^2)^2} \tag{7}$$

Model XI

Remark

- *From* (6) *note that* $\alpha = 0 \Rightarrow \tilde{s}_1^{SB} = 0$ *and* $\alpha = 0 \Rightarrow s_1^{SB} = \frac{1}{1+rc\sigma^2}$. *That is, if the outputs are technologically independent than relative performance evaluation is not optimum. Why?*
- $\alpha > 0 \Rightarrow \tilde{s}_1^{SB} < 0$, *i.e., an agent is penalized[rewarded] when the other individual's performance is higher[lower].*
- *However,* $\alpha < 0 \Rightarrow \tilde{s}_1^{SB} > 0$. *In this case, an agent is compensated[penalized] when the other agent's performance is higher [lower].*

Model XII

Remark

- *From* (6), $(\alpha \to 1) \Rightarrow [\tilde{s}_1^{SB} = -s_1^{SB}]$ *and from* (7), $(\alpha \to 1) \Rightarrow [s_1^{SB} = 1]$.
- *When* $\alpha = 1$, *there is a common shock affects the two performances. In this case, the relative performance evaluation allows filtering out of the common shock.*
- *Therefore, the FB can be implemented even with risk-averse agents.*

Question

Suppose $\alpha = 1$. *Can the agents collude and choose* $e = 0$ *each?*